

1.2 Functions of positive type

Definition 1.2.1. Given a group Γ , a **function of positive type** on Γ is a map $\varphi : \Gamma \rightarrow \mathbb{C}$ such that for all $\Sigma_{\gamma \in \Gamma} \alpha_\gamma u_\gamma \in \mathbb{C}\Gamma$ we have

$$\Sigma_{\gamma, \lambda \in \Gamma} \overline{\alpha_\lambda} \alpha_\gamma \varphi(\lambda^{-1} \gamma) \geq 0.$$

One way in which function of positive types appear is when we have a unitary representation $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{K})$, together with a vector $\xi \in \mathcal{K}$. Then function defined by:

$$\varphi_\xi(\gamma) = \langle \pi(\gamma) \xi, \xi \rangle, \text{ for all } \gamma \in \Gamma,$$

describes a function of positive type on Γ . Indeed, if $\Sigma_{\gamma \in \Gamma} \alpha_\gamma u_\gamma \in \mathbb{C}\Gamma$ then

$$\begin{aligned} \Sigma_{\gamma, \lambda \in \Gamma} \overline{\alpha_\lambda} \alpha_\gamma \varphi(\lambda^{-1} \gamma) &= \Sigma_{\gamma, \lambda \in \Gamma} \overline{\alpha_\lambda} \alpha_\gamma \langle \pi(\lambda^{-1} \gamma) \xi, \xi \rangle \\ &= \|\Sigma_{\gamma \in \Gamma} \alpha_\gamma \pi(\gamma) \xi\|^2 \geq 0. \end{aligned}$$

Every function of positive type arises in this way. Specifically, if $\varphi : \Gamma \rightarrow \mathbb{C}$ is a function of positive type then we may place a pseudo-inner product on $\mathbb{C}\Gamma$ by the formula:

$$\langle \Sigma_{\gamma \in \Gamma} \alpha_\gamma u_\gamma, \Sigma_{\lambda \in \Gamma} \beta_\lambda u_\lambda \rangle_\varphi = \Sigma_{\gamma, \lambda \in \Gamma} \overline{\beta_\lambda} \alpha_\gamma \varphi(\lambda^{-1} \gamma).$$

The positivity of this inner product is given by the fact that φ is of positive type. After quotienting by the kernel and completion we obtain a Hilbert space $\mathcal{K}$. Moreover, there is a natural unitary representation of Γ on $\mathcal{K}$ given by

$$\pi_\varphi(\gamma_0) \Sigma_{\gamma \in \Gamma} \alpha_\gamma u_\gamma = \Sigma_{\gamma \in \Gamma} \alpha_\gamma u_{\gamma_0 \gamma}.$$

The fact that this is well defined and describes a unitary follows from the fact that for all $\gamma_0 \in \Gamma$, and $\Sigma_{\gamma \in \Gamma} \alpha_\gamma u_\gamma \in \mathbb{C}\Gamma$ we have

$$\begin{aligned} \|\Sigma_{\gamma \in \Gamma} \alpha_\gamma u_{\gamma_0 \gamma}\|_\phi^2 &= \Sigma_{\gamma, \lambda \in \Gamma} \overline{\alpha_\lambda} \alpha_\gamma \phi((\gamma_0 \lambda)^{-1} (\gamma_0 \gamma)) \\ &= \Sigma_{\gamma, \lambda \in \Gamma} \overline{\alpha_\lambda} \alpha_\gamma \phi(\lambda^{-1} \gamma) = \|\Sigma_{\gamma \in \Gamma} \alpha_\gamma u_\gamma\|_\phi^2. \end{aligned}$$

One has to check that this action is well defined and preserves the inner product structure, and then the cyclic vector $\xi_\varphi = u_e$ gives the formula above.

If we start with a unitary representation $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$, with a non-zero vector $\xi \in \mathcal{H}$, consider the function of positive type φ_ξ , and then consider the representation π_{φ_ξ} . Then the map $\Sigma_{\gamma \in \Gamma} \alpha_\gamma u_\gamma \mapsto \Sigma_{\gamma \in \Gamma} \alpha_\gamma \pi(\gamma) \xi$ defines an isometry from $\mathcal{K}$ to $\mathcal{H}$ which is Γ equivariant. If ξ is a cyclic vector for π then this gives a natural isomorphism between π_{φ_ξ} and π .

One can easily check that the correspondence described above satisfies the following relationships:

functions of positive type	pointed unitary representations
$\varphi = 1$	the trivial representation
$\varphi = \delta_e$	the left regular representation on $\ell^2\Gamma$, $\xi = \delta_e$
$\varphi = \delta_\Lambda$, $\Lambda < \Gamma$	the quasi-regular representation on $\ell^2(\Gamma/\Lambda)$, $\xi = \delta_\Lambda$
φ a character	the one dimensional representation given by the character
addition	direct sum
product	tensor product