2.1 Examples

Definition 2.1.1. Let $(X, \mathcal{B}, \upsilon)$ be a σ-finite measure space, and let Γ be a countable group, an action $\Gamma \curvearrowright X$ such that $\gamma^{-1}(\mathcal{B}) = \mathcal{B}$, for all $\gamma \in \Gamma$ is measure preserving if for each measurable set $A \in \mathcal{B}$ we have $\upsilon(\gamma^{-1}A) = \upsilon(A)$. Alternately, we will say that υ is an invariant measure for the action $\Gamma \curvearrowright (X, \mathcal{B})$.

We'll say that Γ preserves the measure class of υ, or alternately, υ is a quasi-invariant measure, if for each $\gamma \in \Gamma$, and $A \in \mathcal{B}$ we have that $\upsilon(A) = 0$ if and only if $\upsilon(\gamma^{-1}A) = 0$, i.e., for each $\gamma \in \Gamma$, the push-forward measure $\gamma_*\upsilon$ defined by $\gamma_*\upsilon(A) = \upsilon(\gamma^{-1}A)$ for all $A \in \mathcal{B}$ is absolutely continuous with respect to υ.

Note, that since the action $\Gamma \curvearrowright X$ is measurable we obtain an action $\sigma : \Gamma \to \mathrm{Aut}(\mathcal{M}(X, \mathcal{B}))$ of Γ on the $\mathcal{B}$-measurable functions by the formula $\sigma_\gamma(f) = f \circ \gamma^{-1}$.

If $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$ preserves the measure class of υ, then for each $\gamma \in \Gamma$ there exists the Radon-Nikodym derivative $\frac{d\gamma_*\upsilon}{d\upsilon} : X \to [0, \infty)$, such that for each measurable set $A \in \mathcal{B}$ we have

$$\upsilon(\gamma^{-1}A) = \int 1_A \frac{d\gamma_*\upsilon}{d\upsilon}\, d\upsilon.$$

The Radon-Nikodym derivative is unique up to measure zero for υ.

If $\gamma_1, \gamma_2 \in \Gamma$, and $A \in \mathcal{B}$ then we have

$$\upsilon((\gamma_1\gamma_2)^{-1}A) = \int 1_{\gamma_1^{-1}A} \frac{d\gamma_{2*}\upsilon}{d\upsilon}\, d\upsilon$$

$$= \int 1_A \sigma_{\gamma_1}(\frac{d\gamma_{2*}\upsilon}{d\upsilon})\, d\gamma_{1*}\upsilon = \int 1_A \sigma_{\gamma_1}(\frac{d\gamma_{2*}\upsilon}{d\upsilon})\frac{d\gamma_{1*}\upsilon}{d\upsilon}\, d\upsilon.$$

Hence, we have almost everywhere

$$\frac{d(\gamma_1\gamma_2)_*\upsilon}{d\upsilon} = \sigma_{\gamma_1}(\frac{d\gamma_{2*}\upsilon}{d\upsilon})\frac{d\gamma_{1*}\upsilon}{d\upsilon}.$$

Or, in other words, the map $(\gamma, x) \mapsto \frac{d\gamma_*\upsilon}{d\upsilon}(\gamma x) \in \mathbb{R}^\times$ is a cocycle almost everywhere.

Definition 2.1.2. Let $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ be measure class preserving actions of a countable group Γ on σ-finite measure spaces $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$. The actions are **isomorphic** (or **conjugate**) if there exists an isomorphism of measure spaces[1] such that for all $\gamma \in \Gamma$ and almost every $x \in X$ we have

$$\theta(\gamma x) = \gamma\theta(x).$$

Example 2.1.3. Any action $\Gamma \curvearrowright X$ of a countable group Γ on a countable set X can be viewed as a measure preserving action on X with the counting measure.

[1] A map $\theta : X \to Y$ is an isomorphism of measure spaces if θ is almost everywhere a bijection such that θ and also θ^{-1} is measure preserving.

Example 2.1.4. Consider the torus $\mathbb{T}$ with the Borel σ-algebra, and Lebesgue measure, if $a \in [0, 1)$ then we obtain a measure preserving transformation $T : \mathbb{T} \to \mathbb{T}$ by $T(e^{i\theta}) = e^{i(\theta+2\pi a)}$. This then induces a measure preserving action of $\mathbb{Z}$.

Example 2.1.5 (The odometer action)**.** Consider the space $\{0, 1\}$, with the uniform probability measure, and consider $X = \{0, 1\}^{\mathbb{N}}$ with the product measure. We obtain a measure preserving transformation $T : X \to X$ given by "adding one". That is to say that T applied to a sequence $a_1a_2a_3\cdots$ will be the sequence $000\cdots01a_{n+1}a_{n+2}\cdots$ where a_n is the first position in which a 0 occurs in $a_1a_2a_3\cdots$. Then T induces a probability preserving action of $\mathbb{Z}$ on $\{0, 1\}^{\mathbb{N}}$.

Example 2.1.6 (Bernoulli shift)**.** Let $(X_0, \mathcal{B}_0, \mu_0)$ be a probability space, let Γ be a countable group, and consider $X = X_0^\Gamma$ with the product measure. Then we have a measure preserving action of Γ on X by $\gamma x = x \circ \gamma^{-1}$ for each $x \in X_0^\Gamma$.

Example 2.1.7 (Generalized Bernoulli shift)**.** Let $(X_0, \mathcal{B}_0, \mu_0)$ be a probability space, let $\Gamma \curvearrowright I$ be an action of a countable group Γ on a non-empty countable set I, and consider $X = X_0^I$ with the product measure. Then just as in the case of the Bernoulli shift we have a measure preserving action of Γ on X given by $\gamma x = x \circ \gamma^{-1}$ for each $x \in X_0^\Gamma$.

Exercise 2.1.8 ("The baker's map")**.** Let $X = [0, 1] \times [0, 1]$ with Lebesgue measure, consider the map $T : X \to X$ defined by

$$T(x, y) = \begin{cases} (2x, \frac{y}{2}), & 0 \leq x \leq \frac{1}{2}; \\ (2x - 1, \frac{y+1}{2}), & \frac{1}{2} < x \leq 1. \end{cases}$$

Then T and T^{-1} are both measure preserving and hence give a measure preserving action of $\mathbb{Z}$ on X.

Show that the map $\theta : \{0, 1\}^{\mathbb{Z}} \to [0, 1] \times [0, 1]$ given by

$$\theta(x) = (\Sigma_{n\leq 0}x(n)2^{-(n+1)}, \Sigma_{n>0}x(n)2^n)$$

is an isomorphism of measure spaces which implements a conjugacy between the Bernoulli shift $\mathbb{Z} \curvearrowright \{0, 1\}^{\mathbb{Z}}$ and the baker's action $\mathbb{Z} \curvearrowright [0, 1] \times [0, 1]$.

Example 2.1.9. Let G be a σ-compact, locally compact group, and let Γ be a countable group. Then any homomorphism from Γ to G induces a Haar measure preserving action of Γ on G by left multiplication.

If G is compact, then we obtain a measure preserving action of Γ on a probability space.

Example 2.1.10. Let G be a σ-compact, locally compact group, let $H < G$ be a closed subgroup, and let $\Gamma < G$ be a countable subgroup. There always exists a G-quasi-invariant measure on the homogeneous space G/H, (see for example Section 2.6 in [Fol95]). Thus we always obtain a measure class preserving action of Γ on G/H.

A **lattice** in G is a discrete subgroup Δ such that G/Δ has an invariant probability measure. In this case we obtain a probability measure preserving action of Γ on G/Δ.

Example 2.1.11. Suppose A is an abelian group, and Γ is a group of automorphisms of A, then not only does Γ preserve the Haar measure of A, but also Γ has a Haar measure preserving action of the dual group $\hat{A}$ given by $\gamma x = x\gamma^{-1}$. As an example, we can consider the action of $SL_n\mathbb{Z}$ on $\mathbb{Z}^n$ given by matrix multiplication, this then induces a probability measure preserving action of $SL_n\mathbb{Z}$ on the dual group $\mathbb{T}^n$.

Example 2.1.12 (Uniform ordering [MOP79, Kie75])**.** Let Γ be a countable group, and consider $O(\Gamma)$ the set of all total orders of Γ. By interpreting total orders $<_t$ on Γ with functions from $\Gamma \times \Gamma$ to $\{0,1\}$ which take the value 1 at (γ_1, γ_2) if and only if $\gamma_1 <_t \gamma_2$, we may consider $O(\Gamma)$ as a subset of $\{0,1\}^{\Gamma\times\Gamma}$, and we consider the corresponding σ-algebra.

We have an action of Γ on $O(\Gamma)$ by requiring that $x <_{\gamma t} y$ if and only if $\gamma x <_t \gamma y$. We may place a probability measure μ on $O(\Gamma)$ by requiring that for each pairwise distinct $x_1, x_2, \ldots, x_n \in \Gamma$ we have

$$\mu(\{<_t \in O(\Gamma) \mid x_1 <_t x_2 <_t \cdots <_t x_n\}) = \frac{1}{n!}.$$

By Carathéodory's Extension Theorem, this extends to an invariant measure for the action of Γ. If $\Gamma \curvearrowright X$ is an action of Γ on a countable set X, then we can of course also consider the corresponding probability measure preserving action of Γ on $O(X)$.

Example 2.1.13 (Furstenberg's Correspondence Principle [Fur77])**.** Let Γ be a countable amenable group and let $F_n \subset \Gamma$ be a Følner sequence. Then each set F_n gives rise to a probability measure $\mu_n \in \mathrm{Prob}(\Gamma) \subset \mathrm{Prob}(\beta\Gamma)$, given as the uniform probability measure on F_n. Since the Stone-Čech compactification $\beta\Gamma$ is compact, the Arzelà-Ascoli Theorem implies that the sequence μ_n has a cluster point $\mu \in \mathrm{Prob}(\beta\Gamma)$.

Since F_n is a Følner sequence we have that μ is an invariant probability measure for the action of Γ on $\beta\Gamma$. Moreover, if $A \subset \Gamma \subset \beta\Gamma$ then we have that

$$\liminf_{n\to\infty} \frac{|A \cap F_n|}{|F_n|} \leq \mu(\overline{A}) \leq \limsup_{n\to\infty} \frac{|A \cap F_n|}{|F_n|}.$$

We also remark that if $A_1, A_2, \ldots, A_k \subset \Gamma$ then it is easy to see that $\cap_{j=1}^k \overline{A_j} = \overline{\cap_{j=1}^k A_j}$.

Since $\beta\Gamma$ is not second countable $L^2(\beta\Gamma, \mu)$ will not be separable in general. However, since Γ is countable, if $A \subset \Gamma$ then we can consider the Γ-invariant sub-σ-algebra generated by $\overline{A} \subset \beta\Gamma$, this then gives a separable Hilbert space.

Example 2.1.14. Similar to Furstenberg's Correspondence Principle, suppose G is a locally compact group. The Baire σ-algebra $\mathcal{B}_{aire}$ on G is the σ-algebra generated by the G_δ sets which are compact. Alexanderoff showed that there is a Banach space isomorphism between $C_b(G)^*$ and the space $ba(G, \mathcal{B}_{aire})$ of regular, bounded, finitely additive means m on $\mathcal{B}_{aire}$ with norm given by total valuation (see for example Theorem IV.6.2 in [DS88]).

Hence if m is a regular, finitely additive mean on the Baire σ-algebra of G which is invariant under the action of G. Then m gives a state on $C_b(G) = C(\beta G)$ which by the Riesz Representation Theorem gives a Radon probability measure μ on βG, hence, if Γ is a countable group and we have a homomorphism from Γ to G then left multiplication induces a μ-probability measure preserving action of Γ on βG.

Example 2.1.15 (Randomorphisms [Mon06])**.** Let Γ and Λ be two countable groups and consider the action of Γ on the space $[\Gamma, \Lambda] = \{f \in \Lambda^\Gamma \mid f(e) = e\}$ as described in Example 1.3.3. We consider Λ^Γ with the Polish space structure given by the product topology where Λ is discrete, we then endow Λ^Γ with the Borel σ-algebra. A **randomorphism** from Γ to Λ is a Γ-invariant probability measure μ on Λ^Γ on this σ-algebra. Note that homomorphism from Γ to Λ is just a Γ fixed point in Λ^Γ, hence the Dirac measure at such a point gives a random homomorphism.

If a random homomorphism μ is supported on the space of maps which are injective then we say that μ is a randembedding. There is also of course the corresponding notion of a random surjection, and a random bijection.

Notice that we can identify the space of bijections in $[\Gamma, \Lambda]$ with the space of bijections in $[\Lambda, \Gamma]$ by the inverse map. Under this identification we then obtain an action of Λ on the space of bijections in $[\Gamma, \Lambda]$ given by

$$(\lambda f)(x) = f(xf^{-1}(\lambda))\lambda^{-1}.$$

Example 2.1.16. Let $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ be measure class preserving actions of a countable group Γ on σ-finite measure spaces $(X, \mathcal{B}, \upsilon)$, and $(Y, \mathcal{A}, \nu)$. Then we obtain a diagonal action $\Gamma \curvearrowright (X \times Y, \mathcal{B} \otimes \mathcal{A}, \upsilon \times \nu)$ given by

$$\gamma(x, y) = (\gamma x, \gamma y).$$

If actions $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ are measure preserving, then so is the diagonal action.

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jesse.d.peterson@vanderbilt.edu