2.2 The Koopman representation

Definition 2.2.1. Let $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$ be an action on a measure space which preserves the measure class υ. The **Koopman representation** of Γ associated to this action is the unitary representation $\pi : \Gamma \to \mathcal{U}(L^2(X, \upsilon))$ given by

$$(\pi(\gamma)f)(x) = f(\gamma^{-1}x)(\frac{d\gamma_*\upsilon}{d\upsilon})^{1/2}(x).$$

Note that for all $\gamma_1, \gamma_2 \in \Gamma$, and $f \in L^2(X, \upsilon)$ we have

$$\pi(\gamma_1\gamma_2)f = \sigma_{\gamma_1\gamma_2}(f)(\frac{d(\gamma_1\gamma_2)_*\upsilon}{d\upsilon})^{1/2}$$

$$= \sigma_{\gamma_1}(\sigma_{\gamma_2}(f)(\frac{d\gamma_{2*}\upsilon}{d\upsilon})^{1/2})(\frac{d\gamma_{1*}\upsilon}{d\upsilon})^{1/2} = \pi(\gamma_1)(\pi(\gamma_2)f).$$

Also, for all $\gamma \in \Gamma$, and $f \in L^2(X, \upsilon)$ we have

$$\|\pi(\gamma)f\|_2^2 = \int |\sigma_\gamma(f)|^2 \frac{d\gamma_*\upsilon}{d\upsilon}\, d\upsilon$$

$$= \int |\sigma_\gamma(f)|^2 \, d\gamma_*\upsilon = \|f\|_2^2.$$

Hence, π is indeed a unitary representation.

If $(X, \mathcal{B}, \upsilon)$ is a finite measure space, and $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$ is measure preserving, then $1 \in L^2(X, \upsilon)$ and $\pi(\gamma)(1) = 1$ is an invariant vector. For this reason, in this setting the Koopman representation usually denotes the restriction of the above representation to the orthogonal complement

$$L^2_0(X, \upsilon) = \{f \in L^2(X, \upsilon) \mid \int f \, d\upsilon = 0\}.$$

Exercise 2.2.2. Let $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$ be an action on a measure space which preserves the measure class υ. Show that the Koopman representation π is isomorphic to its conjugate representation $\overline{\pi}$.

Exercise 2.2.3. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ be measure preserving actions of a countable group Γ on probability spaces $(X, \mathcal{B}, \upsilon)$, and $(Y, \mathcal{A}, \nu)$. Show that the Koopman representation $\pi_{X\times Y}$ for the product action decomposes as $\pi_{X\times Y} \cong \pi_X \oplus \pi_Y \oplus (\pi_X \otimes \pi_Y)$ where π_X and π_Y are the Koopman representations for $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$.

2.2.1 Ergodicity

Definition 2.2.4. Let $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$ be a measure class preserving action of a countable group Γ on a σ-finite measure space $(X, \mathcal{B}, \upsilon)$. The action $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$ is **ergodic**, if for any $E \in \mathcal{B}$ which is Γ-invariant, we have that either $\upsilon(E) = 0$ or $\upsilon(X \setminus E) = 0$.

Lemma 2.2.5. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. The action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is ergodic if and only if the Koopman representation is ergodic.*

Proof. If $E \in \mathcal{B}$ is Γ-invariant then $\chi_E - \mu(E) \in L^2_0(X, \mu)$ is also Γ-invariant, which is non-zero if $\mu(E) \neq 0, 1$. On the other hand, if $\xi \in L^2_0(X, \mu)$ is a non-zero Γ-invariant function then $E_t = \{x \in X \mid |\xi(x)| < t\}$ is Γ-invariant for all $t > 0$, and we must have $\mu(E_t) \neq 0, 1$ for some $t > 0$. □

Theorem 1.6.5 can be adapted to the setting of actions on measure spaces as follows.

Theorem 2.2.6. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$ be a measure preserving action of a countable group Γ on an infinite, σ-finite measure space $(X, \mathcal{B}, \upsilon)$. The following conditions are equivalent.*

(1). *There exists a state $\varphi \in (L^\infty(X, \mathcal{B}, \upsilon))^*$ such that for all $\gamma \in \Gamma$, and $f \in L^\infty(X, \mathcal{B}, \upsilon)$ we have $\varphi(\sigma(\gamma)(f)) = \varphi(f)$.*

(2). *For every $\varepsilon > 0$, and $F \subset \Gamma$ finite, there exists $\nu \in \mathrm{Prob}(X, \mathcal{B})$, such that ν is absolutely continuous with respect to υ and*

$$\left\| \frac{d\gamma_* \nu}{d\upsilon} - \frac{d\nu}{d\upsilon} \right\|_1 < \varepsilon.$$

(3). *For every $\varepsilon > 0$, and $F \subset \Gamma$ finite, there exists a measurable set $A \subset X$ such that $\upsilon(A) < \infty$, and*

$$\upsilon(A \Delta \gamma A) < \varepsilon \upsilon(A).$$

(4). *The Koopman representation $\pi : \Gamma \to \mathcal{U}(L^2(X, \mathcal{B}, \upsilon))$ has almost invariant vectors.*

(5). *The Koopman representation $\pi : \Gamma \to \mathcal{U}(L^2(X, \mathcal{B}, \upsilon))$ is amenable.*

Exercise 2.2.7. Adapt the proof of Theorem 1.6.5 to prove Theorem 2.2.6.

If Γ acts by measure preserving transformations on a probability space, then the conditions above are trivially satisfied, however, we still have the following, non-trivial, adaptation.

Theorem 2.2.8. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. The following conditions are equivalent.*

(1). *There exists a state $\varphi \in (L^\infty(X, \mathcal{B}, \mu))^*$, different than $f \mapsto \int f \, d\mu$, such that for all $\gamma \in \Gamma$, and $f \in L^\infty(X, \mathcal{B}, \mu)$ we have $\varphi(\sigma(\gamma)(f)) = \varphi(f)$.*

(2). *There exits $A \subset X$ measurable such that $\mu(A) < 1$, and for every $\varepsilon > 0$, and $F \subset \Gamma$ finite, there exists $\nu \in \mathrm{Prob}(X, \mathcal{B})$, such that ν is absolutely continuous with respect to μ and*

$$\nu(A) > 1 - \varepsilon, \qquad \text{and} \qquad \left\| \frac{d\gamma_* \nu}{d\mu} - \frac{d\nu}{d\mu} \right\|_1 < \varepsilon.$$

(3). *For every* $\varepsilon > 0$, *and* $F \subset \Gamma$ *finite, there exists a measurable set* $A \subset X$ *such that* $\mu(A) \leq 1/2$, *and*

$$\mu(A \Delta \gamma A) < \varepsilon \mu(A).$$

(4). *The Koopman representation* $\pi : \Gamma \to \mathcal{U}(L^2_0(X, \mathcal{B}, \mu))$ *has almost invariant vectors.*

(5). *The Koopman representation* $\pi : \Gamma \to \mathcal{U}(L^2_0(X, \mathcal{B}, \mu))$ *is amenable.*

If $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is an ergodic measure preserving action of a countable group on a probability space $(X, \mathcal{B}, \mu)$ such that none of the conditions of the previous theorem hold, then we say that the action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ has **spectral gap**.

2.2.2 Weak mixing

Definition 2.2.9. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group on a probability space $(X, \mathcal{B}, \mu)$. A sequence $E_n \in \mathcal{B}$ of measurable subsets is an **asymptotically invariant sequence** if $\mu(E_n \Delta \gamma E_n) \to 0$, for all $\gamma \in \Gamma$. Such a sequence is said to be non-trivial if $\liminf_{n \to \infty} \mu(E_n)(1 - \mu(E_n)) > 0$. The action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is **strongly ergodic** if there does not exist a non-trivial asymptotically invariant sequence.

Note that if $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ has a non-trivial invariant set $E \in \mathcal{B}$, then the constant sequence $E_n = E$ is a non-trivial asymptotically invariant sequence, hence strongly ergodic implies ergodic. Also, a non-trivial asymptotically invariant sequence will show that condition (3) in Theorem 2.2.8 is satisfied, hence an ergodic action with spectral gap must be strongly ergodic.

Definition 2.2.10. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. The action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is **weak mixing**, if $|\Gamma| = \infty$, and for all $\mathcal{E} \subset \mathcal{B}$ finite, we have

$$\liminf_{\gamma \to \infty} \Sigma_{A,B \in \mathcal{E}} |\mu(A \cap \gamma B) - \mu(A)\mu(B)| = 0.$$

Proposition 2.2.11. *Let* $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ *be a measure preserving action of a countable group* Γ *on a probability space* $(X, \mathcal{B}, \mu)$. *The following conditions are equivalent:*

(1). *The action* $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ *is weak mixing.*

(2). *The Koopman representation* $\pi : \Gamma \to \mathcal{U}(L^2_0(X, \mathcal{B}, \mu))$ *is weak mixing.*

(3). *The diagonal action* $\Gamma \curvearrowright (X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \times \mu)$ *is ergodic.*

(4). *For any ergodic, measure preserving action* $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ *on a probability space, the diagonal action* $\Gamma \curvearrowright (X \times Y, \mathcal{B} \otimes \mathcal{A}, \mu \times \nu)$ *is ergodic.*

(5). *The diagonal action* $\Gamma \curvearrowright (X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \times \mu)$ *is weak mixing.*

(6). *For any weak mixing, measure preserving action* $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ *on a probability space, the diagonal action* $\Gamma \curvearrowright (X \times Y, \mathcal{B} \otimes \mathcal{A}, \mu \times \nu)$ *is weak mixing.*

Proof. For (1) $\iff$ (2), if $\mathcal{E} \subset \mathcal{B}$ is a finite set then we can consider the finite set $\mathcal{F} = \{\chi_E - \mu(E) \mid E \in \mathcal{E}\} \subset L^2_0(X, \mathcal{B}, \mu)$. For each $\gamma \in \Gamma$ we therefore have

$$\begin{aligned}\Sigma_{\xi,\eta \in \mathcal{F}} |\langle \pi(\gamma)\xi, \eta\rangle| &= \Sigma_{A,B \in \mathcal{E}} |\langle \sigma_\gamma(\chi_B) - \mu(B), \chi_A - \mu(A)\rangle| \\ &= \Sigma_{A,B \in \mathcal{E}} |\langle \chi_{\gamma B} - \mu(B), \chi_A - \mu(A)\rangle| \\ &= \Sigma_{A,B \in \mathcal{E}} |\mu(A \cap \gamma B) - \mu(A)\mu(B)|.\end{aligned}$$

If the Koopman representation is weak mixing then from this we see immediately that the action is weak mixing. A similar calculation shows that if the action is weak mixing and $\mathcal{F} \subset L^2_0(X, \mathcal{B}, \mu)$ is a finite set of simple functions, then

$$\liminf_{\gamma \to \infty} \Sigma_{\xi,\eta \in \mathcal{F}} |\langle \pi(\gamma)\xi, \eta\rangle| = 0.$$

Since simple functions are dense in $L^2_0(X, \mathcal{B}, \mu)$ this shows that the Koopman representation is weak mixing.

The remaining equivalences then easily follow from Exercise 2.2.3, together with the corresponding properties of weak mixing for unitary representations in Sections 1.5 and 1.7. □

Corollary 1.7.7 then shows that weak mixing is preserved under taking finite index.

Corollary 2.2.12. *Let* $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ *be a weak mixing, measure preserving action of a countable group* Γ *on a probability space* $(X, \mathcal{B}, \mu)$. *If* $\Sigma < \Gamma$ *is a finite index subgroup, then the restriction of the action of* Γ *to* Σ *is also weak mixing.*

Corollary 1.7.16 gives the following result for weak mixing actions of amenable groups.

Corollary 2.2.13. *Let* $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ *be a measure preserving action of a countable amenable group* Γ *on a probability space* $(X, \mathcal{B}, \mu)$. *Let* $F_n \subset \Gamma$*, be a Følner sequence for* Γ. *The action* $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ *is weak mixing if and only if for all* $A, B \in \mathcal{B}$*, we have*

$$\frac{1}{|F_n|} \Sigma_{\gamma \in F_n} |\mu(A \cap \gamma B) - \mu(A)\mu(B)| \to 0.$$

Definition 2.2.14. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. The action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is (**strong**) **mixing**, if $|\Gamma| = \infty$, and for any $A, B \subset X$ measurable we have

$$\lim_{\gamma \to \infty} |\mu(A \cap \gamma B) - \mu(A)\mu(B)| = 0.$$

The same proof as in Proposition 2.2.11 yields the following proposition for mixing actions.

Proposition 2.2.15. *Let* $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ *be a measure preserving action of a countable group* Γ *on a probability space* $(X, \mathcal{B}, \mu)$. *The action* $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ *is mixing if and only if the Koopman representation is mixing.*

2.2.3 Compact actions

Definition 2.2.16. Let $(X, \mathcal{B}, \mu)$ be a probability space. We denote by $\mathrm{Aut}(X, \mathcal{B}, \mu)$ the group of automorphisms of $(X, \mathcal{B}, \mu)$, where we identify two automorphisms if they agree almost everywhere. The **weak topology** on $\mathrm{Aut}(X, \mathcal{B}, \mu)$ is the smallest topology such that the maps $T \mapsto \mu(T(A)\Delta B)$ are continuous for all $A, B \in \mathcal{B}$.

Exercise 2.2.17. Show that the weak topology endows $\mathrm{Aut}(X, \mathcal{B}, \mu)$ with a topological group structure.

Exercise 2.2.18. The Koopman representation for $\mathrm{Aut}(X, \mathcal{B}, \mu)$ is defined as before, i.e., $\pi : \mathrm{Aut}(X, \mathcal{B}, \mu) \to \mathcal{U}(L^2_0(X, \mathcal{B}, \mu))$ is defined by $\pi(T)(f) = f \circ T^{-1}$. Show that the image of π is closed and that π is homeomorphism from $\mathrm{Aut}(X, \mathcal{B}, \mu)$ with the the weak topology onto $\pi(\mathrm{Aut}(X, \mathcal{B}, \mu)) \subset \mathcal{U}(L^2_0(X, \mathcal{B}, \mu))$ with the strong operator topology.

Definition 2.2.19. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. The action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is **compact** if the image of Γ in $\mathrm{Aut}(X, \mathcal{B}, \mu)$ is precompact in the weak topology.

An immediate consequence of Exercise 2.2.18 is the following.

Proposition 2.2.20. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. Then $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is compact if and only if the Koopman representation $\pi : \Gamma \to \mathcal{U}(L^2_0(X, \mathcal{B}, \mu))$ is compact.*

Definition 2.2.21. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. A function $f \in L^2(X, \mathcal{B}, \mu)$ is **almost periodic** if the Γ orbit $\{\sigma_\gamma(f) \mid \gamma \in \Gamma\}$ is pre-compact in the $\| \cdot \|_2$-topology.

Proposition 2.2.22. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a discrete group Γ on a standard probability space $(X, \mathcal{B}, \mu)$. Let $\mathcal{A} \subset \mathcal{B}$ be the σ-algebra generated by all almost periodic functions in $L^2(X, \mathcal{B}, \mu)$. Then $\mathcal{A}$ is Γ-invariant, and $f \in L^2(X, \mathcal{B}, \mu)$ is almost periodic if and only if f is $\mathcal{A}$ measurable.*

Proof. For each $f \in L^2(X, \mathcal{B}, \mu)$ denote by S_f the Γ-orbit of f, and consider the set $AP(L^2(X, \mathcal{B}, \mu))$ of all almost periodic functions. Note that it is obvious that $AP(L^2(X, \mathcal{B}, \mu))$ is Γ-invariant, contains the scalars, and is closed under scalar multiplication.

Since addition is continuous with respect to $\| \cdot \|_2$, if $f, g \in AP(L^2(X, \mathcal{B}, \mu))$ we have that $\overline{S_f} + \overline{S_g}$ is compact being the continuous image of the compact set $\overline{S_f} \times \overline{S_g}$. We therefore have that $S_{f+g} \subset \overline{S_f} + \overline{S_g}$ is precompact, hence $AP(L^2(X, \mathcal{B}, \mu))$ is closed under addition. Similarly, if $f, g \in AP(L^2(X, \mathcal{B}, \mu))$ then we have that

$$|f|, \overline{f}, \max\{|f|, |g|\}, \min\{|f|, |g|\} \in AP(L^2(X, \mathcal{B}, \mu)).$$

Also, if we also have that $g \in L^\infty(X, \mathcal{B}, \mu)$ then $fg \in AP(L^\infty(X, \mathcal{B}, \mu))$.

If $f_n \in AP(L^2(X, \mathcal{B}, \mu))$ and $f \in L^2(X, \mathcal{B}, \mu)$ such that $\|f_n - f\|_2 \to 0$, then fix $\varepsilon > 0$, and take $n \in \mathbb{N}$ such that $\|f_n - f\|_2 < \varepsilon/2$. Since f_n is almost periodic we have that S_{f_n} is totally bounded, hence there exists a finite set $\subset L^2(X, \mathcal{B}, \mu)$ such that $\inf_{h \in \mathcal{F}} \|\sigma_\gamma(f_n) - h\|_2 < \varepsilon/2$ for all $\gamma \in \Gamma$. By the triangle inequality we then have $\inf_{h \in \mathcal{F}} \|\sigma_\gamma(f) - h\|_2 < \varepsilon$. This shows that S_f is totally bounded and hence $f \in AP(L^2(X, \mathcal{B}, \mu))$.

The operations above then generate all $\mathcal{A}$-measurable functions and so we obtain the result. □

Definition 2.2.23. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a discrete group Γ on a standard probability space $(X, \mathcal{B}, \mu)$. Suppose $\mathcal{A} \subset \mathcal{B}$ is a Γ-invariant σ-algebra, then we say that the action $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$ is a **factor** of the action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$.

Note that the Koopman representation of a factor $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$ is a subrepresentation of the Koopman representation of $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$. Hence, Proposition 2.2.11 together with Proposition 2.2.22 give the following.

Proposition 2.2.24. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a discrete group Γ on a standard probability space $(X, \mathcal{B}, \mu)$. Then there exists a unique maximal factor $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$ which is compact. Moreover, $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is weak mixing if and only if $\mathcal{A}$ is trivial, i.e., when $\mathcal{A}$ consists only of null or co-null sets.*

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