3.3.2 The basic construction

Relatively independent joinings over the trivial σ-subalgebra corresponds to taking a diagonal action on a product space. We have seen previously that a useful tool in analyzing the structure of product actions $\Gamma \curvearrowright X \times X$ was the identification between $L^2(X \times X)$ and the Hilbert-Schmidt operators on $L^2(X)$. This allowed us to use tools such as functional calculus. There is an analog of the Hilbert-Schmidt operators in the setting of relatively independent joinings which we will now describe.

Suppose $(X, \mathcal{B}, \mu)$ is a probability space and $\mathcal{A} \subset \mathcal{B}$ is a σ-subalgebra. The **basic construction** associated to the inclusion $L^\infty(X, \mathcal{A}, \mu) \subset L^\infty(X, \mathcal{B}, \mu)$ is the algebra $L^\infty(X, \mathcal{A}, \mu)' \subset \mathcal{B}(L^2(X, \mathcal{B}, \mu))$, of operators in $\mathcal{B}(L^2(X, \mathcal{B}, \mu))$ which commute with $L^\infty(X, \mathcal{A}, \mu)$. We will denote the basic construction by $\langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A} \rangle$, where $e_\mathcal{A} \in \mathcal{B}(L^2(X, \mathcal{B}, \mu))$ is the orthogonal projection onto $L^2(X, \mathcal{A}, \mu)$.

Note that we have

$$L^\infty(X, \mathcal{B}, \mu) e_\mathcal{A} L^\infty(X, \mathcal{B}, \mu) = \mathrm{sp}\{\mathrm{f e_\mathcal{A} g} \mid \mathrm{f, g} \in \mathrm{L^\infty(X, \mathcal{B}, \mu)}\} \subset \langle \mathrm{L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}} \rangle,$$

and that this is an algebra since if $f \in L^\infty(X, \mathcal{B}, \mu)$ we have

$$e_\mathcal{A} f e_\mathcal{A} = E_\mathcal{A}(f) e_\mathcal{A}.$$

Note that we distinguish here the the projection $e_\mathcal{A}$, which is an operator on $L^2(X, \mathcal{A}, \mu)$, from the conditional expectation $E_\mathcal{A}$, which is an operator on $L^\infty(X, \mathcal{A}, \mu)$.

Exercise 3.3.1. Suppose $S \in \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A} \rangle$, has polar decomposition $S = V|S|$. Show that $V \in \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A} \rangle$.

Lemma 3.3.2. *Suppose $(X, \mathcal{B}, \mu)$ is a probability space and $\mathcal{A} \subset \mathcal{B}$ is a σ-subalgebra. For each $S \in \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A} \rangle$ there exists a unique $\phi(S) \in L^\infty(X, \mathcal{A}, \mu)$ such that*

$$e_\mathcal{A} S e_\mathcal{A} = \phi(S) e_\mathcal{A}.$$

Moreover, the map $S \mapsto \phi(S)$ is a unital, positivity preserving extension of $E_\mathcal{A}$, which is $L^\infty(X, \mathcal{A}, \mu)$-bimodular, and continuous with respect to the weak operator topology.

Proof. By Lemma **??** $L^\infty(X, \mathcal{A}, \mu)$ is a maximal abelian subalgebra of $\mathcal{B}(L^2(X, \mathcal{A}, \mu))$. Thus, since $e_\mathcal{A} S e_\mathcal{A}$ restricted to $L^2(X, \mathcal{A}, \mu)$ commutes $L^\infty(X, \mathcal{A}, \mu)$, there exists a unique element $\phi(S) \in L^\infty(X, \mathcal{A}, \mu)$ such that $e_\mathcal{A} S e_\mathcal{A} f = \phi(S) f = \phi(S) e_\mathcal{A} f$, for all $f \in L^2(X, \mathcal{A}, \mu)$. If $f \in L^2(X, \mathcal{A}, \mu)^\perp \subset L^2(X, \mathcal{B}, \mu)$ then we have $e_\mathcal{A} S e_\mathcal{A} f = 0 = \phi(S) e_\mathcal{A}$.

That $x \mapsto \phi(x)$ is unital, positivity preserving, and weak operator topology continuous, follows from the fact that it is the composition of the map $x \mapsto e_\mathcal{A} x e_\mathcal{A}$ and the $*$-homomorphism $a e_\mathcal{A} \mapsto a$. $\square$

Exercise 3.3.3 (Generalized Cauchy-Schwartz inequality)**.** Prove that for all $x, y \in \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$ we have

$$|\phi_\mathcal{A}(y^*x)|^2 \leq \phi_\mathcal{A}(y^*y)\phi_\mathcal{A}(x^*x).$$

In the case where the σ-algebra $\mathcal{A}$ is trivial we have that $e_\mathcal{A}$ is the rank 1 projection on to the subspace $\mathbb{C}1 \subset L^2(X, \mathcal{B}, \mu)$, and thus operators of the form $fe_\mathcal{A}g$ were rank 1 projections. Rather than working with a Hilbert space basis $\{\xi_i\} \subset L^2(X, \mathcal{B}, \mu)$ as before, we could have just as easily worked with the family of partial isometries from $\mathbb{C}1$ to $\mathbb{C}\xi_i$. This motivates the following lemma.

Lemma 3.3.4. *Suppose $(X, \mathcal{B}, \mu)$ is a probability space and $\mathcal{A} \subset \mathcal{B}$ is a σ-subalgebra. There exists a family of partial isometries $\{v_i\}_{i\in I} \subset \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$ such that*

$$(a). \quad v_iv_i^* \leq e_\mathcal{A}, \quad \text{for all } i \in I;$$

$$(b). \quad v_iv_j^* = 0, \quad \text{for all } i, j \in I, i \neq j;$$

$$(c). \quad \Sigma_{i\in I} v_i^* e_\mathcal{A} v_i = 1.$$

Proof. A simple argument with Zorn's Lemma shows that there is a maximal (with respect to inclusion) family of partial isometries $\{v_i\}_{i\in I}$ satisfying conditions (a) and (b) above.

Let $P = \Sigma_{i\in I} v_i^* e_\mathcal{A} v_i$, and consider $S \in (1 - P) \cdot \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle \cdot e_\mathcal{A}$. By considering the polar decomposition $S = V|S|$, we have that $V^*V = \text{Proj}_{\overline{\text{Range}(S)}} \leq e_\mathcal{A}$, $VV^* = \text{Proj}_{\overline{\text{Range}(S^*)}} \leq (1 - P)$, and $V \in \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$. Thus by maximality of the family $\{v_i\}_{i\in I}$ we must have that $V = 0$, and hence $S = 0$.

Thus $\{0\} = (1-P)\cdot\langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle\cdot e_\mathcal{A}\cdot L^2(X, \mathcal{B}, \mu) \supset (1-P)\cdot L^\infty(X, \mathcal{B}, \mu)\cdot 1$. Since $L^\infty(X, \mathcal{B}, \mu) \cdot 1$ is dense in $L^2(X, \mathcal{B}, \mu)$ this shows that $P = 1$. □

A family of partial isometries which satisfy the conditions above will be called an **operator basis** for $\langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$.

Definition 3.3.5. Suppose $(X, \mathcal{B}, \mu)$ is a probability space and $\mathcal{A} \subset \mathcal{B}$ is a σ-subalgebra. Let $\{v_i\}_{i\in I}$ be an operator basis for $\langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$. An operator $S \in \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$ is **Hilbert-Schmidt class** with Hilbert-Schmidt norm if

$$\|S\|_{\text{HS}}^2 = \Sigma_{i\in I} \int \phi_\mathcal{A}(v_iS^*Sv_i^*)\, d\mu < \infty.$$

The quantity $\|S\|_{\text{HS}}$ is the Hilbert-Schmidt norm of S.

This definition does not depend on the operator basis $\{v_i\}_{i\in I}$, this can be seen from the following analogue of Perseval's identity. If $\{w_j\}_{j\in J}$ is another operator basis, then we have

$$\Sigma_{i\in I} \int \phi_\mathcal{A}(v_iS^*Sv_i^*)\, d\mu = \Sigma_{i\in I}\Sigma_{j\in J} \int \phi_\mathcal{A}(v_iS^*(w_j^*e_\mathcal{A}w_j)Sv_i^*)\, d\mu$$

$$= \Sigma_{i\in I}\Sigma_{j\in J} \int \phi_\mathcal{A}(v_iS^*w_j^*)\phi_\mathcal{A}(w_jSv_i^*)\, d\mu$$

$$= \Sigma_{j \in J} \Sigma_{i \in I} \int \phi_{\mathcal{A}}(v_j S v_i) \phi_{\mathcal{A}}(v_i S^* w_j^*) \, d\mu$$

$$= \Sigma_{j \in J} \int \phi_{\mathcal{A}}(w_j S^* S w_j^*).$$

Also note that it follows that we may approximate S in the Hilbert-Schmidt norm with finite sums of the form $\Sigma_{i,j} w_j^* (w_j S v_i^*) v_i$.

If $T \in \langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}} \rangle$ then $T^* T \leq \|T\|^2$, and since $\phi_{\mathcal{A}}$ is positivity preserving it follows that

$$\|TS\|_{\text{HS}}^2 = \Sigma_{i \in I} \int \phi_{\mathcal{A}}(v_i S^* T^* T S v_i^*) \, d\mu \leq \|T\|^2 \|S\|_{\text{HS}}^2.$$

Also, it follows from the argument above that the adjoint operator $S \mapsto S^*$ is an anti-linear isometry, and hence we also have

$$\|ST\|_{\text{HS}}^2 \leq \|T\|^2 \|S\|_{\text{HS}}^2.$$

In particular, we see that the Hilbert-Schmidt class in $\langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}} \rangle$ is a two sided ideal.

Exercise 3.3.6. Given a Hilbert-Schmidt class operator $S \in \langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}} \rangle$ show that $\|S\|_{\text{HS}} = 0$ if and only if $S = 0$.

The Hilbert-Schmidt norm has an associated inner product

$$\langle S, T \rangle_{\text{HS}} = \Sigma_{i \in I} \int \phi_{\mathcal{A}}(v_i T^* S v_i^*) \, d\mu,$$

which is well defined by the generalized Cauchy-Schwartz inequality, and does not depend on the operator basis from the arguments above.

Thus, the class of Hilbert-Schmidt operators is an inner-product space. This space is not complete in general[1], we denote the Hilbert space completion by $L^2 \langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}} \rangle$.

Even though the class of Hilbert-Schmidt operators is not a complete space in general we do have that it is complete when we restrict to the Hilbert-Schmidt operators whose uniform norm is bounded by some fixed constant.

Proposition 3.3.7. *Suppose $(X, \mathcal{B}, \mu)$ is a probability space and $\mathcal{A} \subset \mathcal{B}$ is a σ-subalgebra. Suppose $K > 0$ and consider the convex set*

$$B_K = \{ S \in \langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}} \rangle \mid \|S\| \leq K \}.$$

Then B_K is complete in the Hilbert-Schmidt norm.

[1] Consider the case when $\mathcal{A} = \mathcal{B}$, then it is easy to see that the class of Hilbert-Schmidt operators coincides with $L^\infty(X, \mathcal{B}, \mu)$, and the inner-product structure is the usual inner-product on $L^2(X, \mathcal{B}, \mu)$.

Proof. Fix an operator basis $\{v_i\}_{i\in I}$ for $\langle L^\infty(X,\mathcal{B},\mu),e_\mathcal{A}\rangle$. Observe that if $f\in L^\infty(X,\mathcal{A},\mu)$ then we have

$$\|fe_\mathcal{A}\|_2^2=\Sigma_{i\in I}\int\phi_\mathcal{A}(v_ie_\mathcal{A}|f|^2e_\mathcal{A})\,d\mu$$

$$=\Sigma_{i\in I}|f|^2\phi_\mathcal{A}(v_i)\phi_\mathcal{A}(v_i^*)\,d\mu=\Sigma_{i\in I}|f|^2\phi_\mathcal{A}(v_i^*e_\mathcal{A}v_i)\,d\mu=\|f\|_2^2.$$

In particular, this shows that $\phi_\mathcal{A}$ is a contraction with respect to the Hilbert-Schmidt norm. Thus, if $S_n\in B_K$ is Cauchy in the Hilbert-Schmidt norm, then for all $i,j\in I$, $\phi(v_jS_nv_i^*)\in L^\infty(X,\mathcal{A},\mu)$ is Cauchy in $L^2(X,\mathcal{A},\mu)$ and we also have that $\|\phi(v_jS_nv_i^*)\|_\infty\leq K$. Hence, there exists $g_{i,j}\in L^\infty(X,\mathcal{A},\mu)$ such that $\|g_{i,j}\|_\infty\leq K$, and

$$\lim_{n\to\infty}\|v_j^*(v_jS_nv_i^*)v_i-v_j^*g_{i,j}v_i\|_{\mathrm{HS}}=\lim_{n\to\infty}\|v_j^*\phi(v_jS_nv_i^*)e_\mathcal{A}v_i-v_j^*g_{i,j}v_i\|_{\mathrm{HS}}$$

$$\leq\lim_{n\to\infty}\|\phi(v_jS_nv_i^*)e_\mathcal{A}-g_{i,j}e_\mathcal{A}\|_{\mathrm{HS}}=0.$$

Since $\mathrm{Range}(v_j^*)$ are pairwise orthogonal subspaces we may then consider the sum

$$S=\Sigma_{i,j\in I}v_j^*g_{i,j}v_i\in\mathcal{B}(L^2(X,\mathcal{B},\mu)).$$

Then $\|S\|\leq K$ and it is easy to see that $S\in\langle L^\infty(X,\mathcal{B},\mu),e_\mathcal{A}\rangle$. More over for any finite set $I_0\subset I$ it follows from the triangle inequality that

$$\lim_{n\to\infty}\|\Sigma_{i,j\in I_0}v_j^*(v_jS_nv_i^*)v_i-v_j^*(v_jSv_i^*)v_i\|_{\mathrm{HS}}=0.$$

Since S_n is Cauchy in the Hilbert-Schmidt norm this implies that S is Hilbert-Schmidt class and $\|S_n-S\|_{\mathrm{HS}}\to 0$. □

If $\Gamma\curvearrowright(X,\mathcal{B},\mu)$ is a measure preserving action of a countable group Γ, such that $\mathcal{A}$ is Γ-invariant, then we may consider the relatively independent self joining $\Gamma\curvearrowright X\times_\mathcal{A}X$ defined above. We may then define a map $\Xi:L^\infty(X,\mathcal{B},\mu)\otimes_\mathcal{A}L^\infty(X,\mathcal{B},\mu)\to L^\infty(X,\mathcal{B},\mu)e_\mathcal{A}L^\infty(X,\mathcal{B},\mu)$ by linearly extending the formula

$$\Xi(b\otimes a)=be_\mathcal{A}a,$$

for all $a,b\in L^\infty(X,\mathcal{B},\mu)$.

If $\{v_i\}_{i\in I}$ is an operator basis for $\langle L^\infty(X,\mathcal{B},\mu),e_\mathcal{A}\rangle$, we then have that for all $\Sigma_{k=1}^n b_k\otimes a_k\in L^\infty(X,\mathcal{B},\mu)\otimes_\mathcal{A}L^\infty(X,\mathcal{B},\mu)$

$$\|\Sigma_{k=1}^n b_k\otimes a_k\|_{L^2(X\times_\mathcal{A}X)}^2=\int\Sigma_{k,l=1}^n E_\mathcal{A}(b_k^*b_l)a_la_k^*\,d\mu$$

$$=\int\Sigma_{k,l=1}^n\phi_\mathcal{A}(b_k^*b_l)\phi_\mathcal{A}(a_la_k^*)\,d\mu$$

$$=\Sigma_{i,j\in I}\int\Sigma_{k,l=1}^n\phi_\mathcal{A}(b_k^*v_i^*e_\mathcal{A}v_ib_l)\phi_\mathcal{A}(a_lv_j^*e_\mathcal{A}v_ja_k^*)\,d\mu$$

$$= \Sigma_{i,j\in I} \int \Sigma_{k,l=1}^n \phi_{\mathcal{A}}(b_k^* v_i^*)\phi_{\mathcal{A}}(v_i b_l)\phi_{\mathcal{A}}(a_l v_j^*)\phi_{\mathcal{A}}(v_j a_k^*)\, d\mu$$

$$= \Sigma_{i,j\in I} \int \Sigma_{k,l=1}^n \phi_{\mathcal{A}}(v_i b_l e_{\mathcal{A}} a_l v_j^*)\phi_{\mathcal{A}}(v_j a_k^* e_{\mathcal{A}} b_k^* v_i^*)\, d\mu$$

$$= \|\Xi(\Sigma_{k=1}^n b_k \otimes a_k)\|_{\mathrm{HS}}^2.$$

Hence Ξ is well defined and extends to a unitary operator (which we will also denote by Ξ) from $L^2(X \times_{\mathcal{A}} X)$ to $L^2\langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}}\rangle$.

Moreover, this unitary implements an equivalence between the Koopman representation of $\Gamma \curvearrowright X \times_{\mathcal{A}} X$ and the representation of Γ on $L^2\langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}}\rangle$, given by $S \mapsto \sigma_\gamma S \sigma_{\gamma^{-1}}$.

Suppose $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ are two probability measure preserving actions and we have Γ-equivariant embeddings $\alpha : L^\infty(Z_0, \mathcal{C}, \eta) \to L^\infty(X, \mathcal{B}, \mu)$ and $\beta : L^\infty(Z_0, \mathcal{C}, \eta) \to L^\infty(Y, \mathcal{A}, \nu)$. We may consider the class of operators $S \in \mathcal{B}(L^2(X, \mathcal{B}, \mu), L^2(Y, \mathcal{A}, \mu))$ such that $S\alpha(g) = \beta(g)S$ and $|S^*S|^{1/2}$ is in the Hilbert-Schmidt class of $\langle L^\infty(X, \mathcal{B}, \mu), e_{\alpha(\mathcal{C})}\rangle$. In this case we can consider the norm given by $\|S\|_{\mathrm{HS}} = \||S^*S|^{1/2}\|_{\mathrm{HS}}$, and consider the completion under this norm.

It then follows that this is a Hilbert space, and we may consider the map Ξ which linearly extends the formula $\Xi(b \otimes a) = b e_{\beta(\mathcal{C})=\alpha(\mathcal{C})} a$ (here we view $e_{\beta(\mathcal{C})=\alpha(\mathcal{C})}$ as an operator from $L^2(X, \mathcal{B}, \mu)$ to $L^2(Y, \mathcal{A}, \nu)$). Then just as above Ξ extends to a unitary operator which implements an isomorphism between the Koopman representation $\Gamma \curvearrowright Y \times_{\beta(\mathcal{C})=\alpha(\mathcal{C})} X$ and the representation given by $S \mapsto \sigma_\gamma^{\mathcal{A}} S \sigma_{\gamma^{-1}}^{\mathcal{B}}$.

Exercise 3.3.8. Fill in the details to the previous paragraphs.

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