Having established Furstenberg's multiple recurrence for the cases of compact and weak mixing actions of $\mathbb{Z}$, it follows from Proposition 2.2.24 that every probability measure preserving action $\mathbb{Z} \curvearrowright (X, \mathcal{B}, \mu)$ has a non-trivial factor $\mathbb{Z} \curvearrowright (X, \mathcal{A}, \mu)$ for which Furstenberg's multiple recurrence holds. To establish this principle in the general case we must show how to extend this result to larger factors. The strategy for doing this is to replace $\mathbb{C}$ with $L^\infty(X, \mathcal{A}, \mu)$ and to mimic the previous constructions and proofs.

3.3.1 Joinings

Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ be two measure preserving action of a countable group Γ on probability spaces $(X, \mathcal{B}, \mu)$, and $(Y, \mathcal{A}, \nu)$. For the moment let us denote by $B = L^\infty(X, \mathcal{B}, \mu)$ and $A = L^\infty(Y, \mathcal{A}, \nu)$. Suppose $\phi : B \to A$ is a map which is positive ($\phi(f) \geq 0$, whenever $f \geq 0$), unital ($\phi(1) = 1$), preserves the integrals ($\int \phi(f)\, d\nu = \int f\, d\mu$, for all $f \in B$), and is Γ-equivariant ($\phi(\sigma_\gamma(f)) = \sigma_\gamma(\phi(f))$, for all $\gamma \in \Gamma$, and $f \in B$).

Then on the algebra $B \otimes_{\rm alg} A$ we obtain a state τ such that for $\Sigma_{j=1}^n b_j \otimes a_j \in B \otimes_{\rm alg} A$ we have

$$\tau(\Sigma_{j=1}^n b_j \otimes a_j) = \int \Sigma_{j=1}^n \phi(b_j) a_j \, d\mu.$$

Since ϕ is Γ-equivariant, it follows from Proposition 2.3.1 that there exists a probability space $(Z, \mathcal{C}, \eta)$ with a measure preserving action of Γ, and there exists a $*$-homomorphism $\pi : B \otimes_{\rm alg} A \to L^\infty(Z, \mathcal{C}, \eta)$ such that $\pi \circ \sigma_\gamma = \sigma_\gamma \circ \pi$, for all $\gamma \in \Gamma$, and $\int \pi(x)\, d\eta = \tau(x)$ for all $a \in B \otimes_{\rm alg} A$.

Because ϕ is unital, and preserves the integral, it follows that $\pi_{|B \otimes 1}$ and $\pi_{|1 \otimes A}$ injective and integral preserving. Thus, $L^\infty(Z, \mathcal{C}, \eta)$ contains isomorphic copies of $L^\infty(X, \mathcal{B}, \mu)$ and $L^\infty(Y, \mathcal{A}, \nu)$. Specifically, if we denote by $\mathcal{C}_X \subset \mathcal{C}$ (resp. $\mathcal{C}_Y \subset \mathcal{C}$) the σ-subalgebra generated by the image of $\pi_{|B \otimes 1}$ (resp. $\pi_{|1 \otimes A}$) then $\mathcal{C}_X$, and $\mathcal{C}_Y$ are Γ-invariant, and we have that $\pi_{|B \otimes 1}$, and $\pi_{|1 \otimes A}$ give Γ-eqivariant isomrophism of $L^\infty(X, \mathcal{B}, \mu)$ onto $L^\infty(Z, \mathcal{C}_X, \eta)$, and $L^\infty(Y, \mathcal{A}, \nu)$ onto $L^\infty(Z, \mathcal{C}_Y, \eta)$.

Moreover, after this identification, we recover the map ϕ by the formula $\phi(f) = E_{\mathcal{C}_Y}(f) \in L^\infty(Z, \mathcal{C}_Y, \eta)$, for all $f \in L^\infty(Z, \mathcal{C}_X, \eta)$. Indeed, this follows since if $b \in B$, and $a \in A$, then we have

$$\int \phi(b) a \, d\nu = \tau(b \otimes a)$$

$$= \int \pi(b)\pi(a)\, d\eta = \int E_{\mathcal{C}_Y}(\pi(b))\pi(a)\, d\eta.$$

Given $b \in B$, and $a \in A$, we will often abuse notation and denote by $b \otimes a$ the function $\pi(b \otimes a) \in L^\infty(Z, \mathcal{C}, \eta)$. One should be careful however with this abuse of notation because π need not be faithful in general.

Definition 3.3.1. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ be two measure preserving action of a countable group Γ on probability spaces $(X, \mathcal{B}, \mu)$, and $(Y, \mathcal{A}, \nu)$.

A **joining** of these two actions is a measure preserving action $\Gamma \curvearrowright (Z, \mathcal{C}, \eta)$ together with Γ-equivariant, integral preserving embeddings of $L^\infty(X, \mathcal{B}, \mu)$ and $L^\infty(Y, \mathcal{A}, \nu)$ in to $L^\infty(Z, \mathcal{C}, \eta)$, such that $\mathcal{C}$ is the σ-algebra generated by these embeddings.

If $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ is the same action as $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ then we say that a joining is a self joining of the action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$.

From the discussion above, joinings are in 1-1 correspondence with Γ-equivariant, unital, integral preserving, positive maps $\phi : L^\infty(X, \mathcal{B}, \mu) \to L^\infty(Y, \mathcal{A}, \nu)$.

If $\Gamma \curvearrowright (Z_0, \mathcal{C}, \eta)$ is a probability measure preserving action, and we have two probability measure preserving actions $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$, and Γ-equivariant embeddings $\alpha : L^\infty(Z_0, \mathcal{C}, \eta) \to L^\infty(X, \mathcal{B}, \mu)$ and $\beta : L^\infty(Z_0, \mathcal{C}, \eta) \to L^\infty(Y, \mathcal{A}, \nu)$, then we obtain a Γ-equivariant, positive, unital, integral preserving map from $L^\infty(X, \mathcal{B}, \mu)$ to $L^\infty(Y, \mathcal{A}, \nu)$ by first taking the conditional expection from $L^\infty(X, \mathcal{B}, \mu)$ to $\alpha(L^\infty(Z_0, \mathcal{C}, \eta))$ and then applying the isomorphism $\beta \circ \alpha^{-1}$. The joining corresponding to this map is called the **relatively independent joining** over $(Z_0, \mathcal{C}, \eta)$. We denote the new space on which Γ acts by $X \times_{\alpha(\mathcal{C}) = \beta(\mathcal{C})} Y$, or simply by $X \times_{\mathcal{C}} Y$ if the embeddings α and β are clear from the context. We also denote by $L^\infty(X, \mathcal{B}, \mu) \otimes^{\text{alg}}_{\alpha(\mathcal{C}) = \beta(\mathcal{C})} L^\infty(X, \mathcal{B}, \mu)$ the vector space generated by functions of the type $b \otimes a$ where $b \in B$, and $a \in A$.

A special case to consider is when $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ contains a Γ-invariant σ-subalgebra $\mathcal{A}$, and we have $(Z_0, \mathcal{C}, \eta) = (X, \mathcal{A}, \mu)$, then we may consider the relatively independent self joining over $(X, \mathcal{A}, \mu)$. Note, however that a relatively independent joinings consists not only of invariant σ-subalgebras, but also the ways in which we are including these subalgebras into the larger algebras. For instance, if $\alpha \in \text{Aut}(L^\infty(X, \mathcal{A}, \mu))$ is a Γ-equivariant automorphism, then we obtain a new relatively independent joining by considering the alternate embedding $\alpha : L^\infty(X, \mathcal{A}, \mu) \to L^\infty(X, \mathcal{A}, \mu) \subset L^\infty(X, \mathcal{B}, \mu)$. In general, the actions $\Gamma \curvearrowright X \times_{\mathcal{A}} X$ and $\Gamma \curvearrowright X \times_{\mathcal{A} = \alpha(\mathcal{A})} X$ need not be isomorphic.

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