2.3 A remark about measure spaces

So far we have been considering actions of a countable group on a general probability space $(X, \mathcal{B}, \mu)$. For most aspects of ergodic theory this is the proper setting. However, occasionally this level of generality can be problematic. For example, consider the action of $\mathbb{Z}$ by rotation on the circle $\mathbb{T}$. We can consider $\mathbb{T}$ equipped with either the Borel σ-algebra, or the Lebesgue σ-algebra. If we are not concerned with sets of measure 0, then both of these systems contain the same information, however the identity map from the Borel σ-algebra to the Lebesgue σ-algebra is not measurable, and these two systems are not isomorphic under our notion of isomorphism.

For another example, consider the case when $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ are measure preserving actions. Then the diagonal action $\Gamma \curvearrowright (X \times Y, \mathcal{B} \otimes \mathcal{A}, \mu \times \nu)$ has the factor $\Gamma \curvearrowright (X \times Y, \mathcal{B} \otimes \{Y, \emptyset\}, \mu \times \nu)$, which we would like to identify with the action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$. However, a problem arises because the projection map from $X \times Y$ to X is not almost everywhere 1-1 and hence is not an isomorphism of actions.

One way to overcome this problem is to restrict ourselves to only consider actions on nice measure spaces. Specifically, one considers the class of **standard probability spaces**. A standard probability space $(X, \mathcal{B}, \mu)$ is a probability space such that the underlying σ-algebra space $(X, \mathcal{B})$ is isomorphic (as σ-algebra spaces) to a Polish[1] topological space with its Borel σ-algebra. In this setting we do not allow actions on $\mathbb{T}$ with the Lebesgue measure, or actions on the space $(X \times Y, \mathcal{B} \otimes \{Y, \emptyset\}, \mu \times \nu)$ since these are not standard probability spaces, and so the problems above do not arise.

An alternate approach, which we will take here, is to continue to allow general probability spaces, but instead generalize our notion of equivalence so that the spaces above are equivalent under this more general notion. Thus, from now on we will say that two probability spaces $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{A}, \nu)$ are isomorphic if there is an integral preserving unital $*$-isomorphism from $L^\infty(Y, \mathcal{A}, \nu)$ to $L^\infty(X, \mathcal{B}, \mu)$. We will also say that two measure preserving actions $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ are isomorphic (or conjugate) if there exists a $*$-isomorphism $\hat{\theta}$ from $L^\infty(Y, \mathcal{A}, \nu)$ to $L^\infty(X, \mathcal{B}, \mu)$ such that $\hat{\theta} \circ \sigma_\gamma = \sigma_\gamma \circ \hat{\theta}$ for all $\gamma \in \Gamma$. Note that if $\theta : X \to Y$ is a bijection such that θ and θ^{-1} is measure preserving such that $\theta \circ \gamma = \gamma \circ \theta$ for all $\gamma \in \Gamma$, then $\hat{\theta} : L^\infty(Y, \mathcal{A}, \nu) \to L^\infty(X, \mathcal{B}, \mu)$ given by $\hat{\theta}(f) = f \circ \theta^{-1}$ implements an isomorphism between the two actions.

We also have a process (essentially the GNS-construction) which takes us from a general action on a separable measure space to an isomorphic action on a standard Borel space.

Proposition 2.3.1. *Let A be a unital $*$-algebra with a state[2] ϕ such that for all $a \in A$ there exists $K > 0$ such that $\phi(a^*abb^*) \leq K\phi(bb^*)$ for all $b \in A$. Let*

[1] A topological space is Polish if it is separable and has a complete metric which induces the topology

[2] A state on a unital $*$-algebra is a linear map $\phi : A \to \mathbb{C}$ such that $\phi(1) = 1$, and $\phi(a^*a) \geq 0$ for all $a \in A$.

Γ be a countable group and suppose $\sigma : \Gamma \to \mathrm{Aut}(A)$ is an action of Γ on A by unital $*$-homomorphisms such that $\phi(\sigma_\gamma(a)) = \phi(a)$ for all $a \in A$, and $\gamma \in \Gamma$.

Then there exists a compact Hausdorff space X, a continuous action $\Gamma \curvearrowright X$, a Γ-invariant Radon probability measure μ on X, and a unital $*$-homomorphism $\pi : A \to L^\infty(X, \mu)$ such that $\pi \circ \sigma_\gamma = \sigma_\gamma \circ \pi$ for all $\gamma \in \Gamma$, and $\int \pi(a)\, d\mu = \phi(a)$, for all $a \in A$.

Moreover, if A is countable generated as an algebra, then X is separable (and hence a Polish space).

Proof. We endow A with the inner-product $\langle a, b\rangle_\phi = \phi(b^* a)$, by quotienting out by the kernel of this inner-product and then taking the completion, we obtain a Hilbert space $L^2(A, \phi)$. Given $a \in A$ we denote by $\hat{a}$ the equivalence class of a in $L^2(A, \phi)$. If $a, b \in A$, then we may consider a acting on $L^2(A, \phi)$ by left multiplication. The formula

$$\|\widehat{ab}\|_\phi^2 = \phi(a^* a b b^*) \leq K \|\hat{b}\|_\phi^2$$

shows that left multiplication is well defined and bounded, we therefore obtain a $*$-homomorphism $\pi_0 : A \to \mathcal{B}(L^2(A, \phi))$.

We denote by $C^*(A)$ the abelian C^*-algebra generated by $\pi_0(A)$ in $\mathcal{B}(L^2(A, \phi))$, and we denote by X the Gelfand spectrum of $C^*(A)$. By Gelfand's Theorem we have that X is a compact Hausdorff space and we obtain a $*$-isomorphism from $C^*(A)$ to $C(X)$. We therefore obtain a $*$-homomorphism $\pi : A \to C(X)$ by applying the Gelfand transform to the image $\pi_0(A)$.

On $C(X) \cong C^*(A)$ we may consider the state $\hat{\phi}$ given by $\hat{\phi}(x) = \langle x\hat{1}, \hat{1}\rangle_\phi$. By the Riesz Representation Theorem the state $\hat{\phi}$ corresponds to a Radon measure μ on X such that $\hat{\phi}(x) = \int x\, d\mu$ for all $x \in C(X)$. We therefore have that for all $a \in A$

$$\int \pi(a)\, d\mu = \hat{\phi}(\pi(a))$$

$$= \langle \pi_0(a)\hat{1}, \hat{1}\rangle_\phi = \langle \hat{a}, \hat{1}\rangle_\phi = \phi(a).$$

Since $\sigma : \Gamma \to \mathrm{Aut}(A)$ preserves the state ϕ, we have that for all $\gamma \in \Gamma$, and $a, b \in A$

$$\|\widehat{\sigma_\gamma(a)b}\|_\phi^2 = \phi(\sigma_\gamma(a^* a) b b^*) = \phi(a^* a \sigma_{\gamma^{-1}}(b b^*)) = \|\widehat{a\sigma_{\gamma^{-1}}(b)}\|_\phi^2.$$

Hence, if we define σ_γ on $\pi_0(A)$ by $\sigma_\gamma(\pi_0(a)) = \pi_0(\sigma_\gamma(a))$ then this is well defined and preserves the operator norm, hence extends to a map (which we still denote by σ) from Γ to $\mathrm{Aut}(C^*(A))$. The Gelfand transform then gives a continuous action of Γ on X, such that $\sigma_\gamma(x) = x \circ \gamma^{-1}$ for all $x \in C(X)$.

An easy calculation then shows that $\pi \circ \sigma_\gamma = \sigma_\gamma \circ \pi$ for all $\gamma \in \Gamma$, and we have that for all $\gamma \in \Gamma$, and $x \in C(X)$

$$\int x\, d\gamma_* \mu = \int x \circ \gamma^{-1}\, d\mu = \hat{\phi}(\sigma_\gamma(x)) = \hat{\phi}(x) = \int x\, d\mu.$$

Hence μ is Γ-invariant.

Finally, if A is countably generated as an algebra then $C^*(A)$ is a separable C^*-algebra and hence the Gelfand spectrum X is separable. $\square$

Corollary 2.3.2. *Suppose* $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ *is a measure preserving action of a countable group* Γ *on a probability space* $(X, \mathcal{B}, \mu)$ *such that* $L^2(X, \mathcal{B}, \mu)$ *is separable. Then the action* $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ *is isomorphic to a measure preserving action of* Γ *on a standard probability space* $(Y, \mathcal{A}, \nu)$.

Proof. Since $L^2(X, \mathcal{B}, \mu)$ is separable there exists a countably generated $*$-subalgebra $A \subset L^\infty(X, \mathcal{B}, \mu) \subset L^2(X, \mathcal{B}, \mu)$ which is dense in $L^2(X, \mathcal{B}, \mu)$. Since Γ is countable we may also assume that A is Γ-invariant. By Proposition 2.3.1 there exists a measure preserving action of Γ on a standard probability space $(Y, \mathcal{A}, \nu)$ and homomorphism $\pi : A \to L^\infty(Y, \mathcal{A}, \nu)$ which is Γ-equivariant and preserves the integrals.

Since π preserves the integrals we may extend it to a unitary $U : L^2(X, \mathcal{B}, \mu) \to L^2(Y, \mathcal{A}, \nu)$ which also preserves the integrals, and is again Γ-equivariant. The map $f \mapsto U^* f U$ then gives a $*$-isomorphism between $L^\infty(Y, \mathcal{A}, \nu)$ and $L^\infty(X, \mathcal{B}, \nu)$ which preserves the integrals and is again Γ-equivariant. $\square$

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