1.1 Definitions and constructions

Definition 1.1.1. Let Γ be a group, a **unitary representation** (resp. an **orthogonal representation**) of Γ is a homomorphism $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ (resp. $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$) from Γ into the unitary (resp. orthogonal) group of a complex (resp. real) Hilbert space $\mathcal{H}$.

Example 1.1.2. Let $\mathcal{H}$ be a Hilbert space and suppose Γ is a group. The **trivial representation** of Γ on $\mathcal{H}$ is given by the homomorphism which takes any group element to the identity operator on $\mathcal{H}$.

Example 1.1.3. Let Γ be a group, the **left-regular representation** (resp. **right-regular representation**) is given by $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$, (resp. $\rho : \Gamma \to \mathcal{U}(\ell^2\Gamma)$) which is defined by the formula $\lambda(\gamma)(\Sigma_{x\in\Gamma}\alpha_x\delta_x) = \Sigma_{x\in\Gamma}\alpha_x\delta_{\gamma x}$ (resp. $\rho(\gamma)(\Sigma_{x\in\Gamma}\alpha_x\delta_x) = \Sigma_{x\in\Gamma}\alpha_x\delta_{x\gamma^{-1}}$).

Example 1.1.4. More generally, if a group Γ acts on a set I then there is an associated unitary representation on $\ell^2 I$ defined by $\lambda(\gamma)(\Sigma_{x\in I}\alpha_x\delta_x) = \Sigma_{x\in I}\alpha_x\delta_{\gamma x}$. When $\Sigma < \Gamma$ is a subgroup, then Γ acts by left multiplication on the quotient space Γ/Σ and the associated unitary representation is the **quasi-regular representation**.

Suppose Γ is a group and $\Sigma < \Gamma$ is a subgroup. If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation, then we obtain the **reduced representation** $\pi_{|\Sigma}$ of Σ by restricting π to Σ.

Given a complex Hilbert space $\mathcal{H}$ the **adjoint Hilbert space** $\overline{\mathcal{H}}$ is the Hilbert space which agrees with $\mathcal{H}$ as a set, has scalar multiplication given by $\lambda\overline{\xi} = \overline{\overline{\lambda}\xi}$ and whose inner-product is given by $\langle\overline{\xi},\overline{\eta}\rangle_{\overline{\mathcal{H}}} = \overline{\langle\xi,\eta\rangle_{\mathcal{H}}}$.

Given a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ we define the **adjoint representation** $\overline{\pi} : \Gamma \to \mathcal{U}(\overline{\mathcal{H}})$ by setting $\overline{\pi}(\gamma)\overline{\xi} = \overline{\pi(\gamma)\xi}$. Note that we have the natural identification $\pi = \overline{\overline{\pi}}$.

If $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ is an orthogonal representation where $\mathcal{H}$ is a real Hilbert space then we will use the convention $\overline{\mathcal{H}} = \mathcal{H}$ and $\overline{\pi} = \pi$.

If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, and $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ are two unitary representations, then π is contained in ρ if there is an isometry $U : \mathcal{H} \to \mathcal{K}$ such that for all $\gamma \in \Gamma$ we have that $U\pi(\gamma) = \rho(\gamma)U$. If U can be chosen to be a unitary from $\mathcal{H}$ to $\mathcal{K}$ then the representations are isomorphic and we will write $\pi \cong \rho$.

Given a family of unitary representations $\pi_\iota : \Gamma \to \mathcal{U}(\mathcal{H}_\iota)$, with $\iota \in I$, the **direct-sum representation** is given by the map $\oplus_{\iota\in I}\pi_\iota : \Gamma \to \mathcal{U}(\oplus_{\iota\in I}\mathcal{H}_\iota)$ defined by

$$(\oplus_{\iota\in I}\pi_\iota)(\gamma) = \oplus_{\iota\in I}(\pi_\iota(\gamma)).$$

A unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is **reducible** if it contains a non-trivial sub-representations, or equivalently, there exists a closed Γ-invariant subspace $\mathcal{K} \subset \mathcal{H}$ such that $\mathcal{K} \neq \{0\}, \mathcal{H}$. In this case it is easy to see that $\mathcal{K}^\perp$ is also Γ-invariant and the natural isomorphism from $\mathcal{K} \oplus \mathcal{K}^\perp$ induces an equivalence between the representations $\pi_{|\mathcal{K}} \oplus \pi_{|\mathcal{K}^\perp}$ and π. If a unitary representation has no non-trivial invariant subspace we say that it is **irreducible**.

A vector $\xi \in \mathcal{H}$ is a **cyclic vector** if the smallest Γ-invariant subspace of $\mathcal{H}$ which contains ξ is $\mathcal{H}$ itself. Note that if π is irreducible then every non-zero vector is cyclic.

Lemma 1.1.5 (Schur's Lemma)**.** *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, and $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ be two irreducible unitary representations of a group Γ, if $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is Γ-invariant then either $T = 0$, or else T is a scalar multiple of a unitary. In particular, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ has a non-zero Γ-invariant operator if and only if π and ρ are isomorphic.*

Proof. Let π and ρ be as above and suppose $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is Γ-invariant. Thus, $T^*T \in \mathcal{B}(\mathcal{H})$ is Γ-invariant and hence any spectral projection of T^*T gives a Γ-invariant subspace. Since π is irreducible it then follows that $T^*T \in \mathbb{C}$. If $T^*T \neq 0$ then my multiplying T by a scalar we may assume that T is an isometry. Hence, $TT^* \in \mathcal{B}(\mathcal{K})$ is a non-zero Γ-invariant projection, and since ρ is irreducible it follows that $TT^* = TT^* = 1$. □

If I is finite, then the **tensor product representation** is given by the map $\otimes_{\iota \in I} \pi_\iota : \Gamma \to \mathcal{U}(\overline{\otimes}_{\iota \in I} \mathcal{H}_\iota)$ defined by

$$(\otimes_{\iota \in I} \pi_\iota)(\gamma) = \otimes_{\iota \in I} (\pi_\iota(\gamma)).$$

Lemma 1.1.6 (Fell's Absorption Principle)**.** *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a group Γ, let $1_{\mathcal{H}}$ denote the trivial representation of Γ on $\mathcal{H}$, and let $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ denote the left-regular representation. Then the representations $\lambda \otimes \pi$ and $\lambda \otimes 1_{\mathcal{H}}$ are isomorphic.*

Proof. Consider the unitary $U \in \mathcal{U}(\ell^2\Gamma \otimes \mathcal{H})$ determined by $U(\delta_x \otimes \xi) = \delta_x \otimes \pi(x)\xi$, for all $x \in \Gamma$, $\xi \in \mathcal{H}$. Then for all $\gamma, x \in \Gamma$, and $\xi \in \mathcal{H}$ we have that

$$\begin{aligned} U^*(\lambda \otimes \pi)(\gamma)U(\delta_x \otimes \xi) &= U^*(\lambda \otimes \pi)(\gamma)(\delta_x \otimes \pi(x)\xi) \\ &= U^*(\delta_{\gamma x} \otimes \pi(\gamma)\pi(x)\xi) \\ &= \delta_{\gamma x} \otimes \pi((\gamma x)^{-1})\pi(\gamma)\pi(x)\xi = (\lambda \otimes 1_{\mathcal{H}})(\gamma)(\delta_x \otimes \xi). \end{aligned}$$

□

Example 1.1.7 (Hilbert-Schmidt operators)**.** Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ with respective orthonormal bases $\{\xi_i\} \subset \mathcal{H}$ and $\{\eta_j\} \subset \mathcal{K}$ we define the space of Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{K}$ by

$$\mathrm{HS}(\mathcal{H}, \mathcal{K}) = \{S \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \mid \|S\|_{\mathrm{HS}}^2 = \Sigma_i \|S\xi_i\|^2 < \infty\}.$$

Parseval's identity gives

$$\Sigma_i \|S\xi_i\|^2 = \Sigma_{i,j} |\langle S\xi_i, \eta_j\rangle|^2 = \Sigma_j \|S^*\eta_j\|^2,$$

hence we see that this sum is independent of the orthonormal bases chosen. Moreover, this shows that if $A \in \mathcal{B}(\mathcal{H}_0, \mathcal{H})$, $B \in \mathcal{B}(\mathcal{K}, \mathcal{K}_0)$, and $S \in \mathrm{HS}(\mathcal{H}, \mathcal{K})$ then $BSA \in \mathrm{HS}(\mathcal{H}_0, \mathcal{K}_0)$ and

$$\|BSA\|_{\mathrm{HS}} \leq \|B\| \|S\|_{\mathrm{HS}} \|A\|.$$

The space of Hilbert-Schmidt operators is an inner-product space with inner-product given by

$$\langle S, T\rangle_{\rm HS} = \Sigma_{i,j}\langle S\xi_i, \eta_j\rangle\langle \eta_j, T\xi_i\rangle,$$

which is well defined by the Cauchy-Schwarz inequality, and does not depend on the bases by applying Perseval's identity.

The inner-product defined above turns $HS(\mathcal{H}, \mathcal{K})$ into a Hilbert space. To see this suppose $S_n \in {\rm HS}(\mathcal{H}, \mathcal{K})$ is a sequence of Hilbert-Schmidt operators, which is Cauchy in the Hilbert-Schmidt norm, then since the Hilbert-Schmidt norm is the same for any orthonormal basis it follows that for any vector $\xi \in \mathcal{H}$ we have that

$$\|S_n\xi - S_m\xi\| \leq \|\xi\|\|S_n - S_m\|_{\rm HS} \to 0.$$

Hence we may define an operator $S : \mathcal{H} \to \mathcal{K}$ by setting $S\xi = \lim_{n\to\infty} S_n\xi$ for each $\xi \in \mathcal{H}$. Note that by the uniform boundedness principle of we have that S is bounded.

Moreover, for any finite dimensional subspace $\mathcal{K}_0 \subset \mathcal{K}$ it is easy to see that

$$\|P_{\mathcal{K}_0}S - P_{\mathcal{K}_0}S_n\|_{\rm HS} \to 0,$$

and thus we have that

$$\|S\|_{\rm HS} = \sup_{\mathcal{K}_0\subset\mathcal{K},\, \dim(\mathcal{K}_0)<\infty} \|P_{\mathcal{K}_0}S\|_{\rm HS}$$

$$\leq \limsup_{n\to\infty} \|S_n\|_{\rm HS} < \infty.$$

Hence, $S \in {\rm HS}(\mathcal{H}, \mathcal{K})$ and so all that remains to show is that $\|S - S_n\|_{\rm HS} \to 0$.

Suppose $\varepsilon > 0$ is given. Since $S - S_n$ is Cauchy in the Hilbert-Schmidt norm, there exists some $N > 0$ such that for all $n \geq N$ we have that $\|S_N - S_n\|_{\rm HS} < \varepsilon/2$. Take $\mathcal{K}_0 \subset \mathcal{K}$ a finite dimensional subspace such that $\|P_{\mathcal{K}_0^\perp}(S - S_N)\|_{\rm HS} < \varepsilon/2$. Then for all $n \geq N$ we have

$$\|S - S_n\|_{\rm HS}$$

$$\leq \|P_{\mathcal{K}_0}S - P_{\mathcal{K}_0}S_n\|_{\rm HS} + \|P_{\mathcal{K}_0^\perp}S - P_{\mathcal{K}_0^\perp}S_N\|_{\rm HS} + \|P_{\mathcal{K}_0^\perp}S_N - P_{\mathcal{K}_0^\perp}S_n\|_{\rm HS}.$$

The first term above tends to zero while the others are each at most $\varepsilon/2$, hence

$$\limsup_{n\to\infty} \|S - S_n\|_{\rm HS} \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary it follows that $\|S - S_n\|_{\rm HS} \to 0$.

Note that because this does not depend on the choice of bases, we have that unitaries $U \in \mathcal{U}(\mathcal{H})$ and $V \in \mathcal{U}(\mathcal{K})$ define a unitary on ${\rm HS}(\mathcal{H}, \mathcal{K})$ by the map $S \mapsto USV$.

Suppose now that Γ is a group and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ are two unitary representations. We then have a unitary representation of Γ on ${\rm HS}(\mathcal{H}, \mathcal{K})$ given by $T \mapsto \rho(\gamma)T\pi(\gamma^{-1})$, for each $\gamma \in \Gamma$.

Consider the linear map Ξ between $\mathcal{K} \otimes_{\rm Alg} \overline{\mathcal{H}}$ and $\mathrm{HS}(\mathcal{H}, \mathcal{K})$ which takes an elementary tensor $\eta \otimes \xi$ to the rank one operator $\zeta \mapsto \langle \zeta, \overline{\xi} \rangle \eta$. This map preserves the inner-product structure on each space since if $\eta' \otimes \xi', \eta \otimes \xi \in \mathcal{K} \otimes_{\rm Alg} \overline{\mathcal{H}}$ are two elementary tensors then we have that

$$\begin{aligned}
\langle \Xi(\eta' \otimes \xi'), \Xi(\eta \otimes \xi) \rangle_{\rm HS} &= \Sigma_{i,j} \langle \Xi(\eta' \otimes \xi')\xi_i, \eta_j \rangle \langle \eta_j, \Xi(\eta \otimes \xi)\xi_i \rangle \\
&= \Sigma_{i,j} \langle \langle \xi_i, \overline{\xi'} \rangle \eta', \eta_j \rangle \langle \eta_j, \langle \xi_i, \overline{\xi} \rangle \eta \rangle \\
&= \Sigma_{i,j} \langle \overline{\xi}, \xi_i \rangle \langle \xi_i, \overline{\xi'} \rangle \langle \eta', \eta_j \rangle \langle \eta_j, \eta \rangle \\
&= \langle \overline{\xi}, \overline{\xi'} \rangle \langle \eta', \eta \rangle \\
&= \langle \xi' \otimes \eta', \xi \otimes \eta \rangle.
\end{aligned}$$

Since the space of finite rank operators is dense (in the Hilbert-Schmidt norm) in the space of Hilbert-Schmidt operators, it then follows that Ξ extends to a unitary operator from $\mathcal{K} \overline{\otimes} \overline{\mathcal{H}}$ to $HS(\mathcal{H}, \mathcal{K})$.

This isomorphism in turn gives an isomorphism between $\rho \otimes \overline{\pi}$ and the representation of Γ on $\mathrm{HS}(\mathcal{H}, \mathcal{K})$ described above. This follows easily from the formula

$$\rho(\gamma)\Xi(\xi \otimes \eta)\pi(\gamma^{-1}) = \Xi((\rho(\gamma)\xi) \otimes (\overline{\pi}(\gamma)\eta)),$$

for all $\gamma \in \Gamma$, $\xi \in \mathcal{H}$, and $\eta \in \overline{\mathcal{K}}$.

jesse.d.peterson@vanderbilt.edu