2.4 Gaussian actions

Let $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation of a countable group Γ. The aim of this section, which is taken from [PS09], is to describe the construction of a measure-preserving action of Γ on a non-atomic standard probability space (X, μ) such that $\mathcal{H}$ is realized as a subspace of $L^2_{\mathbb{R}}(X, \mu)$ and π is contained in the Koopman representation $\Gamma \curvearrowright L^2_0(X, \mu)$. The action $\Gamma \curvearrowright (X, \mu)$ is referred to as the **Gaussian action** associated to π.

Given a Hilbert space $\mathcal{H}$, the n-symmetric tensor $\mathcal{H}^{\odot n}$ is the subspace of $\mathcal{H}^{\otimes n}$ fixed by the action of the symmetric group S_n by permuting the indices. For $\xi_1, \dots, \xi_n \in \mathcal{H}$, we define their symmetric tensor product $\xi_1 \odot \cdots \odot \xi_n \in \mathcal{H}^{\odot n}$ to be $\frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}$. Denote

$$\mathfrak{S}(\mathcal{H}) = (\mathbb{R}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\odot n}) \otimes_{\mathbb{R}} \mathbb{C},$$

with renormalized inner product such that $\|\xi\|^2_{\mathfrak{S}(\mathcal{H})} = n! \|\xi\|^2$, for $\xi \in \mathcal{H}^{\odot n}$.

For $\xi \in \mathcal{H}$ let x_ξ be the **symmetric creation operator**,

$$x_\xi(\Omega) = \xi, \;\; x_\xi(\eta_1 \odot \cdots \odot \eta_k) = \xi \odot \eta_1 \odot \cdots \odot \eta_k,$$

and its adjoint,

$$x^*_\xi(\Omega) = 0, \;\; x^*_\xi(\eta_1 \odot \cdots \odot \eta_k) = \sum_{i=1}^{k} \langle \eta_i, \xi \rangle \eta_1 \odot \cdots \odot \widehat{\eta_i} \odot \cdots \odot \eta_k.$$

Let

$$s(\xi) = \frac{1}{2}(x_\xi + x^*_\xi),$$

and note that it is an unbounded, self-adjoint operator on $\mathfrak{S}(\mathcal{H})$.

The moment generating function $M(t)$ for the Gaussian distribution is defined to be

$$M(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(tx) \exp(-x^2/2) dx = \exp(t^2/2).$$

It is easy to check that if $\|\xi\| = 1$ then

$$\langle s(\xi)^n \, \Omega, \Omega \rangle = M^{(n)}(0) = \frac{(2k)!}{2^k k!},$$

if $n = 2k$ and 0 if n is odd. Hence, $s(\xi)$ may be regarded as a Gaussian random variable. Note that if $\xi, \eta \in \mathcal{H}$ then $s(\xi)$ and $s(\eta)$ commute, moreover, if $\xi \perp \eta$, then

$$\langle s(\xi)^m s(\eta)^n \Omega, \Omega \rangle = \langle s(\xi)^m \Omega, \Omega \rangle \langle s(\eta)^n \Omega, \Omega \rangle,$$

for all $m, n \in \mathbb{N}$; thus, $s(\xi)$ and $s(\eta)$ are independent random variables.

From now on we will use the convention $\xi_1 \xi_2 \cdots \xi_k$ to denote the symmetric tensor $\xi_1 \odot \xi_2 \odot \cdots \odot \xi_k$. Let Ξ be a basis for $\mathcal{H}$ and

$$\mathcal{S}(\Xi) = \{\Omega\} \cup \{s(\xi_1)s(\xi_2) \cdots s(\xi_k)\Omega \mid \xi_1, \xi_2, \dots, \xi_k \in \Xi\}.$$

Lemma 2.4.1. *The set $\mathcal{S}(\Xi)$ is a (non-orthonormal) basis of $\mathfrak{S}(\mathcal{H})$.*

Proof. We will show that $\xi_1 \cdots \xi_k \in \mathrm{sp}(\mathcal{S}(\Xi))$, for all $\xi_1, \ldots, \xi_k \in \mathcal{H}$. We have $\Omega \in \mathrm{sp}(\mathcal{S}(\Xi))$. Also, since $s(\xi)\Omega = \xi$, $\mathcal{H} \subset \mathrm{sp}(\mathcal{S}(\Xi))$. Now as $s(\xi_1) \cdots s(\xi_k)\Omega = P(\xi_1, \ldots, \xi_k)$ is a polynomial in $\xi_1, \ldots, \xi_k$ of degree k with top term $\xi_1 \cdots \xi_k$, the result follows by induction on k. □

Let $u(\xi_1, \ldots, \xi_k) = \exp(\pi i s(\xi_1) \cdots s(\xi_k))$ and $u(\xi_1, \ldots, \xi_k)^t = \exp(\pi i t s(\xi_1) \cdots s(\xi_k))$. Denote by A the von Neumann algebra generated by all such $u(\xi_1, \ldots, \xi_k)$, which is the same as the von Neumann algebra generated by the spectral projections of the unbounded operators $s(\xi_1) \cdots s(\xi_k)$.

Theorem 2.4.2. *We have that $L^2(A, \tau) \cong \mathfrak{S}(\mathcal{H})$, and A is a maximal abelian $*$-subalgebra of $\mathcal{B}(\mathfrak{S}(\mathcal{H}))$ with faithful state $\tau = \langle \cdot\, \Omega, \Omega \rangle$.*

Proof. By Lemma 2.4.1, $A \mapsto A\Omega$ is an embedding of A into $\mathfrak{S}(\mathcal{H})$. By Stone's Theorem

$$\lim_{t \to 0} \frac{u(\xi_1, \ldots, \xi_k)^t - 1}{\pi i t} \Omega = s(\xi_1) \cdots s(\xi_k)\Omega;$$

hence, $A\Omega$ is dense in $\mathfrak{S}(\mathcal{H})$. This implies that A is maximal abelian in $B(\mathfrak{S}(\mathcal{H}))$. □

There is a natural strong operator topology continuous embedding $\mathcal{O}(\mathcal{H}) \hookrightarrow \mathcal{U}(\mathfrak{S}(\mathcal{H}))$ given by

$$T \mapsto T^{\mathfrak{S}} = 1 \oplus \bigoplus_{n=1}^{\infty} T^{\odot n}.$$

It follows that there is an embedding $\mathcal{O}(\mathcal{H}) \hookrightarrow \mathrm{Aut}(A, \tau)$, $T \mapsto \sigma_T$, which can be identified on the unitaries $u(\xi_1, \ldots, \xi_k)$ by

$$\sigma_T(u(\xi_1, \ldots, \xi_k)) = \mathrm{Ad}(T^{\mathfrak{S}})(u(\xi_1, \ldots, \xi_k)) = u(T(\xi_1), \ldots, T(\xi_k)).$$

Thus for an orthogonal representation $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$, there is a natural action $\sigma^\pi : \Gamma \to \mathrm{Aut}(A, \tau)$ given by $\sigma^\pi_\gamma(u(\xi_1, \ldots, \xi_k)) = u(\pi_\gamma(\xi_1), \ldots, \pi_\gamma(\xi_k)) = \mathrm{Ad}(\pi_\gamma^{\mathfrak{S}})(u(\xi_1, \ldots, \xi_k))$. Applying Proposition 2.3.1 we then obtain a measure preserving probability space $(X, \mathcal{B}, \mu)$ and an action of Γ on this space so that we can identify A with $L^\infty(X, \mu)$ in a state preserving Γ-equivariant manor. The action of Γ on $(X, \mathcal{B}, \mu)$ is the Gaussian action associated to π.

We have an explicit description of the Koopman representation of $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ given by $\pi^{\mathfrak{S}}(\gamma) = (\pi(\gamma))^{\mathfrak{S}} \ominus 1$. Hence, we have that ergodic properties which remain stable with respect to tensor products transfer from π to σ^π.

Proposition 2.4.3. *Let $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation of a countable group Γ. Then the Gaussian action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is ergodic, if and only if it is weak mixing, if and only if π is weak mixing.*

Proof. If π is not weak mixing, then there exists $\xi \in \mathcal{H}^{\otimes 2}$ such that for all $\gamma \in \Gamma$, $\pi^{\otimes 2}(\gamma)(\xi) = \xi$. Viewing ξ as a Hilbert-Schmidt operator on $\mathcal{H}$, let $|\xi| = (\xi\xi^*)^{1/2}$. Since the map $\xi \otimes \eta \mapsto \eta \otimes \xi$ is the same as taking the adjoint of the corresponding Hilbert-Schmidt operator, we have that $|\xi| \in \mathcal{H}^{\odot 2}$ and $\pi^{\odot 2}(\gamma)(|\xi|) = |\xi|$. Since $\pi^{\odot 2}$ embeds into the Koopman representation of the Gaussian action, it follows from Lemma 2.2.5 that $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is not ergodic.

Conversely, if $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is not ergodic then neither is the Koopman representation $\pi^{\mathfrak{S}}$. But $\pi^{\mathfrak{S}}$ is a sub-representation of $\bigoplus_{n=1}^{\infty} \pi^{\otimes n} = \pi \otimes (1 \oplus \bigoplus_{n=1}^{\infty} \pi^{\otimes n})$. Hence if $\pi^{\mathfrak{S}}$ has non-trivial invariant vectors then π is not weak mixing by Proposition 1.5.6. $\square$

Bibliography

[PS09] J. Peterson and T. Sinclair, *On cocycle superrigidity for gaussian actions*, Preprint 2009. arXiv:0910.3958. To appear in Erg. Theory & Dyn. Sys.

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