Lecture notes on ergodic theory

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March 17, 2011

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Chapter 1

Representations of groups

1.1 Definitions and constructions

Definition 1.1.1. Let Γ be a group, a **unitary representation** (resp. an **orthogonal representation**) of Γ is a homomorphism $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ (resp. $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$) from Γ into the unitary (resp. orthogonal) group of a complex (resp. real) Hilbert space $\mathcal{H}$.

Example 1.1.2. Let $\mathcal{H}$ be a Hilbert space and suppose Γ is a group. The **trivial representation** of Γ on $\mathcal{H}$ is given by the homomorphism which takes any group element to the identity operator on $\mathcal{H}$.

Example 1.1.3. Let Γ be a group, the **left-regular representation** (resp. **right-regular representation**) is given by $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$, (resp. $\rho : \Gamma \to \mathcal{U}(\ell^2\Gamma)$) which is defined by the formula $\lambda(\gamma)(\Sigma_{x\in\Gamma}\alpha_x\delta_x) = \Sigma_{x\in\Gamma}\alpha_x\delta_{\gamma x}$ (resp. $\rho(\gamma)(\Sigma_{x\in\Gamma}\alpha_x\delta_x) = \Sigma_{x\in\Gamma}\alpha_x\delta_{x\gamma^{-1}}$).

Example 1.1.4. More generally, if a group Γ acts on a set I then there is an associated unitary representation on $\ell^2 I$ defined by $\lambda(\gamma)(\Sigma_{x\in I}\alpha_x\delta_x) = \Sigma_{x\in I}\alpha_x\delta_{\gamma x}$. When $\Sigma < \Gamma$ is a subgroup, then Γ acts by left multiplication on the quotient space Γ/Σ and the associated unitary representation is the **quasi-regular representation**.

Suppose Γ is a group and $\Sigma < \Gamma$ is a subgroup. If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation, then we obtain the **reduced representation** $\pi_{|\Sigma}$ of Σ by restricting π to Σ.

Given a complex Hilbert space $\mathcal{H}$ the **adjoint Hilbert space** $\overline{\mathcal{H}}$ is the Hilbert space which agrees with $\mathcal{H}$ as a set, has scalar multiplication given by $\lambda\overline{\xi} = \overline{\overline{\lambda}\xi}$ and whose inner-product is given by $\langle\overline{\xi},\overline{\eta}\rangle_{\overline{\mathcal{H}}} = \overline{\langle\xi,\eta\rangle_{\mathcal{H}}}$.

Given a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ we define the **adjoint representation** $\overline{\pi} : \Gamma \to \mathcal{U}(\overline{\mathcal{H}})$ by setting $\overline{\pi}(\gamma)\overline{\xi} = \overline{\pi(\gamma)\xi}$. Note that we have the natural identification $\pi = \overline{\overline{\pi}}$.

If $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ is an orthogonal representation where $\mathcal{H}$ is a real Hilbert space then we will use the convention $\overline{\mathcal{H}} = \mathcal{H}$ and $\overline{\pi} = \pi$.

If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, and $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ are two unitary representations, then π is contained in ρ if there is an isometry $U : \mathcal{H} \to \mathcal{K}$ such that for all $\gamma \in \Gamma$ we have that $U\pi(\gamma) = \rho(\gamma)U$. If U can be chosen to be a unitary from $\mathcal{H}$ to $\mathcal{K}$ then the representations are isomorphic and we will write $\pi \cong \rho$.

Given a family of unitary representations $\pi_\iota : \Gamma \to \mathcal{U}(\mathcal{H}_\iota)$, with $\iota \in I$, the **direct-sum representation** is given by the map $\oplus_{\iota \in I}\pi_\iota : \Gamma \to \mathcal{U}(\oplus_{\iota\in I}\mathcal{H}_\iota)$ defined by

$$(\oplus_{\iota\in I}\pi_\iota)(\gamma) = \oplus_{\iota\in I}(\pi_\iota(\gamma)).$$

A unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is **reducible** if it contains a non-trivial sub-representations, or equivalently, there exists a closed Γ-invariant subspace $\mathcal{K} \subset \mathcal{H}$ such that $\mathcal{K} \neq \{0\}, \mathcal{H}$. In this case it is easy to see that $\mathcal{K}^\perp$ is also Γ-invariant and the natural isomorphism from $\mathcal{K} \oplus \mathcal{K}^\perp$ induces an equivalence between the representations $\pi_{|\mathcal{K}} \oplus \pi_{|\mathcal{K}^\perp}$ and π. If a unitary representation has no non-trivial invariant subspace we say that it is **irreducible**.

A vector $\xi \in \mathcal{H}$ is a **cyclic vector** if the smallest Γ-invariant subspace of $\mathcal{H}$ which contains ξ is $\mathcal{H}$ itself. Note that if π is irreducible then every non-zero vector is cyclic.

Lemma 1.1.5 (Schur's Lemma)**.** *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, and $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ be two irreducible unitary representations of a group Γ, if $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is Γ-invariant then either $T = 0$, or else T is a scalar multiple of a unitary. In particular, $\mathcal{B}(\mathcal{H}, \mathcal{K})$ has a non-zero Γ-invariant operator if and only if π and ρ are isomorphic.*

Proof. Let π and ρ be as above and suppose $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is Γ-invariant. Thus, $T^*T \in \mathcal{B}(\mathcal{H})$ is Γ-invariant and hence any spectral projection of T^*T gives a Γ-invariant subspace. Since π is irreducible it then follows that $T^*T \in \mathbb{C}$. If $T^*T \neq 0$ then my multiplying T by a scalar we may assume that T is an isometry. Hence, $TT^* \in \mathcal{B}(\mathcal{K})$ is a non-zero Γ-invariant projection, and since ρ is irreducible it follows that $TT^* = TT^* = 1$. □

If I is finite, then the **tensor product representation** is given by the map $\otimes_{\iota\in I}\pi_\iota : \Gamma \to \mathcal{U}(\overline{\otimes}_{\iota\in I}\mathcal{H}_\iota)$ defined by

$$(\otimes_{\iota\in I}\pi_\iota)(\gamma) = \otimes_{\iota\in I}(\pi_\iota(\gamma)).$$

Lemma 1.1.6 (Fell's Absorption Principle)**.** *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a group Γ, let $1_\mathcal{H}$ denote the trivial representation of Γ on $\mathcal{H}$, and let $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ denote the left-regular representation. Then the representations $\lambda \otimes \pi$ and $\lambda \otimes 1_\mathcal{H}$ are isomorphic.*

Proof. Consider the unitary $U \in \mathcal{U}(\ell^2\Gamma \otimes \mathcal{H})$ determined by $U(\delta_x \otimes \xi) = \delta_x \otimes \pi(x)\xi$, for all $x \in \Gamma$, $\xi \in \mathcal{H}$. Then for all $\gamma, x \in \Gamma$, and $\xi \in \mathcal{H}$ we have that

$$\begin{aligned} U^*(\lambda \otimes \pi)(\gamma)U(\delta_x \otimes \xi) &= U^*(\lambda \otimes \pi)(\gamma)(\delta_x \otimes \pi(x)\xi) \\ &= U^*(\delta_{\gamma x} \otimes \pi(\gamma)\pi(x)\xi) \\ &= \delta_{\gamma x} \otimes \pi((\gamma x)^{-1})\pi(\gamma)\pi(x)\xi = (\lambda \otimes 1_\mathcal{H})(\gamma)(\delta_x \otimes \xi). \end{aligned}$$

□

Example 1.1.7 (Hilbert-Schmidt operators)**.** Given two Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$ with respective orthonormal bases $\{\xi_i\} \subset \mathcal{H}$ and $\{\eta_j\} \subset \mathcal{K}$ we define the space of Hilbert-Schmidt operators from $\mathcal{H}$ to $\mathcal{K}$ by

$$\mathrm{HS}(\mathcal{H}, \mathcal{K}) = \{S \in \mathcal{B}(\mathcal{H}, \mathcal{K}) \mid \|S\|_{\mathrm{HS}}^2 = \Sigma_i \|S\xi_i\|^2 < \infty\}.$$

Parseval's identity gives

$$\Sigma_i \|S\xi_i\|^2 = \Sigma_{i,j} |\langle S\xi_i, \eta_j\rangle|^2 = \Sigma_j \|S^*\eta_j\|^2,$$

hence we see that this sum is independent of the orthonormal bases chosen. Moreover, this shows that if $A \in \mathcal{B}(\mathcal{H}_0, \mathcal{H})$, $B \in \mathcal{B}(\mathcal{K}, \mathcal{K}_0)$, and $S \in \mathrm{HS}(\mathcal{H}, \mathcal{K})$ then $BSA \in \mathrm{HS}(\mathcal{H}_0, \mathcal{K}_0)$ and

$$\|BSA\| \leq \|BSA\|_{\mathrm{HS}} \leq \|B\| \|S\|_{\mathrm{HS}} \|A\|.$$

The space of Hilbert-Schmidt operators is an inner-product space with inner-product given by

$$\langle S, T\rangle_{\mathrm{HS}} = \Sigma_{i,j} \langle S\xi_i, \eta_j\rangle \langle \eta_j, T\xi_i\rangle,$$

which is well defined by the Cauchy-Schwarz inequality, and does not depend on the bases by applying Perseval's identity.

The inner-product defined above turns $HS(\mathcal{H}, \mathcal{K})$ into a Hilbert space. To see this suppose $S_n \in \mathrm{HS}(\mathcal{H}, \mathcal{K})$ is a sequence of Hilbert-Schmidt operators, which is Cauchy in the Hilbert-Schmidt norm, then S_n is also Cauchy in the operator norm and hence converges in the operator norm to an operator $S : \mathcal{H} \to \mathcal{K}$.

For any finite dimensional subspace $\mathcal{K}_0 \subset \mathcal{K}$ it is easy to see that

$$\|P_{\mathcal{K}_0} S - P_{\mathcal{K}_0} S_n\|_{\mathrm{HS}} \to 0,$$

and thus we have that

$$\begin{aligned} \|S\|_{\mathrm{HS}} &= \sup_{\mathcal{K}_0 \subset \mathcal{K},\, \dim(\mathcal{K}_0) < \infty} \|P_{\mathcal{K}_0} S\|_{\mathrm{HS}} \\ &\leq \limsup_{n \to \infty} \|S_n\|_{\mathrm{HS}} < \infty. \end{aligned}$$

Hence, $S \in \mathrm{HS}(\mathcal{H}, \mathcal{K})$ and so all that remains to show is that $\|S - S_n\|_{\mathrm{HS}} \to 0$.

Suppose $\varepsilon > 0$ is given. Since $S - S_n$ is Cauchy in the Hilbert-Schmidt norm, there exists some $N > 0$ such that for all $n \geq N$ we have that $\|S_N - S_n\|_{\mathrm{HS}} < \varepsilon/2$. Take $\mathcal{K}_0 \subset \mathcal{K}$ a finite dimensional subspace such that $\|P_{\mathcal{K}_0^\perp}(S - S_N)\|_{\mathrm{HS}} < \varepsilon/2$. Then for all $n \geq N$ we have

$$\begin{aligned} &\|S - S_n\|_{\mathrm{HS}} \\ \leq \|P_{\mathcal{K}_0} S - P_{\mathcal{K}_0} S_n\|_{\mathrm{HS}} &+ \|P_{\mathcal{K}_0^\perp} S - P_{\mathcal{K}_0^\perp} S_N\|_{\mathrm{HS}} + \|P_{\mathcal{K}_0^\perp} S_N - P_{\mathcal{K}_0^\perp} S_n\|_{\mathrm{HS}}. \end{aligned}$$

The first term above tends to zero while the others are each at most $\varepsilon/2$, hence

$$\limsup_{n \to \infty} \|S - S_n\|_{\mathrm{HS}} \leq \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary it follows that $\|S - S_n\|_{\mathrm{HS}} \to 0$.

Note that because this does not depend on the choice of bases, we have that unitaries $U \in \mathcal{U}(\mathcal{H})$ and $V \in \mathcal{U}(\mathcal{K})$ define a unitary on $\mathrm{HS}(\mathcal{H}, \mathcal{K})$ by the map $S \mapsto USV$.

Suppose now that Γ is a group and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ are two unitary representations. We then have a unitary representation of Γ on $\mathrm{HS}(\mathcal{H}, \mathcal{K})$ given by $T \mapsto \rho(\gamma)T\pi(\gamma^{-1})$, for each $\gamma \in \Gamma$.

Consider the linear map Ξ between $\mathcal{K}\overline{\otimes}\overline{\mathcal{H}}$ and $\mathrm{HS}(\mathcal{H}, \mathcal{K})$ which takes a vector $\zeta \in \mathcal{K}\overline{\otimes}\overline{\mathcal{H}}$ to the operator $\Xi(\zeta)$ determined by the formula

$$\langle \Xi(\zeta)\xi, \eta \rangle = \langle \zeta, \eta \otimes \overline{\xi} \rangle,$$

for all $\xi \in \mathcal{H}$ and $\eta \in \mathcal{K}$.

Then Ξ is a unitary since if $\zeta, \zeta' \in \mathrm{HS}(\mathcal{H}, \mathcal{K})$ we have that

$$\begin{aligned}\langle \Xi(\zeta), \Xi(\zeta') \rangle_{\mathrm{HS}} &= \Sigma_{i,j} \langle \Xi(\zeta)\xi_i, \eta_j \rangle \langle \eta_j, \Xi(\zeta')\xi_i \rangle \\ &= \Sigma_{i,j} \langle \zeta, \eta_j \otimes \xi_i \rangle \langle \eta_j \otimes \xi_i, \zeta' \rangle = \langle \zeta, \zeta' \rangle.\end{aligned}$$

It is easy to see that this isomorphism in turn gives an isomorphism between $\rho \otimes \overline{\pi}$ and the representation of Γ on $\mathrm{HS}(\mathcal{H}, \mathcal{K})$ described above.

1.2 Functions of positive type

Definition 1.2.1. Given a group Γ, a **function of positive type** on Γ is a map $\varphi : \Gamma \to \mathbb{C}$ such that for all $\Sigma_{\gamma \in \Gamma} \alpha_\gamma u_\gamma \in \mathbb{C}\Gamma$ we have

$$\Sigma_{\gamma, \lambda \in \Gamma} \overline{\alpha_\lambda} \alpha_\gamma \varphi(\lambda^{-1}\gamma) \geq 0.$$

One way in which function of positive types appear is when we have a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{K})$, together with a vector $\xi \in \mathcal{K}$. Then function defined by:

$$\varphi_\xi(\gamma) = \langle \pi(\gamma)\xi, \xi \rangle, \text{ for all } \gamma \in \Gamma,$$

describes a function of positive type on Γ. Indeed, if $\Sigma_{\gamma \in \Gamma} \alpha_\gamma u_\gamma \in \mathbb{C}\Gamma$ then

$$\begin{aligned}\Sigma_{\gamma, \lambda \in \Gamma} \overline{\alpha_\lambda} \alpha_\gamma \varphi(\lambda^{-1}\gamma) &= \Sigma_{\gamma, \lambda \in \Gamma} \overline{\alpha_\lambda} \alpha_\gamma \langle \pi(\lambda^{-1}\gamma)\xi, \xi \rangle \\ &= \|\Sigma_{\gamma \in \Gamma} \alpha_\gamma \pi(\gamma)\xi\|^2 \geq 0.\end{aligned}$$

Every function of positive type arrises in this way. Specifically, if $\varphi : \Gamma \to \mathbb{C}$ is a function of positive type then we may place a pseudo-inner product on $\mathbb{C}\Gamma$ by the formula:

$$\langle \Sigma_{\gamma \in \Gamma} \alpha_\gamma u_\gamma, \Sigma_{\lambda \in \Gamma} \beta_\lambda u_\lambda \rangle_\varphi = \Sigma_{\gamma, \lambda \in \Gamma} \overline{\beta_\lambda} \alpha_\gamma \varphi(\lambda^{-1}\gamma).$$

The positivity of this inner product is given by the fact that φ is of positive type. After quotienting by the kernel and completion we obtain a Hilbert space $\mathcal{K}$. Moreover, there is a natural unitary representation of Γ on $\mathcal{K}$ given by

$$\pi_\varphi(\gamma_0) \Sigma_{\gamma \in \Gamma} \alpha_\gamma u_\gamma = \Sigma_{\gamma \in \Gamma} \alpha_\gamma u_{\gamma_0 \gamma}.$$

The fact that this is well defined and describes a unitary follows from the fact that for all $\gamma_0 \in \Gamma$, and $\Sigma_{\gamma\in\Gamma}\alpha_\gamma u_\gamma \in \mathbb{C}\Gamma$ we have

$$\|\Sigma_{\gamma\in\Gamma}\alpha_\gamma u_{\gamma_0\gamma}\|_\phi^2 = \Sigma_{\gamma,\lambda\in\Gamma}\overline{\alpha}_\lambda\alpha_\gamma\phi((\gamma_0\lambda)^{-1}(\gamma_0\gamma))$$
$$= \Sigma_{\gamma,\lambda\in\Gamma}\overline{\alpha}_\lambda\alpha_\gamma\phi(\lambda^{-1}\gamma) = \|\Sigma_{\gamma\in\Gamma}\alpha_\gamma u_\gamma\|_\phi^2.$$

One has to check that this action is well defined and preserves the inner product structure, and then the cyclic vector $\xi_\varphi = u_e$ gives the formula above.

If we start with a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, with a non-zero vector $\xi \in \mathcal{H}$, consider the function of positive type φ_ξ, and then consider the representation π_{φ_ξ}. Then the map $\Sigma_{\gamma\in\Gamma}\alpha_\gamma u_\gamma \mapsto \Sigma_{\gamma\in\Gamma}\alpha_\gamma\pi(\gamma)\xi$ defines an isometry from $\mathcal{K}$ to $\mathcal{H}$ which is Γ equivariant. If ξ is a cyclic vector for π then this gives a natural isomorphism between π_{φ_ξ} and π.

One can easily check that the correspondence described above satisfies the following relationships:

functions of positive type	pointed unitary representations
$\varphi = 1$	the trivial representation
$\varphi = \delta_e$	the left regular representation on $\ell^2\Gamma$, $\xi = \delta_e$
$\varphi = \delta_\Lambda$, $\Lambda < \Gamma$	the quasi-regular representation on $\ell^2(\Gamma/\Lambda)$, $\xi = \delta_\Lambda$
φ a character	the one dimensional representation given by the character
addition	direct sum
product	tensor product

1.3 Cocycles

Definition 1.3.1. Suppose $\Gamma \curvearrowright X$ is an action of a group Γ on a set X, and Λ is a group. A **cocycle** for the action into Λ is a map $\alpha : \Gamma \times X \to \Lambda$ such that

$$\alpha(\gamma_1\gamma_2, x) = \alpha(\gamma_1, \gamma_2 x)\alpha(\gamma_2, x),$$

for all $\gamma_1, \gamma_2 \in \Gamma$, and $x \in X$. Two cocycles $\alpha, \beta : \Gamma \times X \to \Lambda$ are **cohomologous** if there is a map $\xi : X \to \Lambda$ such that

$$\alpha(\gamma, x) = \xi(\gamma x)\beta(\gamma, x)\xi(x)^{-1},$$

for all $\gamma \in \Gamma$, $x \in X$. A cocycle α is **trivial** if it is cohomologous to the cocycle which takes constant value $e \in \Lambda$. A cocycle α **untwists** if there is a homomorphism $\delta : \Gamma \to \Lambda$ such that α is cohomologous to the cocycle given by $(\gamma, x) \mapsto \delta(\gamma)$, for all $\gamma \in \Gamma$, and $x \in X$.

The set of all cocycles for the action $\Gamma \curvearrowright X$ with values in Λ is denoted by $Z^1(\Gamma \curvearrowright X; \Lambda)$, the set of trivial cocycles is denoted by $B^1(\Gamma \curvearrowright X; \Lambda)$, and the set of equivalence classes of cohomologous cocycles is denoted by $H^1(\Gamma \curvearrowright X; \Lambda)$. Note that if Λ is abelian then $Z^1(\Gamma \curvearrowright X; \Lambda)$ is an abelian group under pointwise multiplication, $B^1(\Gamma \curvearrowright X; \Lambda)$ is a subgroup and $H^1(\Gamma \curvearrowright X; \Lambda)$ is the quotient group.

Example 1.3.2. If $\alpha : \Gamma \times X \to \Lambda$ is a cocycle for an action $\Gamma \curvearrowright X$ on a group Λ, and if $S \subset \Gamma$ is a generating set, then the cocycle relation implies that the cocycle is completely determined by its values on the set $S \times X \subset \Gamma \times X$. In particular, for the group $\mathbb{Z}$, a cocycle $\alpha : \mathbb{Z} \times X \to \Lambda$ for an action $\mathbb{Z} \curvearrowright X$ is completely determined by the function $x \mapsto \alpha(1, x)$. Moreover, any function $\xi : X \to \Lambda$ determines a cocycle α such that $\alpha(1, x) = \xi(x)$.

For instance if $T : X \to X$ is a bijection and $\mathbb{Z} \curvearrowright X$ by $n \cdot x = T^n(x)$, and if $\xi : X \to \mathbb{R}$ is a function, then the corresponding cocycle

$$\alpha(n, x) = \Sigma_{k=0}^{n-1} \xi \circ T^k(x),$$

for $n > 0$. Hence, if we denote by $S_n : X \to \mathcal{R}$ the function given by

$$S_n(x) = \alpha(n, T^{-n}(x)),$$

then $\frac{1}{n} S_n$ is the average of the functions $\xi \circ T^{-k}$ for $1 \leq k \leq n$.

Example 1.3.3. Suppose Γ and Λ are two groups and consider the space

$$[\Gamma, \Lambda] = \{f : \Gamma \to \Lambda \mid f(e) = e\} \subset \Lambda^\Gamma.$$

We have an action of Γ on this set by

$$(\gamma \cdot f)(x) = f(x\gamma) f(\gamma)^{-1}.$$

Note that the fixed points of this action are precisely the set of homomorphisms from Γ to Λ.

We then have a cocycle for this action $\alpha : \Gamma \times [\Gamma, \Lambda] \to \Lambda$ given by

$$\alpha(\gamma, f) = f(\gamma).$$

Note that this action preserves the subsets of injective, surjective, and bijective maps.

Exercise 1.3.4. Think about the previous example and verify all the claims.

Example 1.3.5. One natural way in which cocycles arise is if we have a set X together with a pair of actions $\Gamma \curvearrowright X$, and $\Lambda \curvearrowright X$ of groups Γ, and Λ.

If the action of Λ is free (if $\lambda \neq e$ then $\lambda x \neq x$ for all $x \in X$), and if the orbits of Γ are contained in the orbits of Λ ($\Gamma x \subset \Lambda x$ for each $x \in X$), then we can define a cocycle $\alpha : \Gamma \times X \to \Lambda$ by setting $\alpha(\gamma, x)$ to be the unique element in Λ such that

$$\gamma x = \alpha(\gamma, x) x.$$

We can verify that α is a cocycle since for each $\gamma_1, \gamma_2 \in \Gamma$ and $x \in X$ we have

$$\alpha(\gamma_1 \gamma_2, x) x = \gamma_1 \gamma_2 x$$

$$= \alpha(\gamma_1, \gamma_2 x)(\gamma_1 x) = \alpha(\gamma_1, \gamma_2 x) \alpha(\gamma_2, x) x.$$

Since the action $\Lambda \curvearrowright X$ is free this then implies that

$$\alpha(\gamma_1 \gamma_2, x) = \alpha(\gamma_1, \gamma_2 x) \alpha(\gamma_2, x).$$

Definition 1.3.6. If $\Gamma \curvearrowright Y$ is an action of a group Γ on a set Y then a **fundamental domain** for the action is a subset $D \subset Y$ such that each orbit of Γ contains exactly one element in D, i.e., for all $x \in Y$ we have $|\Gamma x \cap D| = 1$.

Note that if $D \subset Y$ is a fundamental domain for the action $\Gamma \curvearrowright Y$ then we have that the map from $\Gamma \times D$ to Y given by $(\gamma, x) \mapsto \gamma x$ is sujective. Moreover, if the action $\Gamma \curvearrowright Y$ is free then the map is also injective and hence the space Y decomposes as a disjoint union of orbits

$$Y = \sqcup_{x \in D} \Gamma x.$$

We thus obtain a bijection between the fundamental domain D and the orbit space $\Gamma \backslash X$ given by $x \mapsto \Gamma x$.

Example 1.3.7. Another way in which cocycles arise is if Y is a set and we have a left action of Γ on Y and a right action of Λ such that the actions commute, i.e., for each $\gamma \in \Gamma$, $\lambda \in \Lambda$, and $x \in Y$ we have

$$(\gamma x)\lambda = \gamma(x\lambda).$$

As in Example 1.3.5 we will require the action of Λ to be free.

Since the actions of Γ and Λ commute, the action of Γ on Y passes to the space of Λ orbits Y/Λ. If $D \subset Y$ is a fundamental domain for the action $Y \curvearrowleft \Lambda$ then as mentioned above the map from D to Y/Λ given by $x \mapsto x\Lambda$ is a bijection. We will denote by $\Phi : Y/\Lambda \to D$ the inverse map.

We then obtain a cocycle $\alpha : \Gamma \times (Y/\Lambda) \to \Lambda$ by assigning to each $\gamma \in \Gamma$ and $x\Lambda \in Y/\Lambda$ the value of $\alpha(\gamma, x\Lambda)$ to be the unique element in Λ such that

$$\gamma\Phi(x\Lambda) = \Phi(\gamma x\Lambda)\alpha(\gamma, x\Lambda).$$

If $\gamma_1, \gamma_2 \in \Gamma$ and $x\Lambda \in Y/\Lambda$ then we have

$$\begin{aligned} \Phi(\gamma_1\gamma_2 x\Lambda)\alpha(\gamma_1\gamma_2, x\Lambda) &= \gamma_1\gamma_2\Phi(x\Lambda) \\ &= \gamma_1\Phi(\gamma_2 x\Lambda)\alpha(\gamma_2, x\Lambda) \\ &= \Phi(\gamma_1\gamma_2 x\Lambda)\alpha(\gamma_1, \gamma_2 x\Lambda)\alpha(\gamma_2, x\Lambda). \end{aligned}$$

Since the action $Y \curvearrowleft \Lambda$ is free this shows that α is indeed a cocycle.

Suppose $D' \subset Y$ is also a fundamental domain for the action $Y \curvearrowleft \Lambda$ and $\beta : \Gamma \times (Y/\Lambda) \to \Lambda$ is the corresponding cocycle. If we let $\Phi' : Y/\Lambda \to D'$ be the corresponding selection map for D', and we define $\xi : Y/\Lambda \to \Lambda$ so that for all $x\Lambda \in Y/\Lambda$ we have

$$\xi(x\Lambda) = \Phi'(x\Lambda)\Phi(x\Lambda)^{-1},$$

then it's easy to see that for each $\gamma \in \Gamma$ and $x\Lambda \in Y/\Lambda$ we have

$$\xi(\gamma x\Lambda)\alpha(\gamma, x\Lambda)\xi(x\Lambda)^{-1} = \beta(\gamma, x\Lambda).$$

Hence, the cohomology class of α is independent of the fundamental domain D.

Example 1.3.8. A special case of the above example to keep in mind is when Γ is a group and $\Sigma < \Gamma$ is a subgroup. Then $\Gamma \curvearrowright \Gamma \curvearrowleft \Sigma$ is a pair of commuting actions given by group multiplication. Thus, we obtain a cocycle (unique up to cohomology) $\alpha : \Gamma \times (\Gamma/\Sigma) \to \Sigma$.

Exercise 1.3.9. Show that in Example 1.3.8 if the action $\Gamma \curvearrowleft \Sigma$ has a fundamental domain $\Gamma_0 \subset \Gamma$ which is a subgroup of Γ, then we have that $\Sigma \lhd \Gamma$ is a normal subgroup and Γ splits as a semidirect product $\Gamma = \Gamma_0 \ltimes \Sigma$. Also, compute the corresponding cocycle for this fundamental domain.

1.4 Induced representations

Given an action $\Gamma \curvearrowright X$ of a group Γ on a set X, a cocycle $\alpha : \Gamma \times X \to \Lambda$ into a group Λ, and a unitary representation $\pi : \Lambda \to \mathcal{U}(\mathcal{K})$, we obtain an induced representation $\mathrm{Ind}_\Lambda^\alpha \pi : \Gamma \to \mathcal{U}(\ell^2 X \overline{\otimes} \mathcal{K})$ by linearly extending the formula

$$\mathrm{Ind}_\Lambda^\alpha \pi(\gamma)(\delta_x \otimes \eta) = \delta_{\gamma x} \otimes (\pi(\alpha(\gamma, x))\eta).$$

We can easily check that this is a representation since for all $\gamma_1, \gamma_2 \in \Gamma$, $x \in X$, and $\eta \in \mathcal{K}$ we have

$$\begin{aligned}
\mathrm{Ind}_\Lambda^\alpha \pi(\gamma_1\gamma_2)(\delta_x \otimes \eta) &= \delta_{\gamma_1\gamma_2 x} \otimes (\pi(\alpha(\gamma_1\gamma_2, x))\eta) \\
&= \delta_{\gamma_1\gamma_2 x} \otimes (\pi(\alpha(\gamma_1, \gamma_2 x))\pi(\alpha(\gamma_2, x))\eta) \\
&= \mathrm{Ind}_\Lambda^\alpha \pi(\gamma_1)(\delta_{\gamma_2 x} \otimes (\pi(\alpha(\gamma_2, x))\eta) \\
&= \mathrm{Ind}_\Lambda^\alpha \pi(\gamma_1)\,\mathrm{Ind}_\Lambda^\alpha \pi(\gamma_2)(\delta_x \otimes \eta).
\end{aligned}$$

If $\beta : \Gamma \times X \to \Lambda$ is cocycle which is cohomologous to α, and $\xi : X \to \Lambda$ such that $\alpha(\gamma, x) = \xi(\gamma x)\beta(\gamma, x)\xi(x)^{-1}$, for all $\gamma \in \Gamma$, and $x \in X$, then we obtain a unitary $U_\xi \in \mathcal{U}(\ell^2 X \overline{\otimes} \mathcal{K})$, by linearly extending the formula

$$U_\xi(\delta_x \otimes \eta) = \delta_x \otimes \pi(\xi(x)^{-1})\eta.$$

We then see easily that for all $\gamma \in \Gamma$, $x \in X$, and $\eta \in \mathcal{K}$ we have

$$\begin{aligned}
U_\xi^* \,\mathrm{Ind}_\Lambda^\beta \pi(\gamma) U_\xi(\delta_x \otimes \eta) &= U_\xi^* \,\mathrm{Ind}_\Lambda^\beta \pi(\gamma)(\delta_x \otimes \pi(\xi(x)^{-1})\eta) \\
&= U_\xi^*(\delta_{\gamma x} \otimes \pi(\beta(\gamma, x))\pi(\xi(x)^{-1})\eta) \\
&= \delta_{\gamma x} \otimes \pi(\xi(\gamma x))\pi(\beta(\gamma, x))\pi(\xi(x)^{-1})\eta \\
&= \mathrm{Ind}_\Lambda^\alpha \pi(\gamma)(\delta_x \otimes \eta).
\end{aligned}$$

Hence, the representations $\mathrm{Ind}_\Lambda^\beta \pi$ and $\mathrm{Ind}_\Lambda^\alpha \pi$ are isomorphic.

Lemma 1.4.1. *Let $\Gamma \curvearrowright X$ be an action of a group Γ on a set X, let Λ be a group, and suppose $\alpha : \Gamma \times X \to \Lambda$ is a cocycle. If $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ and $\rho : \Lambda \to \mathcal{U}(\mathcal{K})$ are unitary representations such that $\pi \cong \rho$ then $\mathrm{Ind}_\Lambda^\alpha \pi \cong \mathrm{Ind}_\Lambda^\alpha \rho$.*

Proof. If $W : \mathcal{H} \to \mathcal{K}$ is a unitary such that $W\pi(\gamma) = \rho(\gamma)W$ for all $\gamma \in \Gamma$. Then $\mathrm{id} \otimes W : \ell^2 X \otimes \mathcal{H} \to \ell^2 X \otimes K$, and it follows easily that

$$(\mathrm{id} \otimes W) \operatorname{Ind}_\Lambda^\alpha \pi(\gamma) = \operatorname{Ind}_\Lambda^\alpha \rho(\gamma)(\mathrm{id} \otimes W).$$

□

Example 1.4.2. Consider a group Γ acting on itself by left multiplication, and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then we obtain cocycles $\alpha, \beta : \Gamma \times \Gamma \to \Gamma$ given by $\alpha(\gamma, x) = \gamma$, and $\beta(\gamma, x) = e$.

By considering the function $\xi(x) = x$ we see that these cocycles are cohomologous, for every $\gamma, x \in \Gamma$ we have

$$\alpha(\gamma, x) = \gamma = (\gamma x)x^{-1} = \xi(\gamma x)\beta(\gamma, x)\xi(x)^{-1}.$$

We therefore recover Fell's absorption principle by obtaining an isomorphism between the representations $\lambda \otimes \pi = \operatorname{Ind}_\Gamma^\alpha \pi$ and $\lambda \otimes 1_{\mathcal{H}} = \operatorname{Ind}_\Gamma^\beta \pi$.

If Γ is a group, and $\Sigma < \Gamma$ is a subgroup, then we may consider the cocycle $\alpha : \Gamma \times (\Gamma/\Sigma) \to \Sigma$ associated to some fundamental domain for Σ as is described in Example 1.3.8. Hence if $\pi : \Sigma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Σ then we obtain the **induced representation** $\operatorname{Ind}_\Sigma^\Gamma \pi : \Gamma \to \mathcal{U}(\ell^2(\Gamma/\Sigma)\overline{\otimes}\mathcal{H})$ as the induced representation associated to the cocycle α. Since the cohomology class of α does not depend on the fundamental domain, we have that the induced representation is well defined up to unitary equivalence.

Observe that if $\xi_0 \in \mathcal{H}$, then we may consider the vector $\xi' = \delta_\Sigma \otimes \xi_0 \in \ell^2(\Gamma/\Sigma)\overline{\otimes}\mathcal{H}$. We then have that the positive definite function $\phi_{\xi'} : \Gamma \to \mathbb{C}$ is given by

$$\phi_{\xi'}(\gamma) = \begin{cases} \phi_\xi(\gamma) & \text{if } \gamma \in \Sigma; \\ 0 & \text{otherwise.} \end{cases}$$

Also observe that if $\pi = 1_{\mathcal{H}} : \Sigma \to \mathcal{U}(\mathcal{H})$ is the trivial representation then $\operatorname{Ind}_\Sigma^\Gamma \pi = \lambda_{\Gamma/\Sigma} \otimes 1_{\mathcal{H}}$ is a multiple of the quasi-regular representation corresponding to Σ.

Remark 1.4.3. Suppose Y is a set and we have commuting left and right actions $\Gamma \curvearrowright Y \curvearrowleft \Lambda$ of groups Γ and Λ such that the action of Λ is free. If $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ is a unitary representation, and $\alpha : \Gamma \times (Y/\Lambda) \to \Lambda$ is the cocycle coming from a fundamental domain D for $Y \curvearrowleft$ as explained in Example 1.3.7, then we obtain the induced representation $\operatorname{Ind}_\Lambda^\alpha \pi$. Here we will explain an alternate way to obtain this representation which makes no reference to a fundamental domain.

Consider a function $\xi : X \to \mathcal{H}$ which is Λ-equivariant, i.e., $\xi(x\lambda^{-1}) = \pi(\lambda)\xi(x)$, for all $x \in X$, and $\lambda \in \Lambda$. Because, ξ is Λ-equivariant we have that the function $x \mapsto \|\xi(x)\|$ is constant on the Λ-orbits. We may therefore consider the well defined space

$$L^2(Y; \mathcal{H})^\Lambda = \{\xi : Y \to \mathcal{H} \mid \xi \text{ is } \Lambda - \text{equivariant, and } \Sigma_{x\Lambda \in Y/\Lambda} \|\xi(x)\|^2 < \infty\}.$$

Since this space consists of Λ-equivariant functions, this becomes a Hilbert space when it is endowed with the well defined inner-product

$$\langle \xi, \eta \rangle = \Sigma_{x\Lambda \in Y/\Lambda} \langle \xi(x), \eta(x) \rangle.$$

We then obtain the induced representation of π on $L^2(Y;\mathcal{H})^\Lambda$ by requiring that for all $\gamma \in \Gamma$ and $x \in Y$ we have

$$(\mathrm{Ind}_\Lambda^\Gamma \pi(\gamma)\xi)(x) = \xi(\gamma^{-1}x).$$

If $D \subset Y$ is a fundamental domain for the action $Y \curvearrowleft \Lambda$, then any function $\xi \in \ell^2 D \overline{\otimes} \mathcal{H} \cong \ell^2(D;\mathcal{H})$ extends uniquely to a Λ-equivariant function $\tilde{\xi} \in \ell^2(Y;\mathcal{H})^\Lambda$ by setting $\tilde{\xi}(x\lambda^{-1}) = \pi(\lambda)\xi(x)$ for each $\lambda \in \Lambda$ and $x \in D$. By identifying Y/Λ with D by the map Φ we then obtain a unitary $W : \ell^2(Y/\Lambda)\overline{\otimes}\mathcal{H} \to \ell^2(Y;\mathcal{H})$. If $\alpha : \Gamma \times (Y/\Lambda) \to \Lambda$ is the cocycle from Example 1.3.7, then unwinding the definitions gives

$$W\,\mathrm{Ind}_\Lambda^\alpha(\gamma) = \mathrm{Ind}_\Lambda^\Gamma(\gamma) W,$$

for all $\gamma \in \Gamma$.

Lemma 1.4.4. *Let Γ be a group and $\Delta < \Sigma < \Gamma$ subgroups. If $\pi : \Delta \to \mathcal{U}(\mathcal{H})$ is a unitary representation then*

$$\mathrm{Ind}_\Delta^\Gamma \pi \cong \mathrm{Ind}_\Sigma^\Gamma \mathrm{Ind}_\Delta^\Sigma \pi.$$

Proof. Using the equivalence in the previous remark, we may consider the map $W : L^2(\Gamma; L^2(\Sigma;\mathcal{H})^\Delta)^\Sigma \to L^2(\Gamma;\mathcal{H})^\Delta$ given by

$$(Wf)(\gamma) = (f(\gamma))(e).$$

Note that if $\delta \in \Delta$ then we have

$$\begin{aligned}(Wf)(\gamma\delta^{-1}) = (f(\gamma\delta^{-1}))(e) = (\mathrm{Ind}_\Delta^\Sigma \pi(\delta) f(\gamma))(e) \\ = (f(\gamma))(\delta^{-1}) \\ = \pi(\delta)((f(\gamma))(e)) = \pi(\delta)((Wf)(\gamma)).\end{aligned}$$

Also, if $D \subset \Sigma$ is a set of coset representatives for Δ, and $E \subset \Gamma$ is a set of coset representatives for Σ then $ED \subset \Gamma$ is a set of coset representatives for Δ, hence just as above we see that for all $f \in L^2(\Gamma; L^2(\Sigma;\mathcal{H})^\Delta)^\Sigma$ we have

$$\|Wf\|^2 = \Sigma_{(\gamma,\lambda)\in E\times D} \|(Wf)(\gamma\lambda)\|^2$$

$$= \Sigma_{\gamma\in E}\Sigma_{\lambda\in D}\|((Wf)(\gamma))(\lambda)\|^2 = \|f\|^2.$$

Thus, W is a well defined isometry, which is easy to see is a unitary. An easy calculation then shows that for all $\gamma \in \Gamma$ we have

$$\mathrm{Ind}_\Delta^\Gamma \pi(\gamma) W = W\,\mathrm{Ind}_\Sigma^\Gamma \mathrm{Ind}_\Delta^\Sigma \pi(\gamma).$$

□

Proposition 1.4.5. *If $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a unitary representation of Γ and $\rho : \Sigma \to \mathcal{U}(\mathcal{K})$ is a unitary representation of Σ, then*

$$\pi \otimes \mathrm{Ind}_\Sigma^\Gamma \rho \cong \mathrm{Ind}_\Sigma^\Gamma(\pi_{|\Sigma} \otimes \rho).$$

Proof. Consider the map $W : \mathcal{H}\overline{\otimes}L^2(\Gamma; \mathcal{K})^\Sigma \to L^2(\Gamma; \mathcal{H}\overline{\otimes}\mathcal{K})^\Sigma$ such that for each $\gamma \in \Gamma$, $\xi \in \mathcal{H}$, and $f \in L^2(\Gamma; \mathcal{K})^\Sigma$ we have

$$(W(\xi \otimes f))(\gamma) = (\pi(\gamma^{-1})\xi) \otimes f(\gamma).$$

Then, if $\sigma \in \Sigma$ we have

$$\begin{aligned}(W(\xi \otimes f))(\gamma\sigma^{-1}) &= (\pi(\sigma\gamma)\xi) \otimes f(\gamma\sigma^{-1})\\ &= (\pi \otimes \rho)(\sigma)(W(\xi \otimes f)).\end{aligned}$$

Hence, it follows easily that W is a well defined unitary operator. A routine check then shows that

$$\mathrm{Ind}_\Sigma^\Gamma(\pi_{|\Sigma} \otimes \rho)(\gamma)W = W(\pi \otimes \mathrm{Ind}_\Sigma^\Gamma \rho)(\gamma),$$

for all $\gamma \in \Gamma$. □

If Γ, Λ, and Υ are groups, $\Gamma \curvearrowright X$, and $\Lambda \curvearrowright Y$ are actions, and $\alpha : \Gamma \times X \to \Lambda$, and $\beta : \Lambda \times Y \to \Upsilon$ are cocycles then just as we induced representations above we may induce the action $\Gamma \curvearrowright X$ to an action $\Gamma \curvearrowright X \times Y$ by the formula

$$\gamma(x, y) = (\gamma x, \alpha(\gamma, x)y).$$

We then may define the **composition of cocycles** $\beta\alpha : \Gamma \times (X \times Y) \to \Upsilon$ by the formula

$$\beta\alpha(\gamma, (x, y)) = \beta(\alpha(\gamma, x), y).$$

We can verify that this is indeed a cocycle since for all $\gamma_1, \gamma_2 \in \Gamma$, and $(x, y) \in X \times Y$ we have

$$\begin{aligned}\beta\alpha(\gamma_1\gamma_2, (x, y)) &= \beta(\alpha(\gamma_1, \gamma_2 x)\alpha(\gamma_2, x), y)\\ &= \beta(\alpha(\gamma_1, \gamma_2 x), \alpha(\gamma_2, x)y)\beta(\alpha(\gamma_2, x), y)\\ &= \beta\alpha(\gamma_1, (\gamma_2 x, \alpha(\gamma_2, x)y))\beta\alpha(\gamma_2, (x, y))\\ &= \beta\alpha(\gamma_1, \gamma_2(x, y))\beta\alpha(\gamma_2, (x, y)).\end{aligned}$$

Exercise 1.4.6. If we have an inclusion of groups $\Delta < \Sigma < \Gamma$ and we consider the cocycles $\alpha_{\Sigma<\Gamma}$, $\alpha_{\Delta<\Sigma}$, and $\alpha_{\Delta<\Gamma}$ as described in Example 1.3.8, then show that by identifying the sets $(\Gamma/\Sigma) \times (\Sigma/\Delta)$ and Γ/Δ we obtain an identification of $\alpha_{\Delta<\Sigma}\alpha_{\Sigma<\Gamma}$ and $\alpha_{\Delta<\Gamma}$.

In light of the previous exercise we then have that the following lemma is an extension of Lemma 1.4.4.

Lemma 1.4.7. *Let Γ, Λ, and Υ be groups, $\Gamma \curvearrowright X$, and $\Lambda \curvearrowright Y$ be actions, and $\alpha : \Gamma \times X \to \Lambda$, and $\beta : \Lambda \times Y \to \Upsilon$ be cocycles. If $\pi : \Upsilon \to \mathcal{U}(\mathcal{H})$ is a unitary representation, then*

$$\mathrm{Ind}_{\Upsilon}^{\beta\alpha} \pi \cong \mathrm{Ind}_{\Upsilon}^{\beta} \mathrm{Ind}_{\Lambda}^{\alpha} \pi.$$

Proof. If we consider the natural identification $V : \ell^2 X \otimes \ell^2 Y \to \ell^2(X \times Y)$, then it is easy to see that $V \otimes \mathrm{id}_{\mathcal{H}}$ implements an equivalence between the representations $\mathrm{Ind}_{\Upsilon}^{\beta} \mathrm{Ind}_{\Lambda}^{\alpha} \pi$ and $\mathrm{Ind}_{\Upsilon}^{\beta\alpha} \pi$. □

Lemma 1.4.8. *Let $\Gamma \curvearrowright X$ be a action of a group Γ on a set X, let Λ be a group, and let $\alpha : \Gamma \times X \to \Lambda$ be a cocycle, if $\pi_i : \Lambda \to \mathcal{U}(\mathcal{H}_i)$, $i \in I$ is a family of unitary representations, then*

$$\mathrm{Ind}_{\Lambda}^{\alpha}(\oplus_i \pi_i) \cong \oplus_{i \in I} \mathrm{Ind}_{\Lambda}^{\alpha}(\pi_i).$$

Proof. This follows easily by considering the natural unitary from $\oplus_{i \in I}(\ell^2 X \otimes \mathcal{H}_i)$ to $\ell^2 X \otimes (\oplus_{i \in I} \mathcal{H}_i)$. □

1.5 Invariant vectors

Definition 1.5.1. Let Γ be a group, a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ contains **invariant vectors** if there exists a non-zero vector $\xi \in \mathcal{H}$ such that $\pi(\gamma)\xi = \xi$ for all $\gamma \in \Gamma$. The representation contains **almost invariant vectors** if for each $F \subset \Gamma$, and $\varepsilon > 0$, there exists $\xi \in \mathcal{H}$, such that

$$\|\pi(\gamma)\xi - \xi\| < \varepsilon \|\xi\|, \text{ for all } \gamma \in F.$$

We will also say that a representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is **ergodic** if it does not contain invariant vectors. In general, if we denote by $\mathcal{H}_0 \subset \mathcal{H}$ the subspace of Γ-invariant vectors, then we say the representation π has **spectral gap** if $\mathcal{H}_0 \neq \mathcal{H}$ and the sub-representation $\pi_{|\mathcal{H}_0^{\perp}}$ does not contain almost invariant vectors.

Proposition 1.5.2. *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. If there exists $\xi \in \mathcal{H}$ and $c > 0$ such that $\Re(\langle \pi(\gamma)\xi, \xi \rangle) > c\|\xi\|^2$ for all $\gamma \in \Gamma$, then π contains an invariant vector.*

Proof. Let K be the closed convex hull of the orbit $\pi(\Gamma)\xi$. We therefore have that K is Γ-invariant and $\Re(\langle \eta, \xi \rangle) \geq c\|\xi\|^2$ for every $\eta \in K$. Let $\xi_0 \in K$ be the unique element of minimal norm, then since Γ acts isometrically we have that for each $\gamma \in \Gamma$, $\pi(\gamma)\xi_0$ is the unique element of minimal norm for $\pi(\gamma)K = K$, and hence $\pi(\gamma)\xi_0 = \xi_0$ for each $\gamma \in \Gamma$. Since $\xi_0 \in K$ we have that $\Re(\langle \xi_0, \xi \rangle) \neq 0$, and hence $\xi_0 \neq 0$. □

Corollary 1.5.3. *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. If there exists $\xi \in \mathcal{H}$ and $c < \sqrt{2}$ such that $\|\pi(\gamma)\xi - \xi\| < c\|\xi\|$ for all $\gamma \in \Gamma$, then π contains an invariant vector.*

Proof. For each $\gamma \in \Gamma$ we have

$$2\Re(\langle \pi(\gamma)\xi, \xi\rangle) = 2\|\xi\|^2 - \|\pi(\gamma)\xi - \xi\|^2 \geq (2 - c^2)\|\xi\|^2.$$

Hence, we may apply Proposition 1.5.2. □

Lemma 1.5.4. *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. Then π contains almost invariant vectors if and only if $\pi^{\oplus n}$ contains almost invariant vector, where $n \geq 1$ is any cardinal number.*

Proof. If π does not contain almost invariant vectors then there exists $c > 0$, and $S \subset \Gamma$ finite, such that for all $\xi \in \mathcal{H}$ we have

$$c\|\xi\|^2 \leq \Sigma_{\gamma\in S}\|\pi(\gamma)\xi - \xi\|^2.$$

If I is a set $|I| = n$, and $\xi_i \in \mathcal{H}$ for $i \in I$, such that $\Sigma_{i\in I}\|\xi_i\|^2 < \infty$, then

$$c\| \oplus_{i\in I} \xi_i\|^2 = \Sigma_{i\in I} c\|\xi_i\|^2$$

$$\leq \Sigma_{i\in I}\Sigma_{\gamma\in S}\|\pi(\gamma)\xi_i - \xi_i\|^2 = \Sigma_{\gamma\in S}\|\pi^{\oplus n}(\gamma)(\oplus_{i\in I}\xi_i) - \oplus_{i\in I}\xi_i\|^2.$$

Hence, $\pi^{\oplus n}$ does not contain almost invariant vectors. The converse is trivial since π is contained in $\pi^{\oplus\infty}$. □

Proposition 1.5.5. *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. Then π contains almost invariant vectors if and only if for any finite symmetric set $S \subset \Gamma$ the operator $T_S = \frac{1}{|S|}\Sigma_{\gamma\in S}\pi(\gamma)$ satisfies $\|T_S\| = 1$.*

Proof. If π contains almost invariant vectors then the triangle inequality easily implies that 1 is in the spectrum of T_S, for each finite symmetric set $S \subset \Gamma$.

Conversely, if $S \subset \Gamma$ is a finite symmetric set with $e \in S$ and $\|T_S\| = 1$, then since T_S is self-adjoint either 1 or -1 is contained in its spectrum, however since $e \in S$ it is easy to see that $-1 \notin \sigma(T_S)$, hence for any $\varepsilon > 0$ there exists $\xi \in \mathcal{H}$, $\|\xi\| = 1$ such that

$$|1 - \langle T_S\xi, \xi\rangle| < \varepsilon^2/2|S|.$$

We then have that for each $\gamma \in S$

$$\|\xi - \pi(\gamma)\xi\|^2 = 2|1 - \Re\langle\pi(\gamma)\xi, \xi\rangle|$$

$$\leq 2|S||1 - \langle T_S\xi, \xi\rangle| < \varepsilon^2.$$

Since, ε and S were arbitrary this shows that π contains almost invariant vectors. □

Proposition 1.5.6. *Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ a unitary representation. The following are equivalent:*

(1). *The representation $\pi \otimes \overline{\pi}$ contains invariant vectors.*

(2). *The representation $\pi \otimes \lambda$ contains invariant vectors for some unitary representation $\lambda : \Gamma \to \mathcal{U}(\mathcal{K})$.*

(3). *The representation π contains a finite dimensional sub-representation.*

Proof. (1) $\implies$ (2) is obvious.

To show (2) $\implies$ (3) suppose $\lambda : \Gamma \to \mathcal{U}(\mathcal{K})$ is a unitary representation such that $\pi \otimes \lambda$ contains invariant vectors. Identifying $\mathcal{H}\overline{\otimes}\mathcal{K}$ with the space of Hilbert-Schmidt operators $\mathrm{HS}(\overline{\mathcal{K}}, \mathcal{H})$ we then have that there exists $T \in \mathrm{HS}(\overline{\mathcal{K}}, \mathcal{H})$, non-zero, such that $\pi(\gamma)T\overline{\rho}(\gamma^{-1}) = T$, for all $\gamma \in \Gamma$. Then $TT^* \in \mathcal{B}(\mathcal{H}, \mathcal{H})$ is positive, non-zero, compact, and $\pi(\gamma)TT^*\pi(\gamma^{-1}) = TT^*$, for all $\gamma \in \Gamma$. By taking the range of a non-trivial spectral projection of TT^* we then obtain a finite dimensional invariant subspace of π.

(3) $\implies$ (1) follows because if $\mathcal{L} \subset \mathcal{H}$ is a finite dimensional invariant subspace then $\mathrm{Proj}_{\mathcal{L}}$ is a finite rank projection such that $\pi(\gamma)\,\mathrm{Proj}_{\mathcal{L}}\,\pi(\gamma^{-1}) = \mathrm{Proj}_{\mathcal{L}}$, for all $\gamma \in \Gamma$. By identifying $\mathrm{HS}(\mathcal{H}, \mathcal{H})$ with $\mathcal{H}\overline{\otimes}\overline{\mathcal{H}}$, we then obtain a non-zero invariant vector for $\pi \otimes \overline{\pi}$. □

1.6 Amenability

Much of the material of this section has been taken from Section 2.6 in the book of Brown and Ozawa [BO08].

Definition 1.6.1. Let Γ be a group, a **Følner net** is a net of non-empty finite subsets $F_i \subset \Gamma$ such that $|F_i \Delta \gamma F_i|/|F_i| \to 0$, for all $\gamma \in \Gamma$.

Note that we do not require that $\Gamma = \cup_i F_i$, nor do we require that F_i are increasing, however, if $|\Gamma| = \infty$ then it is easy to see that any Følner net $\{F_i\}_i$ must satisfy $|F_i| \to \infty$.

Exercise 1.6.2. Let Γ be a group, show that if Γ has a Følner net, then for each finite set $E \subset \Gamma$, and $\varepsilon > 0$, there exists a finite set $F \subset \Gamma$ such that $E \subset F$, $F = F^{-1}$, and $|F \Delta \gamma F|/|F|, |F \Delta F \gamma|/|F| < \varepsilon$.

Definition 1.6.3. A **mean** m on a non-empty set X is a finitely additive probability measure on 2^X, i.e., $m : 2^X \to [0, 1]$ such that $m(X) = 1$, and if $A_1, \ldots, A_n \subset X$ are disjoint then $m(\cup_{j=1}^n A_n) = \Sigma_{j=1}^n m(A_n)$.

Given a mean m on X it is possible to define an integral over X just as in the case if m were a measure. We therefore obtain a state $\phi_m \in (\ell^\infty X)^*$ by the formula $\phi_m(f) = \int_X f\, dm$. Conversely, if $\phi \in (\ell^\infty X)^*$ is a state, then restricting ϕ to characteristic functions defines a corresponding mean.

If $\Gamma \curvearrowright X$ is an action, then an invariant mean m on X is a mean such that $m(\gamma A) = m(A)$ for all $A \subset X$. An **approximately invariant mean** is a net of probability measures $\mu_i \in \mathrm{Prob}(X)$, such that $\|\gamma_* \mu_i - \mu_i\|_1 \to 0$, for all $\gamma \in \Gamma$.

Definition 1.6.4. Let Γ be a group, Γ is **amenable** if it Γ has a mean which is invariant under the action of left multiplication, or equivalently, if there is a state on $\ell^\infty \Gamma$ which is invariant under the action of left multiplication.

Amenable groups were first introduced by von Neumann [vN29]. The term amenable was coined by M. M. Day.

Theorem 1.6.5. *Let* Γ *be a group. The following conditions are equivalent.*

(1). Γ *is amenable.*

(2). Γ *has an approximate invariant mean under the action of left multiplication.*

(3). Γ *has a Følner net.*

(4). *The left regular representation* $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$ *contains almost invariant vectors.*

(5). *For any finite symmetric set* $S \subset \Gamma$ *the operator* $T_S = \frac{1}{|S|}\Sigma_{\gamma\in S}\lambda(\gamma)$ *satisfies* $\|T_S\| = 1$.

(6). *There exists a state* $\Phi \in (\mathcal{B}(\ell^2\Gamma))^*$ *such that* $\Phi(\lambda(\gamma)T) = \Phi(T\lambda(\gamma))$ *for all* $\gamma \in \Gamma$, $T \in \mathcal{B}(\ell^2\Gamma)$.

(7). *The continuous action of* Γ *on its Stone-Čech compactification* $\beta\Gamma$ *which is induced by left-multiplication admits an invariant Radon probability measure.*

(8). *Any continuous action* $\Gamma \curvearrowright K$ *on a compact Hausdorff space* K *admits an invariant Radon probability measure.*

Proof. We show (1) $\implies$ (2) using the method of Day [Day57]. Since $\ell^\infty\Gamma = (\ell^1\Gamma)^*$, the unit ball in $\ell^1\Gamma$ is wk*-dense in the unit ball of $(\ell^\infty\Gamma)^* = (\ell^1\Gamma)^{**}$. It follows that $\mathrm{Prob}(\Gamma) \subset \ell^1\Gamma$ is wk*-dense in the state space of $\ell^\infty\Gamma$.

Let $S \subset \Gamma$, be finite and let $K \subset \oplus_{\gamma\in S}\ell^1\Gamma$ be the wk-closure of the set $\{\oplus_{\gamma\in S}(\gamma_*\mu - \mu) \mid \mu \in \mathrm{Prob}(\Gamma)\}$. From the remarks above, since Γ has a left invariant state on $\ell^\infty\Gamma$, we have that $0 \in K$. However, K is convex and so by the Hahn-Banach Separation Theorem the wk-closure coincides with the norm closure. Thus, for any $\varepsilon > 0$ there exists $\mu \in \mathrm{Prob}(\Gamma)$ such that

$$\Sigma_{\gamma\in S}\|\gamma_*\mu - \mu\|_1 < \varepsilon.$$

We show (2) $\implies$ (3) using the method of Namioka [Nam64]. Let $S \subset \Gamma$ be a finite set, and denote by E_r the characteristic function on the set (r, ∞). If $\mu \in \mathrm{Prob}(\Gamma)$ then we have

$$\Sigma_{\gamma\in S}\|\gamma_*\mu - \mu\|_1 = \Sigma_{\gamma\in S}\Sigma_{x\in\Gamma}|\gamma_*\mu(x) - \mu(x)|$$

$$= \Sigma_{\gamma\in S}\Sigma_{x\in\Gamma}\int_{\mathbb{R}_{\geq 0}} |E_r(\gamma_*\mu(x)) - E_r(\mu(x))|\, dr$$

$$= \Sigma_{\gamma\in S}\int_{\mathbb{R}_{\geq 0}} \Sigma_{x\in\Gamma}|E_r(\gamma_*\mu(x)) - E_r(\mu(x))|\, dr$$

$$= \Sigma_{\gamma\in S}\int_{\mathbb{R}_{\geq 0}} \|E_r(\gamma_*\mu) - E_r(\mu)\|_1\, dr.$$

By hypothesis, if $\varepsilon > 0$ then there exists $\mu \in \mathrm{Prob}(\Gamma)$ such that $\Sigma_{\gamma \in S} \|\gamma_* \mu - \mu\|_1 < \varepsilon$, and hence for this μ we have

$$\Sigma_{\gamma \in S} \int_{\mathbb{R}_{\geq 0}} \|E_r(\gamma_* \mu) - E_r(\mu)\|_1 \, dr < \varepsilon = \varepsilon \int_{\mathbb{R}_{\geq 0}} \|E_r(\mu)\|_1 \, dr.$$

Hence, if we denote by $F_r \subset \Gamma$ the (finite) support of $E_r(\mu)$, then for some particular $r > 0$ we must have

$$\Sigma_{\gamma \in S} |\gamma F_r \Delta F_r| = \Sigma_{\gamma \in S} \|E_r(\gamma_* \mu) - E_r(\mu)\|_1 < \varepsilon \|E_r(\mu)\|_1 = \varepsilon |F_r|.$$

For (3) $\implies$ (4) just notice that if $F_i \subset \Gamma$ is a Følner net, then $\frac{1}{|F_i|^{1/2}} \Sigma_{x \in F_i} \delta_x \in \ell^2 \Gamma$ is a net of almost invariant vectors.

(4) $\iff$ (5) follows from Proposition 1.5.5.

For (4) $\implies$ (6) let $\xi_i \in \ell^2 \Gamma$ be a net of almost invariant vectors for λ. We define states Φ_i on $\mathcal{B}(\ell^2 \Gamma)$ by $\Phi_i(T) = \langle T \xi_i, \xi_i \rangle$. By wk-compactness of the state space, we may take a subnet and assume that this converges in the weak topology to $\Phi \in \mathcal{B}(\ell^2 \Gamma)^*$. We then have that for all $T \in \mathcal{B}(\ell^2 \Gamma)$ and $\gamma \in \Gamma$,

$$|\Phi(\lambda(\gamma) T - T \lambda(\gamma))| = \lim_i |\langle (\lambda(\gamma) T - T \lambda(\gamma)) \xi_i, \xi_i \rangle|$$

$$= \lim_i |\langle T \xi_i, \lambda(\gamma^{-1}) \xi_i \rangle - \langle T \lambda(\gamma) \xi_i, \xi_i \rangle|$$

$$\leq \lim_i \|T\| (\|\lambda(\gamma^{-1}) \xi_i - \xi_i\| + \|\lambda(\gamma) \xi_i - \xi_i\|) = 0.$$

For (6) $\implies$ (1), we have a natural embedding $M : \ell^\infty \Gamma \to \mathcal{B}(\ell^2 \Gamma)$ as "diagonal matrices", i.e., for a function $f \in \ell^\infty \Gamma$ we have $M_f(\Sigma_{x \in \Gamma} \alpha_x \delta_x) = \Sigma_{x \in \Gamma} \alpha_x f(x) \delta_x$. Moreover, for $f \in \ell^\infty \Gamma$ and $\gamma \in \Gamma$ we have $\lambda(\gamma) M_f \lambda(\gamma^{-1}) = M_{\gamma \cdot f}$. Thus, if $\Phi \in \mathcal{B}(\ell^2 \Gamma)^*$ is a state which is invariant under the conjugation by $\lambda(\gamma)$, then restricting this state to $\ell^\infty \Gamma$ gives a state on $\ell^\infty \Gamma$ which is Γ-invariant.

For (1) $\implies$ (7), the map $\beta : \ell^\infty \Gamma \to C(\beta \Gamma)$ which takes a bounded function on Γ to its unique continuous extension on $\beta \Gamma$, is a C^*-algebra isomorphism, which is Γ-equivariant. Hence amenability of Γ implies the existence of a Γ-invariant state on $C(\beta \Gamma)$. The Riesz Representation Theorem then gives an invariant probability measure on $\beta \Gamma$.

For (7) $\iff$ (8), suppose Γ acts continuously on a compact Hausdorff space K, and fix a point $x_0 \in K$. Then the map $f(\gamma) = \gamma x_0$ on Γ extends uniquely to a continuous map $\beta f : \beta \Gamma \to K$, moreover since f is Γ-equivariant, so is βf. If μ is an invariant Radon probability measure for the action on $\beta \Gamma$ then we obtain the invariant Radon probability measure $f_* \mu$ on K. Since $\beta \Gamma$ itself is compact, the converse is trivial.

For (7) $\implies$ (1), if there is a Γ-invariant Radon probability measure μ on $\beta \Gamma$, then we obtain an invariant mean m on Γ my setting $m(A) = \mu(\overline{A})$. □

Example 1.6.6. Any finite group is amenable, and any group which is locally amenable (each finitely generated subgroup is amenable) is also amenable. The group of integers $\mathbb{Z}$ is amenable (consider the Følner sequence $F_n = \{1, \ldots, n\}$ for example). From this it follows easily that all finitely generated abelian groups are amenable, and hence all abelian groups are.

It is also easy to see from the definition that subgroups of amenable groups are amenable. A bit more difficult is to show that if $1 \to \Sigma \to \Gamma \to \Lambda \to 1$ is an exact sequence of groups then Γ is amenable if and only if both Σ and Λ are amenable. Thus all solvable groups, and even all nilpotent groups are amenable.

There are also finitely generated amenable groups which cannot be constructed from finite, and abelian groups using only the operations above [Gri84].

Example 1.6.7. Let $\mathbb{F}_2$ be the free group on two generators a, and b. Let A^+ be the set of all elements in $\mathbb{F}_2$ whose leftmost entry in reduced form is a, let A^- be the set of all elements in $\mathbb{F}_2$ whose leftmost entry in reduced form is a^{-1}, let B^+, and B^- be defined analogously, and consider $C = \{e, b, b^2, \ldots\}$. Then we have that

$$\begin{aligned} \mathbb{F}_2 &= A^+ \sqcup A^- \sqcup (B^+ \setminus C) \sqcup (B^- \cup C) \\ &= A^+ \sqcup aA^- \\ &= b^{-1}(B^+ \setminus C) \sqcup (B^- \cup C). \end{aligned}$$

If m were a left-invariant mean on $\mathbb{F}_2$ then we would have

$$\begin{aligned} m(\mathbb{F}_2) &= m(A^+) + m(A^-) + m(B^+ \setminus C) + m(B^- \cup C) \\ &= m(A^+) + m(aA^-) + m(b^{-1}(B^+ \setminus C)) + m(B^- \cup C) \\ &= m(A^+ \sqcup aA^-) + m(b^{-1}(B^+ \setminus C) \sqcup (B^- \cup C)) = 2m(\mathbb{F}_2). \end{aligned}$$

Hence, $\mathbb{F}_2$ and also any group containing $\mathbb{F}_2$ is not-amenable. There are also finitely generated nonamenable groups which do not contain $\mathbb{F}_2$[Ols80], and even finitely presented nonamenable groups which do not contain $\mathbb{F}_2$ [OS02].

1.6.1 Von Neumann's Mean Ergodic Theorem

Amenable groups allow for nice averaging properties, we give such an example here.

Theorem 1.6.8. *Let Γ be an amenable group with Følner net $F_i \subset \Gamma$, and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Let $P_0 \in \mathcal{B}(\mathcal{H})$ be the projection onto the subspace of Γ invariant vectors. Then for each $\xi \in \mathcal{H}$ we have that*

$$\|\frac{1}{|F_i|}\Sigma_{\gamma \in F_i^{-1}} \pi(\gamma)\xi - P_0(\xi)\| \to 0.$$

Proof. By considering the vector $\xi - P_0(\xi)$ instead, we may assume that $P_0(\xi) = 0$.

Note that if $\gamma \in \Gamma$, and $\eta \in \mathcal{H}$ then $\eta - \pi(\gamma)\eta$ is orthogonal to the space of invariant vectors. Indeed, if $\zeta \in \mathcal{H}$ is an invariant vector, then $\langle \eta - \pi(\gamma)\eta, \zeta \rangle = \langle \eta, \zeta - \pi(\gamma^{-1})\zeta \rangle = 0$.

Denote by $\mathcal{L}$ the closure of the subspace of $\mathcal{H}$ spanned by vectors of the form $\eta - \pi(\gamma)\eta$ for $\eta \in \mathcal{H}$, $\gamma \in \Gamma$. Then if $\zeta \in \mathcal{L}^\perp$ we have that $0 = \langle \zeta, \eta - \pi(\gamma)\eta \rangle = \langle \zeta - \pi(\gamma^{-1})\zeta, \eta \rangle$ for all $\gamma \in \Gamma$, and $\eta \in \mathcal{H}$, thus $\zeta - \pi(\gamma)\zeta = 0$, for all $\gamma \in \Gamma$. Therefore we have shown that $\mathcal{L}^\perp$ is precisely the space of invariant vectors.

Fix $\gamma_0 \in \Gamma$, and $\eta \in \mathcal{H}$, then if $\xi = \eta - \pi(\gamma_0)\eta$ we have that

$$\| \frac{1}{|F_i|} \Sigma_{\gamma \in F_i^{-1}} \pi(\gamma)\xi \| = \| \frac{1}{|F_i|} \Sigma_{\gamma \in F_i^{-1}} (\pi(\gamma)\eta - \pi(\gamma\gamma_0)\eta) \|$$

$$\leq (|F_i \Delta \gamma_0 F_i| / |F_i|) \|\eta\| \to 0.$$

Since $\frac{1}{|F_i|} \Sigma_{\gamma \in F_i^{-1}} \pi(\gamma) \in \mathcal{B}(\mathcal{H})$ is always a contraction we may then pass to the closure of the span to conclude that for all $\xi \in \mathcal{L}$ we have

$$\| \frac{1}{|F_i|} \Sigma_{\gamma \in F_i^{-1}} \pi(\gamma)\xi \| \to 0.$$

□

In the case when the representation π is ergodic, $\Gamma = \mathbb{Z}$, and the we consider the Følner sequence $F_n = \{0, -1, \ldots, -n+1\}$, the above theorem then gives the following corollary.

Corollary 1.6.9. *Let $\pi : \mathbb{Z} \to \mathcal{U}(\mathcal{H})$ be a unitary representation. If π is ergodic, then for each $\xi \in \mathcal{H}$ we have that*

$$\lim_{n\to\infty} \| \frac{1}{n} \Sigma_{k=0}^{n-1} \pi(k)\xi \| = 0.$$

We remark that from the perspective of Example 1.3.2 another possible generalization of Corollary 1.6.9 could be given in terms of cocycles. This perspective would then lead to Corollary 3.3 in [dCTV07] the proof of which is quite similar to the proof of von Neumann's Ergodic Theorem given above.

1.7 Mixing properties

Definition 1.7.1. Let Γ be a group, a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is **weak mixing** if for each finite set $\mathcal{F} \subset \mathcal{H}$, and $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that

$$|\langle \pi(\gamma)\xi, \xi \rangle| < \varepsilon,$$

for all $\xi \in \mathcal{F}$.

The representation π is (**strong**) **mixing** if $|\Gamma| = \infty$, and for each finite set $\mathcal{F} \subset \mathcal{H}$, we have

$$\lim_{\gamma\to\infty} |\langle \pi(\gamma)\xi, \xi \rangle| = 0.$$

These definitions should be compared with the characterization of ergodicity found in Proposition 1.5.2. Hence we see that mixing implies weak mixing, which in turn implies ergodic. It is also easy to see that if $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is mixing (resp. weak mixing) then so is $\pi^{\oplus\infty}$, and if π is mixing then so is $\pi \otimes \rho$ for any representation ρ. We'll see below in Corollary 1.7.6 that weak mixing is also stable under tensoring.

Exercise 1.7.2. Let Γ be a group and $\Sigma < \Gamma$ a subgroup. Show that the quasi-regular representation of Γ on $\ell^2(\Gamma/\Sigma)$ is weak mixing if and only if it is ergodic, if and only if $[\Gamma : \Sigma] = \infty$. Show that it is mixing if and only if $|\Gamma| = \infty$ and $|\Sigma| < \infty$.

Exercise 1.7.3. Let $\Gamma \curvearrowright X$ be an action of a group Γ on a set X, let Λ be a group and let $\alpha : \Gamma \times X \to \Lambda$ be a cocycle. Suppose $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$ is a unitary representation. Find necessary and sufficient conditions for the induced representation of Section 1.4 to be mixing.

Lemma 1.7.4. *Let Γ be a group, a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is weak mixing if and only if for each finite set $\mathcal{F} \subset \mathcal{H}$, and $\varepsilon > 0$ there exists $\gamma \in \Gamma$ such that*

$$|\langle \pi(\gamma)\xi, \eta\rangle| < \varepsilon,$$

for all $\xi, \eta \in \mathcal{F}$.

The representation π is mixing if $|\Gamma| = \infty$ and for each finite set $\mathcal{F} \subset \mathcal{H}$, we have

$$\lim_{\gamma\to\infty} |\langle \pi(\gamma)\xi, \eta\rangle| = 0,$$

for all $\xi, \eta \in \mathcal{F}$.

Proof. This follows from the polarization identity. For each $\xi, \eta \in \mathcal{H}$, and $\gamma \in \Gamma$ we have

$$\langle \pi(\gamma)\xi, \eta\rangle = \frac{1}{4}\Sigma_{k=0}^{3} i^k \langle \pi(\gamma)(\xi + i^k\eta), (\xi + i^k\eta)\rangle.$$

□

We now add another equivalent condition to Proposition 1.5.6.

Proposition 1.7.5. *Let Γ be a group, and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then π is weak mixing if and only if π contains no finite dimensional sup-representations.*

Proof. If π is weak mixing then if $\mathcal{L} \subset \mathcal{H}$ is a non-trivial, finite dimensional subspace with orthonormal basis $\mathcal{F} \subset \mathcal{H}$, there exists $\gamma \in \Gamma$ such that $|\langle \pi(\gamma)\xi, \eta\rangle| < 1/\sqrt{\dim(\mathcal{L})}$, for all $\xi, \eta \in \mathcal{F}$. Hence, if $\xi \in \mathcal{F}$ then $\| \operatorname{Proj}_{\mathcal{L}}(\pi(\gamma)\xi)\| < 1 = \|\xi\|$. Thus, $\mathcal{L}$ is not an invariant subspace.

Conversely, If π has no finite dimensional invariant subspaces, $\mathcal{L} \subset \mathcal{H}$ is a finite dimensional subspace, and $\varepsilon > 0$, then there exists $\gamma \in \Gamma$ such that for all $\xi \in \mathcal{L}$ we have that

$$\| \operatorname{Proj}_{\mathcal{L}}(\pi(\gamma)\xi)\| < \varepsilon.$$

Indeed, if this is not the case then we would have that there exists $c > 0$ such that

$$\langle \pi(\gamma^{-1}) \operatorname{Proj}_{\mathcal{L}} \pi(\gamma), \operatorname{Proj}_{\mathcal{L}} \rangle_{\rm HS} \geq \sup_{\xi \in \mathcal{L}, \|\xi\|=1} \| \operatorname{Proj}_{\mathcal{L}}(\pi(\gamma)\xi)\|^2 \geq c.$$

It would then follow Proposition 1.5.2 that there is a non-zero Hilbert-Schmidt operator T such that $\pi(\gamma)T\pi(\gamma^{-1}) = T$, for all $\gamma \in \Gamma$. Proposition 1.5.6 would then give a contradiction.

Thus, if $\mathcal{F} \subset \mathcal{H}$ is finite such that $\|\xi\| \leq 1$ for each $\xi \in \mathcal{F}$, then by considering the finite dimensional subspace $\mathcal{L}$ spanned by $\mathcal{F}$ we have shown that there exists $\gamma \in \Gamma$ such that for all $\xi, \eta \in \mathcal{F}$ we have

$$|\langle \pi(\gamma)\xi, \eta \rangle| \leq \| \operatorname{Proj}_{\mathcal{L}}(\pi(\gamma)\xi)\| \|\eta\| < \varepsilon.$$

□

From the above proposition together with the equivalence between conditions 2) and 3) in Proposition 1.5.6 we obtain the following.

Corollary 1.7.6. *Let Γ be a group and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then π is weak mixing if and only if $\pi \otimes \overline{\pi}$ is weak mixing, if and only if $\pi \overline{\otimes} \rho$ is weak mixing for all unitary representations ρ.*

Corollary 1.7.7. *Let Γ be a group and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a weak mixing unitary representation. If $\Sigma < \Gamma$ is a finite index subgroup then $\pi_{|\Sigma}$ is also weak mixing.*

Proof. Let $D \subset \Gamma$ be a set of coset representatives for Σ. If $\pi_{|\Sigma}$ is not mixing, then by Proposition 1.7.5 there is a finite dimensional subspace $\mathcal{L} \subset \mathcal{H}$ which is Σ-invariant. We then have that $\Sigma_{\gamma \in D} \pi(\gamma)(\mathcal{L}) \subset \mathcal{H}$ is finite dimensional and Γ-invariant. Hence, again by Proposition 1.7.5, π is not weak mixing. □

1.7.1 Compact representations

If $\mathcal{H}$ is a Hilbert space the strong operator topology on $\mathcal{B}(\mathcal{H})$ is such that $\lim_i T_i = T$ if and only if $\lim_i \|(T_i - T)\xi\| = 0$, for all $\xi \in \mathcal{H}$. The weak operator topology on $\mathcal{B}(\mathcal{H})$ is such that $\lim_i T_i = T$ if and only if $\lim_i \langle (T_i - T)\xi, \eta \rangle = 0$, for all $\xi, \eta \in \mathcal{H}$. The unitary group $\mathcal{U}(\mathcal{H})$ then becomes a topological group when endowed with the strong operator topology. We also note that if $U_i \in \mathcal{U}(\mathcal{H})$ is a net of unitaries such that $U_i \to U \in \mathcal{U}(\mathcal{H})$ in the weak operator topology, then we also have that $U_i \to U$ in the strong operator topology. Indeed, for any $\xi \in \mathcal{H}$ we have

$$\|(U_i - U)\xi\|^2 = 2\|\xi\|^2 - 2\Re(\langle U_i \xi, U\xi \rangle) \to 0.$$

Definition 1.7.8. Let Γ be a group, a unitary representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is **compact** if $\pi(\Gamma) \subset \mathcal{U}(\mathcal{H})$ is pre-compact in the strong operator topology.

Lemma 1.7.9. *Let (X, d) be a compact metric space, then* $\mathrm{Isom}(X, d)$ *with the topology of point-wise convergence is compact.*

Proof. By Tychonoff's Theorem X^X is compact, thus we need only to show that $\mathrm{Isom}(X, d) \subset X^X$ is closed. Suppose $g \in \overline{\mathrm{Isom}(X, d)}$, for all $x, y \in X$ we have that $\{f \in X^X \mid d(f(x), f(y)) = d(x, y)\}$ is closed and contains $\mathrm{Isom}(X, d)$, hence g is isometric.

For each $x \in X$ denote by d_x the distance from x to $g(X)$. Then we have that for all $m \in \mathbb{N}$, $d(x, g^m(x)) \geq d_x$, hence for all $n, m \in \mathbb{N}$, $n < m$ we have $d(g^n(x), g^m(x)) = d(x, g^{m-n}(x)) \geq d_x$. Since X is compact, it must be totally bounded and hence we must have that $d_x = 0$. Hence g is surjective and thus $\mathrm{Isom}(X, d)$ is compact. □

Lemma 1.7.10. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a group Γ. Then π is compact if and only if for each $\xi \in \mathcal{H}$, the orbit $\pi(\Gamma)\xi$ is pre-compact in $\mathcal{H}$.*

Proof. If $\pi(\Gamma)$ is pre-compact in the strong operator topology and $G = \overline{\pi(\Gamma)}$, then for each $\xi \in \mathcal{H}$ we have that $\overline{\pi(\Gamma)\xi} = G\xi$ is compact, being the continuous image of the compact set G.

Conversely, suppose that each orbit $\pi(\Gamma)\xi \subset \mathcal{H}$ is pre-compact. By Zorn's Lemma we can find a collection of vectors $\mathcal{J} \subset \mathcal{H}$ such that $\mathcal{H} = \oplus_{\xi \in \mathcal{J}} \overline{\mathrm{sp}} \pi(\Gamma)\xi$.

We therefore have a strong operator topology continuous embedding of $\pi(\Gamma)$ into the compact space $\Pi_{\xi \in \mathcal{J}} \mathrm{Isom}(\overline{\pi(\Gamma)\xi}, d_{\mathcal{H}}) \subset \mathcal{U}(\oplus_{\xi \in \mathcal{J}} \overline{\mathrm{sp}} \pi(\Gamma)\xi)$, hence $\pi(\Gamma)$ is pre-compact. □

The following is part of the Peter-Weyl Theorem.

Theorem 1.7.11. *Let G be a compact group, and let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a strong operator topology continuous unitary representation. Then π decomposes as a direct sum of finite dimensional representations.*

Proof. Let λ denote the Haar measure on G. Since every representation decomposes into a direct sum of cyclic representations we may assume that the representation π has a cyclic vector $\zeta \in \mathcal{H}$. We then define an operator $K \in \mathcal{B}(\mathcal{H})$ such that for all $\xi, \eta \in \mathcal{H}$ we have

$$\langle K\xi, \eta \rangle = \int_G \langle \pi(g)\xi, \zeta \rangle \langle \zeta, \pi(g)\eta \rangle \, d\lambda(g).$$

First, note that from left invariance of the Haar measure we have that $\pi(g)K\pi(g^{-1}) = K$, for all $g \in G$. Also, if $\xi \in \mathcal{H}$, then we have

$$\langle K\xi, \xi \rangle = \int_G |\langle \pi(g)\xi, \zeta \rangle|^2 \, d\lambda(g) \geq 0.$$

Thus, K is positive.

Moreover, if $\langle K\xi, \xi \rangle = 0$ then we have that $\langle \pi(g)\xi, \zeta \rangle = \langle \xi, \pi(g^{-1})\zeta \rangle = 0$, for all $g \in G$. This then implies that $\xi = 0$ since ζ is a cyclic vector. Thus K is strictly positive.

If $\xi_n \in \mathcal{H}$ is a bounded sequence which weakly converges to 0, then

$$\lim_{n\to\infty} \|K\xi_n\|^2 = \lim_{n\to\infty} \int_G \langle \pi(g)\xi_n, \zeta\rangle\langle\zeta, \pi(g)K\xi_n\rangle \, d\lambda(g)$$

$$= \lim_{n\to\infty} \int_G \int_G \langle \pi(g)\xi_n, \zeta\rangle\langle\zeta, \pi(h)\xi_n\rangle\langle\pi(h)\pi(g^{-1})\zeta, \zeta\rangle \, d\lambda(g)\, d\lambda(h) = 0.$$

We therefore have shown that K is a compact operator. Hence, $\mathcal{H}$ decomposes as a direct sum of the (finite dimensional) eigenspaces of K, each of which is G-invariant. □

Lemma 1.7.10 and Theorem 1.7.11 then give us a structural result for compact representations.

Corollary 1.7.12. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a representation of a group Γ. Then π is compact if and only if π decomposes as a direct sum of finite dimensional representations.*

Proposition 1.7.13. *Let Γ be a group and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then there is a unique Γ-invariant closed subspace $\mathcal{K} \subset \mathcal{H}$ such that $\pi_{|\mathcal{K}}$ is compact and $\pi_{|\mathcal{K}^\perp}$ is weak mixing.*

Proof. Let $\mathcal{Z}$ be the set of all orthonormal sets $\mathcal{J} \subset \mathcal{H}$ such that $\mathrm{sp}(\pi(\Gamma)\xi)$ is finite dimensional for all $\xi \in \mathcal{J}$ and $\mathrm{sp}(\pi(\Gamma)\xi) \perp \mathrm{sp}(\pi(\Gamma)\eta)$ for all $\xi, \eta \in \mathcal{J}$, $\xi \neq \eta$. If we order $\mathcal{Z}$ by inclusion then it is easy to see that the union of any increasing chain in $\mathcal{Z}$ is again in $\mathcal{Z}$.

If π is not weakly mixing then by Proposition 1.7.5 we have that $\mathcal{Z} \neq \emptyset$, hence by Zorn's Lemma there is a maximal element $\mathcal{J} \in \mathcal{Z}$. Let $\mathcal{K} = \Sigma_{\xi\in\mathcal{J}}\mathrm{sp}(\pi(\Gamma)\xi) \subset \mathcal{H}$, then $\pi_{|\mathcal{K}} \cong \oplus_{\xi\in\mathcal{J}}\pi_{|\mathrm{sp}(\pi(\Gamma)\xi)}$ is compact, and by maximality of $\mathcal{J}$ we have that $\pi_{|\mathcal{K}^\perp}$ contains no finite dimensional sub-representation, and hence is weakly mixing.

If $\mathcal{K}_0 \subset \mathcal{H}$ is a finite dimensional Γ-invariant subspace, then $\mathrm{Proj}_{\mathcal{K}^\perp}(\mathcal{K}_0) \subset \mathcal{K}^\perp$ is also a finite dimensional Γ-invariant subspace. Since $\pi_{|\mathcal{K}^\perp}$ is weak mixing it then follows that $\mathrm{Proj}_{\mathcal{K}^\perp}(\mathcal{K}_0) = \{0\}$ and hence $\mathcal{K}_0 \subset \mathcal{K}$. This then implies uniqueness of the decomposition. □

1.7.2 Weak mixing for amenable groups

Proposition 1.7.14. *Let Γ be an amenable group with a Følner net $F_i \subset \Gamma$, and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then π is weak mixing if and only if for each $\xi, \eta \in \mathcal{H}$ we have*

$$\frac{1}{|F_i|}\Sigma_{\gamma\in F_i^{-1}}|\langle\pi(\gamma)\xi, \eta\rangle|^2 \to 0.$$

Proof. If π is not weak mixing, then there exists a finite set $\mathcal{F} \subset \mathcal{H}$, and $\varepsilon > 0$ such that $\Sigma_{\xi,\eta\in\mathcal{F}}|\langle\pi(\gamma)\xi, \eta\rangle|^2 > \varepsilon$, for all $\gamma \in \Gamma$. It then follows that

$$\liminf_{i\to\infty} \frac{1}{|F_i|}\Sigma_{\gamma\in F_i^{-1}}\Sigma_{\xi,\eta\in\mathcal{F}}|\langle\pi(\gamma)\xi, \eta\rangle|^2 \geq \varepsilon.$$

Hence, by the pigeonhole principle, for some $\xi, \eta \in \mathcal{F}$ we have that

$$\limsup_{i \to \infty} \frac{1}{|F_i|} \Sigma_{\gamma \in F_i^{-1}} |\langle \pi(\gamma)\xi, \eta \rangle|^2 \geq \varepsilon / |\mathcal{F}|^2 > 0.$$

Conversely, if π is weak mixing then by Propositions 1.5.6 and 1.7.5 we have that $\pi \otimes \overline{\pi}$ is ergodic. Hence by von Neumann's Ergodic Theorem the operators $\frac{1}{|F_i|} \Sigma_{\gamma \in F_i^{-1}} (\pi \otimes \overline{\pi})(\gamma)$ converge in the strong operator topology (and hence also in the weak operator topology) to 0. If $\xi, \eta \in \mathcal{H}$ we then have

$$\frac{1}{|F_i|} \Sigma_{\gamma \in F_i^{-1}} |\langle \pi(\gamma)\xi, \eta \rangle|^2 = \frac{1}{|F_i|} \Sigma_{\gamma \in F_i^{-1}} \langle (\pi \otimes \overline{\pi})(\gamma)(\xi \otimes \overline{\xi}), \eta \otimes \overline{\eta} \rangle \to 0.$$

□

Lemma 1.7.15 (Koopman, von Neumann 1932)**.** *Let X be a set, and let $F_i \subset X$, $i \in I$ be a net of non-empty finite subsets of X. Suppose $a \in \ell^\infty X$, then*

$$\frac{1}{|F_i|} \Sigma_{x \in F_i} |a(x)| \to 0$$

if and only if

$$\frac{1}{|F_i|} \Sigma_{x \in F_i} |a(x)|^2 \to 0.$$

Proof. By the Cauchy-Schwarz inequality we have

$$\frac{1}{|F_i|} \Sigma_{x \in F_i} |a(x)| \leq \left(\frac{1}{|F_i|} \Sigma_{x \in F_i} |a(x)|^2\right)^{1/2} \leq \|a\|_\infty^{1/2} \left(\frac{1}{|F_i|} \Sigma_{x \in F_i} |a(x)|\right)^{1/2}.$$

□

Corollary 1.7.16. *Let Γ be an amenable group with a Følner net $F_i \subset \Gamma$, and let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then π is weak mixing if and only if for each $\xi, \eta \in \mathcal{H}$ we have*

$$\frac{1}{|F_i|} \Sigma_{\gamma \in F_i^{-1}} |\langle \pi(\gamma)\xi, \eta \rangle| \to 0.$$

Specializing to representations of the integers we also have the following spectral characterization of weak mixing.

Proposition 1.7.17. *Let $\pi : \mathbb{Z} \to \mathcal{U}(\mathcal{H})$ be a unitary representation. Then π is weak mixing if and only if $\pi(1)$ has no eigenvalues.*

Proof. If π is weak mixing then by Proposition 1.7.5 there is no finite dimensional subspace of $\mathcal{H}$ which is $\mathbb{Z}$ invariant. In particular, this shows that $\pi(1)$ has no eigenvalues.

Conversely, suppose $\pi(1)$ has no eigenvalues. Then π has no 1-dimensional invariant subspaces. However, since any unitary matrix in $\mathbb{M}_n(\mathbb{C})$ has an eigenvalue, it then follows that π has no finite dimensional invariant subspaces. Hence, π is weak mixing by Proposition 1.7.5. □

1.8 Weak containment

Definition 1.8.1. Let Γ be a group, and $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$, $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ two unitary representations. The representation π is **weakly contained** in the representation ρ (written $\pi \prec \rho$) if for each $\xi \in \mathcal{H}$, $F \subset \Gamma$ finite, and $\varepsilon > 0$, there exists $\eta_1, \ldots, \eta_n \in \mathcal{K}$ such that

$$|\langle \pi(\gamma)\xi, \xi \rangle - \Sigma_{j=1}^n \langle \rho(\gamma)\eta_j, \eta_j \rangle| < \varepsilon,$$

for all $\gamma \in F$. The representations π and ρ are weakly equivalent (written $\pi \sim \rho$) if $\pi \prec \rho$ and $\rho \prec \pi$.

It follows easily from the definition that $\pi \sim \pi^{\oplus\infty}$ for any representation π. Also, it is easy to see that containment implies weak containment, and weak containment is a partial order. We also have that if π_i, and ρ_i, $i \in I$ are families of representations such that $\pi_i \prec \rho_i$ for each $i \in I$, then $\oplus_{i \in I} \pi_i \prec \oplus_{i \in I} \rho_i$.

Exercise 1.8.2. Show that if π_1, π_2, ρ_1, and ρ_2 are unitary representations of a group Γ such that $\pi_i \prec \rho_i$, for $i \in \{1, 2\}$, then $\pi_1 \otimes \pi_2 \prec \rho_1 \otimes \rho_2$.

Example 1.8.3. If a representation $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ weakly contains the trivial representation then for any finite symmetric set $S \subset \Gamma$ and $\varepsilon > 0$ there exists $\eta_1, \ldots, \eta_n \in \mathcal{H}$ such that

$$|1 - \Sigma_{j=1}^n \langle \pi(\gamma)\eta_j, \eta_j \rangle| < \varepsilon,$$

for all $\gamma \in S \cup \{e\}$. Hence, if we denote by $\eta = \oplus_{j=1}^n \eta_j \in \mathcal{H}^{\oplus n} \subset \mathcal{H}^{\oplus\infty}$, then we have $|1 - \|\eta\|^2| < \varepsilon$, and hence for each $\gamma \in S$ we have

$$\|\eta - \pi^{\oplus\infty}(\gamma)\eta\|^2 = 2(\|\eta\|^2 - \Re(\langle \pi^{\oplus\infty}(\gamma)\eta, \eta \rangle)) < 4\varepsilon.$$

It follows that π contains almost invariant vectors by Lemma 1.5.4.

Conversely, if π contains almost invariant vectors, then it is easy to see that π weakly contains the trivial representation.

The following lemma from [Fel63] is a useful tool for checking if one representation is weakly contained in another.

Lemma 1.8.4. *Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ and $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ be two unitary representations of a group Γ. Let $\mathcal{L} \subset \mathcal{H}$ be a set such that $\overline{\mathrm{sp}}\pi(\Gamma)\mathcal{L} = \mathcal{H}$. Then $\pi \prec \rho$ if and only if for each $\xi \in \mathcal{L}$, $F \subset \Gamma$ finite, and $\varepsilon > 0$, there exists $\eta_1, \ldots, \eta_n \in \mathcal{K}$ such that*

$$|\langle \pi(\gamma)\xi, \xi \rangle - \Sigma_{j=1}^n \langle \rho(\gamma)\eta_j, \eta_j \rangle| < \varepsilon,$$

for all $\gamma \in F$.

Proof. Suppose $\mathcal{L} \subset \mathcal{H}$ is as above, and consider $\mathcal{X} \subset \mathcal{H}$ the set of vectors $\xi \in \mathcal{H}$ such that the positive definite function $\gamma \mapsto \langle \pi(\gamma)\xi, \xi \rangle$ can be approximated arbitrarily well on finite sets by sums of positive definite functions associated to ρ. By hypothesis $\mathcal{L} \subset \mathcal{X}$, and we need to show that $\mathcal{X} = \mathcal{H}$.

If $\xi \in \mathcal{X}$, $\eta \in \mathcal{K}^{\oplus\infty}$, and $\Sigma_{x\in\Gamma}\alpha_x u_x \in \mathbb{C}\Gamma$, then from the formula

$$|\langle \pi(\gamma)(\Sigma_{x\in\Gamma}\alpha_x\pi(x)\xi), \Sigma_{x\in\Gamma}\alpha_x\pi(x)\xi\rangle - \langle \rho(\gamma)(\Sigma_{x\in\Gamma}\alpha_x\pi(x)\eta), \Sigma_{x\in\Gamma}\alpha_x\pi(x)\eta\rangle|$$

$$\leq \Sigma_{x,y\in\Gamma}|\alpha_y\alpha_x||\langle \pi(y^{-1}\gamma x)\xi, \xi\rangle - \langle \rho(y^{-1}\gamma x)\eta, \eta\rangle|,$$

we see that $\Sigma_{x\in\Gamma}\alpha_x\pi(x)\xi \in \mathcal{X}$. In particular, $\mathcal{X}$ is Γ-invariant.

It is also easy to see that $\mathcal{X}$ is a closed set. Moreover, if $\xi, \xi' \in \mathcal{X}$ are such that $\overline{\text{sp}}(\pi(\Gamma)\xi) \perp \overline{\text{sp}}(\pi(\Gamma)\xi')$, then it is easy to see that $\xi + \xi' \in \mathcal{X}$.

In general, we then have that if $\xi, \xi' \in \mathcal{X}$ then

$$\xi + \xi' = (\xi + \text{Proj}_{\overline{\text{sp}}(\pi(\Gamma)\xi)}(\xi')) + (\xi' - \text{Proj}_{\overline{\text{sp}}(\pi(\Gamma)\xi)}(\xi')) \in \mathcal{X}.$$

We therefore have shown that $\mathcal{X}$ is a closed Γ-invariant subspace which contains $\mathcal{L}$ and hence $\mathcal{X} = \mathcal{H}$. □

If $\varphi : \Gamma \to \mathbb{C}$ is a function of positive type and $\pi_\varphi : \Gamma \to \mathcal{U}(\mathcal{H}_\varphi)$ is the corresponding representation described in Section 1.2, then π_φ is generated by a single vector. We therefore obtain the following corollary.

Corollary 1.8.5. *If $\rho : \Gamma \to \mathcal{U}(\mathcal{K})$ is a representation of a group Γ, $\varphi : \Gamma \to \mathbb{C}$ is a function of positive type, and $\pi_\varphi : \Gamma \to \mathcal{U}(\mathcal{H}_\varphi)$ is the corresponding representation. Then $\pi_\varphi \prec \rho$ if and only if $F \subset \Gamma$ finite, and $\varepsilon > 0$, there exists $\eta_1, \ldots, \eta_n \in \mathcal{K}$ such that*

$$|\phi(\gamma) - \Sigma_{j=1}^n \langle \rho(\gamma)\eta_j, \eta_j\rangle| < \varepsilon,$$

for all $\gamma \in F$.

Exercise 1.8.6. Suppose $\Gamma \curvearrowright X$ is an action of a group Γ on a set X, and $\alpha : \Gamma \times X \to \Lambda$ is a cocycle into a group Λ. If $\pi : \Lambda \to \mathcal{U}(\mathcal{H})$, and $\rho : \Lambda \to \mathcal{U}(\mathcal{K})$ are two representations such that $\pi \prec \rho$, then show that $\text{Ind}_\Lambda^\alpha \pi \prec \text{Ind}_\Lambda^\alpha \rho$.

Conclude that if $\Sigma < \Gamma$ is an amenable subgroup of a group Γ then $\lambda_{\Gamma/\Sigma} \prec \lambda_\Gamma$.

For further properties of weak containment a good place to look is Appendix F in [BdlHV08].

Definition 1.8.7. [Bek90] Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a group Γ, then π is **amenable** if there exists a tate $\Phi \in (\mathcal{B}(\mathcal{H}))^*$ such that $\Phi(\pi(\gamma)T) = \Phi(T\pi(\gamma))$ for all $\gamma \in \Gamma$, $T \in \mathcal{B}(\mathcal{H})$.

Note that Γ is amenable if and only if the left-regular representation is amenable. We also have an analogue of Theorem 1.6.5 for amenable representations. The proof is similar, however we will not present it here.

Theorem 1.8.8. *[Bek90] Let $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ be a unitary representation of a group Γ, then the following conditions are equivalent.*

(1). *π is amenable.*

(2). *There exists a net of trace class operators* $T_i \in \mathcal{B}(\mathcal{H})$ *such that* $\|T_i\|_{\mathrm{Tr}} = 1$, *and* $\|T_i\pi(\gamma) - \pi(\gamma)T_i\|_{\mathrm{Tr}} \to 0$, *for all* $\gamma \in \Gamma$.

(3). *There exists a net of finite rank projections* $P_i \in \mathcal{B}(\mathcal{H})$ *such that* $\frac{1}{\|P_i\|_{\mathrm{HS}}}\|P_i\pi(\gamma) - \pi(\gamma)P_i\|_{\mathrm{HS}} \to 0$, *for all* $\gamma \in \Gamma$.

(4). $\pi \otimes \overline{\pi}$ *contains almost invariant vectors.*

(5). *For any finite symmetric set* $S \subset \Gamma$ *the operator* $T_S = \frac{1}{|S|}\Sigma_{\gamma \in S}\pi \otimes \overline{\pi}(\gamma)$ *satisfies* $\|T_S\| = 1$.

Chapter 2

Group actions on measure spaces

2.1 Examples

Definition 2.1.1. Let $(X, \mathcal{B}, \upsilon)$ be a σ-finite measure space, and let Γ be a countable group, an action $\Gamma \curvearrowright X$ such that $\gamma^{-1}(\mathcal{B}) = \mathcal{B}$, for all $\gamma \in \Gamma$ is measure preserving if for each measurable set $A \in \mathcal{B}$ we have $\upsilon(\gamma^{-1}A) = \upsilon(A)$. Alternately, we will say that υ is an invariant measure for the action $\Gamma \curvearrowright (X, \mathcal{B})$.

We'll say that Γ preserves the measure class of υ, or alternately, υ is a quasi-invariant measure, if for each $\gamma \in \Gamma$, and $A \in \mathcal{B}$ we have that $\upsilon(A) = 0$ if and only if $\upsilon(\gamma^{-1}A) = 0$, i.e., for each $\gamma \in \Gamma$, the push-forward measure $\gamma_*\upsilon$ defined by $\gamma_*\upsilon(A) = \upsilon(\gamma^{-1}A)$ for all $A \in \mathcal{B}$ is absolutely continuous with respect to υ.

Note, that since the action $\Gamma \curvearrowright X$ is measurable we obtain an action $\sigma : \Gamma \to \mathrm{Aut}(\mathcal{M}(X, \mathcal{B}))$ of Γ on the $\mathcal{B}$-measurable functions by the formula $\sigma_\gamma(f) = f \circ \gamma^{-1}$.

If $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$ preserves the measure class of υ, then for each $\gamma \in \Gamma$ there exists the Radon-Nikodym derivative $\frac{d\gamma_*\upsilon}{d\upsilon} : X \to [0, \infty)$, such that for each measurable set $A \in \mathcal{B}$ we have

$$\upsilon(\gamma^{-1}A) = \int 1_A \frac{d\gamma_*\upsilon}{d\upsilon}\, d\upsilon.$$

The Radon-Nikodym derivative is unique up to measure zero for υ.

If $\gamma_1, \gamma_2 \in \Gamma$, and $A \in \mathcal{B}$ then we have

$$\upsilon((\gamma_1\gamma_2)^{-1}A) = \int 1_{\gamma_1^{-1}A} \frac{d\gamma_{2*}\upsilon}{d\upsilon}\, d\upsilon$$

$$= \int 1_A \sigma_{\gamma_1}(\frac{d\gamma_{2*}\upsilon}{d\upsilon})\, d\gamma_{1*}\upsilon = \int 1_A \sigma_{\gamma_1}(\frac{d\gamma_{2*}\upsilon}{d\upsilon}) \frac{d\gamma_{1*}\upsilon}{d\upsilon}\, d\upsilon.$$

Hence, we have almost everywhere

$$\frac{d(\gamma_1\gamma_2)_*\upsilon}{d\upsilon} = \sigma_{\gamma_1}(\frac{d\gamma_{2*}\upsilon}{d\upsilon})\frac{d\gamma_{1*}\upsilon}{d\upsilon}.$$

Or, in other words, the map $(\gamma, x) \mapsto \frac{d\gamma_*\upsilon}{d\upsilon}(\gamma x) \in \mathbb{R}^\times$ is a cocycle almost everywhere.

Definition 2.1.2. Let $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ be measure class preserving actions of a countable group Γ on σ-finite measure spaces $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$. The actions are **isomorphic** (or **conjugate**) if there exists an isomorphism of measure spaces[1] such that for all $\gamma \in \Gamma$ and almost every $x \in X$ we have

$$\theta(\gamma x) = \gamma\theta(x).$$

Example 2.1.3. Any action $\Gamma \curvearrowright X$ of a countable group Γ on a countable set X can be viewed as a measure preserving action on X with the counting measure.

Example 2.1.4. Consider the torus $\mathbb{T}$ with the Borel σ-algebra, and Lebesgue measure, if $a \in [0, 1)$ then we obtain a measure preserving transformation $T : \mathbb{T} \to \mathbb{T}$ by $T(e^{i\theta}) = e^{i(\theta+2\pi a)}$. This then induces a measure preserving action of $\mathbb{Z}$.

Example 2.1.5 (The odometer action)**.** Consider the space $\{0, 1\}$, with the uniform probability measure, and consider $X = \{0, 1\}^{\mathbb{N}}$ with the product measure. We obtain a measure preserving transformation $T : X \to X$ given by "adding one". That is to say that T applied to a sequence $a_1a_2a_3\cdots$ will be the sequence $000\cdots01a_{n+1}a_{n+2}\cdots$ where a_n is the first position in which a 0 occurs in $a_1a_2a_3\cdots$. Then T induces a probability preserving action of $\mathbb{Z}$ on $\{0, 1\}^{\mathbb{N}}$.

Example 2.1.6 (Bernoulli shift)**.** Let $(X_0, \mathcal{B}_0, \mu_0)$ be a probability space, let Γ be a countable group, and consider $X = X_0^\Gamma$ with the product measure. Then we have a measure preserving action of Γ on X by $\gamma x = x \circ \gamma^{-1}$ for each $x \in X_0^\Gamma$.

Example 2.1.7 (Generalized Bernoulli shift)**.** Let $(X_0, \mathcal{B}_0, \mu_0)$ be a probability space, let $\Gamma \curvearrowright I$ be an action of a countable group Γ on a non-empty countable set I, and consider $X = X_0^I$ with the product measure. Then just as in the case of the Bernoulli shift we have a measure preserving action of Γ on X given by $\gamma x = x \circ \gamma^{-1}$ for each $x \in X_0^\Gamma$.

Exercise 2.1.8 ("The baker's map")**.** Let $X = [0, 1] \times [0, 1]$ with Lebesgue measure, consider the map $T : X \to X$ defined by

$$T(x, y) = \begin{cases} (2x, \frac{y}{2}), & 0 \leq x \leq \frac{1}{2}; \\ (2x - 1, \frac{y+1}{2}), & \frac{1}{2} < x \leq 1. \end{cases}$$

[1] A map $\theta : X \to Y$ is an isomorphism of measure spaces if θ is almost everywhere a bijection such that θ and also θ^{-1} is measure preserving.

Then T and T^{-1} are both measure preserving and hence give a measure preserving action of $\mathbb{Z}$ on X.

Show that the map $\theta : \{0,1\}^{\mathbb{Z}} \to [0,1] \times [0,1]$ given by

$$\theta(x) = (\Sigma_{n\leq 0} x(n) 2^{-(n+1)}, \Sigma_{n>0} x(n) 2^n)$$

is an isomorphism of measure spaces which implements a conjugacy between the Bernoulli shift $\mathbb{Z} \curvearrowright \{0,1\}^{\mathbb{Z}}$ and the baker's action $\mathbb{Z} \curvearrowright [0,1] \times [0,1]$.

Example 2.1.9. Let G be a σ-compact, locally compact group, and let Γ be a countable group. Then any homomorphism from Γ to G induces a Haar measure preserving action of Γ on G by left multiplication.

If G is compact, then we obtain a measure preserving action of Γ on a probability space.

Example 2.1.10. Let G be a σ-compact, locally compact group, let $H < G$ be a closed subgroup, and let $\Gamma < G$ be a countable subgroup. There always exists a G-quasi-invariant measure on the homogeneous space G/H, (see for example Section 2.6 in [Fol95]). Thus we always obtain a measure class preserving action of Γ on G/H.

A **lattice** in G is a discrete subgroup Δ such that G/Δ has an invariant probability measure. In this case we obtain a probability measure preserving action of Γ on G/Δ.

Example 2.1.11. Suppose A is an abelian group, and Γ is a group of automorphisms of A, then not only does Γ preserve the Haar measure of A, but also Γ has a Haar measure preserving action of the dual group $\hat{A}$ given by $\gamma x = x\gamma^{-1}$. As an example, we can consider the action of $SL_n\mathbb{Z}$ on $\mathbb{Z}^n$ given by matrix multiplication, this then induces a probability measure preserving action of $SL_n\mathbb{Z}$ on the dual group $\mathbb{T}^n$.

Example 2.1.12 (Uniform ordering [MOP79, Kie75])**.** Let Γ be a countable group, and consider $O(\Gamma)$ the set of all total orders of Γ. By interpreting total orders $<_t$ on Γ with functions from $\Gamma \times \Gamma$ to $\{0,1\}$ which take the value 1 at (γ_1, γ_2) if and only if $\gamma_1 <_t \gamma_2$, we may consider $O(\Gamma)$ as a subset of $\{0,1\}^{\Gamma\times\Gamma}$, and we consider the corresponding σ-algebra.

We have an action of Γ on $O(\Gamma)$ by requiring that $x <_{\gamma t} y$ if and only if $\gamma x <_t \gamma y$. We may place a probability measure μ on $O(\Gamma)$ by requiring that for each pairwise distinct $x_1, x_2, \ldots, x_n \in \Gamma$ we have

$$\mu(\{<_t \in O(\Gamma) \mid x_1 <_t x_2 <_t \cdots <_t x_n\}) = \frac{1}{n!}.$$

By Carathéodory's Extension Theorem, this extends to an invariant measure for the action of Γ. If $\Gamma \curvearrowright X$ is an action of Γ on a countable set X, then we can of course also consider the corresponding probability measure preserving action of Γ on $O(X)$.

Example 2.1.13 (Furstenberg's Correspondence Principle [Fur77])**.** Let Γ be a countable amenable group and let $F_n \subset \Gamma$ be a Følner sequence. Then each set F_n gives rise to a probability measure $\mu_n \in \mathrm{Prob}(\Gamma) \subset \mathrm{Prob}(\beta\Gamma)$, given as the uniform probability measure on F_n. Since the Stone-Čech compactification $\beta\Gamma$ is compact, the Arzelà-Ascoli Theorem implies that the sequence μ_n has a cluster point $\mu \in \mathrm{Prob}(\beta\Gamma)$.

Since F_n is a Følner sequence we have that μ is an invariant probability measure for the action of Γ on $\beta\Gamma$. Moreover, if $A \subset \Gamma \subset \beta\Gamma$ then we have that

$$\liminf_{n\to\infty} \frac{|A \cap F_n|}{|F_n|} \leq \mu(\overline{A}) \leq \limsup_{n\to\infty} \frac{|A \cap F_n|}{|F_n|}.$$

We also remark that if $A_1, A_2, \ldots, A_k \subset \Gamma$ then it is easy to see that $\cap_{j=1}^k \overline{A_j} = \overline{\cap_{j=1}^k A_j}$.

Since $\beta\Gamma$ is not second countable $L^2(\beta\Gamma, \mu)$ will not be separable in general. However, since Γ is countable, if $A \subset \Gamma$ then we can consider the Γ-invariant sub-σ-algebra generated by $\overline{A} \subset \beta\Gamma$, this then gives a separable Hilbert space.

Example 2.1.14. Similar to Furstenberg's Correspondence Principle, suppose G is a locally compact group. The Baire σ-algebra $\mathcal{B}_{aire}$ on G is the σ-algebra generated by the G_δ sets which are compact. Alexanderoff showed that there is a Banach space isomorphism between $C_b(G)^*$ and the space $ba(G, \mathcal{B}_{aire})$ of regular, bounded, finitely additive means m on $\mathcal{B}_{aire}$ with norm given by total valuation (see for example Theorem IV.6.2 in [DS88]).

Hence if m is a regular, finitely additive mean on the Baire σ-algebra of G which is invariant under the action of G. Then m gives a state on $C_b(G) = C(\beta G)$ which by the Riesz Representation Theorem gives a Radon probability measure μ on βG, hence, if Γ is a countable group and we have a homomorphism from Γ to G then left multiplication induces a μ-probability measure preserving action of Γ on βG.

Example 2.1.15 (Randomorphisms [Mon06])**.** Let Γ and Λ be two countable groups and consider the action of Γ on the space $[\Gamma, \Lambda] = \{f \in \Lambda^\Gamma \mid f(e) = e\}$ as described in Example 1.3.3. We consider Λ^Γ with the Polish space structure given by the product topology where Λ is discrete, we then endow Λ^Γ with the Borel σ-algebra. A **randomorphism** from Γ to Λ is a Γ-invariant probability measure μ on Λ^Γ on this σ-algebra. Note that homomorphism from Γ to Λ is just a Γ fixed point in Λ^Γ, hence the Dirac measure at such a point gives a random homomorphism.

If a random homomorphism μ is supported on the space of maps which are injective then we say that μ is a randembedding. There is also of course the corresponding notion of a random surjection, and a random bijection.

Notice that we can identify the space of bijections in $[\Gamma, \Lambda]$ with the space of bijections in $[\Lambda, \Gamma]$ by the inverse map. Under this identification we then obtain an action of Λ on the space of bijections in $[\Gamma, \Lambda]$ given by

$$(\lambda f)(x) = f(x f^{-1}(\lambda))\lambda^{-1}.$$

Example 2.1.16. Let $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ be measure class preserving actions of a countable group Γ on σ-finite measure spaces $(X, \mathcal{B}, \upsilon)$, and $(Y, \mathcal{A}, \nu)$. Then we obtain a diagonal action $\Gamma \curvearrowright (X \times Y, \mathcal{B} \otimes \mathcal{A}, \upsilon \times \nu)$ given by

$$\gamma(x, y) = (\gamma x, \gamma y).$$

If actions $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ are measure preserving, then so is the diagonal action.

2.2 The Koopman representation

Definition 2.2.1 ([Koo31])**.** Let $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$ be an action on a measure space which preserves the measure class υ. The **Koopman representation** of Γ associated to this action is the unitary representation $\pi : \Gamma \to \mathcal{U}(L^2(X, \upsilon))$ given by

$$(\pi(\gamma)f)(x) = f(\gamma^{-1}x)(\frac{d\gamma_*\upsilon}{d\upsilon})^{1/2}(x).$$

Note that for all $\gamma_1, \gamma_2 \in \Gamma$, and $f \in L^2(X, \upsilon)$ we have

$$\pi(\gamma_1\gamma_2)f = \sigma_{\gamma_1\gamma_2}(f)(\frac{d(\gamma_1\gamma_2)_*\upsilon}{d\upsilon})^{1/2}$$

$$= \sigma_{\gamma_1}(\sigma_{\gamma_2}(f)(\frac{d\gamma_{2*}\upsilon}{d\upsilon})^{1/2})(\frac{d\gamma_{1*}\upsilon}{d\upsilon})^{1/2} = \pi(\gamma_1)(\pi(\gamma_2)f).$$

Also, for all $\gamma \in \Gamma$, and $f \in L^2(X, \upsilon)$ we have

$$\|\pi(\gamma)f\|_2^2 = \int |\sigma_\gamma(f)|^2 \frac{d\gamma_*\upsilon}{d\upsilon}\, d\upsilon$$

$$= \int |\sigma_\gamma(f)|^2 \, d\gamma_*\upsilon = \|f\|_2^2.$$

Hence, π is indeed a unitary representation.

If $(X, \mathcal{B}, \upsilon)$ is a finite measure space, and $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$ is measure preserving, then $1 \in L^2(X, \upsilon)$ and $\pi(\gamma)(1) = 1$ is an invariant vector. For this reason, in this setting the Koopman representation usually denotes the restriction of the above representation to the orthogonal complement

$$L_0^2(X, \upsilon) = \{f \in L^2(X, \upsilon) \mid \int f \, d\upsilon = 0\}.$$

Exercise 2.2.2. Let $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$ be an action on a measure space which preserves the measure class υ. Show that the Koopman representation π is isomorphic to its conjugate representation $\overline{\pi}$.

Exercise 2.2.3. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ be measure preserving actions of a countable group Γ on probability spaces $(X, \mathcal{B}, \upsilon)$, and $(Y, \mathcal{A}, \nu)$. Show that the Koopman representation $\pi_{X \times Y}$ for the product action decomposes as $\pi_{X \times Y} \cong \pi_X \oplus \pi_Y \oplus (\pi_X \otimes \pi_Y)$ where π_X and π_Y are the Koopman representations for $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$.

2.2.1 Ergodicity

Definition 2.2.4. Let $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$ be a measure class preserving action of a countable group Γ on a σ-finite measure space $(X, \mathcal{B}, \upsilon)$. The action $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$ is **ergodic**, if for any $E \in \mathcal{B}$ which is Γ-invariant, we have that either $\upsilon(E) = 0$ or $\upsilon(X \setminus E) = 0$.

Lemma 2.2.5. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. The action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is ergodic if and only if the Koopman representation is ergodic.*

Proof. If $E \in \mathcal{B}$ is Γ-invariant then $\chi_E - \mu(E) \in L^2_0(X, \mu)$ is also Γ-invariant, which is non-zero if $\mu(E) \neq 0, 1$. On the other hand, if $\xi \in L^2_0(X, \mu)$ is a non-zero Γ-invariant function then $E_t = \{x \in X \mid |\xi(x)| < t\}$ is Γ-invariant for all $t > 0$, and we must have $\mu(E_t) \neq 0, 1$ for some $t > 0$. $\square$

Theorem 1.6.5 can be adapted to the setting of actions on measure spaces as follows.

Theorem 2.2.6. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \upsilon)$ be a measure preserving action of a countable group Γ on an infinite, σ-finite measure space $(X, \mathcal{B}, \upsilon)$. The following conditions are equivalent.*

(1). *There exists a state $\varphi \in (L^\infty(X, \mathcal{B}, \upsilon))^*$ such that for all $\gamma \in \Gamma$, and $f \in L^\infty(X, \mathcal{B}, \upsilon)$ we have $\varphi(\sigma(\gamma)(f)) = \varphi(f)$.*

(2). *For every $\varepsilon > 0$, and $F \subset \Gamma$ finite, there exists $\nu \in \mathrm{Prob}(X, \mathcal{B})$, such that ν is absolutely continuous with respect to υ and*

$$\| \frac{d\gamma_* \nu}{d\upsilon} - \frac{d\nu}{d\upsilon} \|_1 < \varepsilon.$$

(3). *For every $\varepsilon > 0$, and $F \subset \Gamma$ finite, there exists a measurable set $A \subset X$ such that $\upsilon(A) < \infty$, and*

$$\upsilon(A \Delta \gamma A) < \varepsilon \upsilon(A).$$

(4). *The Koopman representation $\pi : \Gamma \to \mathcal{U}(L^2(X, \mathcal{B}, \upsilon))$ has almost invariant vectors.*

(5). *The Koopman representation $\pi : \Gamma \to \mathcal{U}(L^2(X, \mathcal{B}, \upsilon))$ is amenable.*

Exercise 2.2.7. Adapt the proof of Theorem 1.6.5 to prove Theorem 2.2.6.

If Γ acts by measure preserving transformations on a probability space, then the conditions above are trivially satisfied, however, we still have the following, non-trivial, adaptation.

Theorem 2.2.8. *Let* $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ *be a measure preserving action of a countable group* Γ *on a probability space* $(X, \mathcal{B}, \mu)$*. The following conditions are equivalent.*

(1). *There exists a state* $\varphi \in (L^\infty(X, \mathcal{B}, \mu))^*$*, different than* $f \mapsto \int f \, d\mu$*, such that for all* $\gamma \in \Gamma$*, and* $f \in L^\infty(X, \mathcal{B}, \mu)$ *we have* $\varphi(\sigma(\gamma)(f)) = \varphi(f)$*.*

(2). *There exits* $A \subset X$ *measurable such that* $\mu(A) < 1$*, and for every* $\varepsilon > 0$*, and* $F \subset \Gamma$ *finite, there exists* $\nu \in \mathrm{Prob}(X, \mathcal{B})$*, such that* ν *is absolutely continuous with respect to* μ *and*

$$\nu(A) > 1 - \varepsilon, \qquad \text{and} \qquad \| \frac{d\gamma_* \nu}{d\mu} - \frac{d\nu}{d\mu} \|_1 < \varepsilon.$$

(3). *For every* $\varepsilon > 0$*, and* $F \subset \Gamma$ *finite, there exists a measurable set* $A \subset X$ *such that* $\mu(A) \leq 1/2$*, and*

$$\mu(A \Delta \gamma A) < \varepsilon \mu(A).$$

(4). *The Koopman representation* $\pi : \Gamma \to \mathcal{U}(L^2_0(X, \mathcal{B}, \mu))$ *has almost invariant vectors.*

(5). *The Koopman representation* $\pi : \Gamma \to \mathcal{U}(L^2_0(X, \mathcal{B}, \mu))$ *is amenable.*

If $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is an ergodic measure preserving action of a countable group on a probability space $(X, \mathcal{B}, \mu)$ such that none of the conditions of the previous theorem hold, then we say that the action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ has **spectral gap**.

2.2.2 Weak mixing

Definition 2.2.9. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group on a probability space $(X, \mathcal{B}, \mu)$. A sequence $E_n \in \mathcal{B}$ of measurable subsets is an **asymptotically invariant sequence** if $\mu(E_n \Delta \gamma E_n) \to 0$, for all $\gamma \in \Gamma$. Such a sequence is said to be non-trivial if $\liminf_{n \to \infty} \mu(E_n)(1 - \mu(E_n)) > 0$. The action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is **strongly ergodic** if there does not exist a non-trivial asymptotically invariant sequence.

Note that if $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ has a non-trivial invariant set $E \in \mathcal{B}$, then the constant sequence $E_n = E$ is a non-trivial asymptotically invariant sequence, hence strongly ergodic implies ergodic. Also, a non-trivial asymptotically invariant sequence will show that condition (3) in Theorem 2.2.8 is satisfied, hence an ergodic action with spectral gap must be strongly ergodic.

Definition 2.2.10. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. The action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is **weak mixing**, if $|\Gamma| = \infty$, and for all $\mathcal{E} \subset \mathcal{B}$ finite, we have

$$\liminf_{\gamma \to \infty} \Sigma_{A, B \in \mathcal{E}} |\mu(A \cap \gamma B) - \mu(A)\mu(B)| = 0.$$

Proposition 2.2.11. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. The following conditions are equivalent:*

(1). *The action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is weak mixing.*

(2). *The Koopman representation $\pi : \Gamma \to \mathcal{U}(L^2_0(X, \mathcal{B}, \mu))$ is weak mixing.*

(3). *The diagonal action $\Gamma \curvearrowright (X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \times \mu)$ is ergodic.*

(4). *For any ergodic, measure preserving action $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ on a probability space, the diagonal action $\Gamma \curvearrowright (X \times Y, \mathcal{B} \otimes \mathcal{A}, \mu \times \nu)$ is ergodic.*

(5). *The diagonal action $\Gamma \curvearrowright (X \times X, \mathcal{B} \otimes \mathcal{B}, \mu \times \mu)$ is weak mixing.*

(6). *For any weak mixing, measure preserving action $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ on a probability space, the diagonal action $\Gamma \curvearrowright (X \times Y, \mathcal{B} \otimes \mathcal{A}, \mu \times \nu)$ is weak mixing.*

Proof. For (1) $\iff$ (2), if $\mathcal{E} \subset \mathcal{B}$ is a finite set then we can consider the finite set $\mathcal{F} = \{\chi_E - \mu(E) \mid E \in \mathcal{E}\} \subset L^2_0(X, \mathcal{B}, \mu)$. For each $\gamma \in \Gamma$ we therefore have

$$\Sigma_{\xi,\eta \in \mathcal{F}} |\langle \pi(\gamma)\xi, \eta \rangle| = \Sigma_{A,B \in \mathcal{E}} |\langle \sigma_\gamma(\chi_B) - \mu(B), \chi_A - \mu(A) \rangle|$$

$$= \Sigma_{A,B \in \mathcal{E}} |\langle \chi_{\gamma B} - \mu(B), \chi_A - \mu(A) \rangle|$$

$$= \Sigma_{A,B \in \mathcal{E}} |\mu(A \cap \gamma B) - \mu(A)\mu(B)|.$$

If the Koopman representation is weak mixing then from this we see immediately that the action is weak mixing. A similar calculation shows that if the action is weak mixing and $\mathcal{F} \subset L^2_0(X, \mathcal{B}, \mu)$ is a finite set of simple functions, then

$$\liminf_{\gamma \to \infty} \Sigma_{\xi,\eta \in \mathcal{F}} |\langle \pi(\gamma)\xi, \eta \rangle| = 0.$$

Since simple functions are dense in $L^2_0(X, \mathcal{B}, \mu)$ this shows that the Koopman representation is weak mixing.

The remaining equivalences then easily follow from Exercise 2.2.3, together with the corresponding properties of weak mixing for unitary representations in Sections 1.5 and 1.7. ☐

Corollary 1.7.7 then shows that weak mixing is preserved under taking finite index.

Corollary 2.2.12. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a weak mixing, measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. If $\Sigma < \Gamma$ is a finite index subgroup, then the restriction of the action of Γ to Σ is also weak mixing.*

Corollary 1.7.16 gives the following result for weak mixing actions of amenable groups.

Corollary 2.2.13. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable amenable group Γ on a probability space $(X, \mathcal{B}, \mu)$. Let $F_n \subset \Gamma$, be a Følner sequence for Γ. The action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is weak mixing if and only if for all $A, B \in \mathcal{B}$, we have*

$$\frac{1}{|F_n|} \Sigma_{\gamma \in F_n} |\mu(A \cap \gamma B) - \mu(A)\mu(B)| \to 0.$$

Definition 2.2.14. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. The action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is (**strong**) **mixing**, if $|\Gamma| = \infty$, and for any $A, B \subset X$ measurable we have

$$\lim_{\gamma \to \infty} |\mu(A \cap \gamma B) - \mu(A)\mu(B)| = 0.$$

The same proof as in Proposition 2.2.11 yields the following proposition for mixing actions.

Proposition 2.2.15. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. The action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is mixing if and only if the Koopman representation is mixing.*

2.2.3 Compact actions

Definition 2.2.16. Let $(X, \mathcal{B}, \mu)$ be a probability space. We denote by $\mathrm{Aut}(X, \mathcal{B}, \mu)$ the group of automorphisms of $(X, \mathcal{B}, \mu)$, where we identify two automorphisms if they agree almost everywhere. The **weak topology** on $\mathrm{Aut}(X, \mathcal{B}, \mu)$ is the smallest topology such that the maps $T \mapsto \mu(T(A)\Delta B)$ are continuous for all $A, B \in \mathcal{B}$.

Exercise 2.2.17. Show that the weak topology endows $\mathrm{Aut}(X, \mathcal{B}, \mu)$ with a topological group structure.

Exercise 2.2.18. The Koopman representation for $\mathrm{Aut}(X, \mathcal{B}, \mu)$ is defined as before, i.e., $\pi : \mathrm{Aut}(X, \mathcal{B}, \mu) \to \mathcal{U}(L^2_0(X, \mathcal{B}, \mu))$ is defined by $\pi(T)(f) = f \circ T^{-1}$. Show that the image of π is closed and that π is homeomorphism from $\mathrm{Aut}(X, \mathcal{B}, \mu)$ with the the weak topology onto $\pi(\mathrm{Aut}(X, \mathcal{B}, \mu)) \subset \mathcal{U}(L^2_0(X, \mathcal{B}, \mu))$ with the strong operator topology.

Definition 2.2.19. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. The action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is **compact** if the image of Γ in $\mathrm{Aut}(X, \mathcal{B}, \mu)$ is precompact in the weak topology.

An immediate consequence of Exercise 2.2.18 is the following.

Proposition 2.2.20. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. Then $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is compact if and only if the Koopman representation $\pi : \Gamma \to \mathcal{U}(L^2_0(X, \mathcal{B}, \mu))$ is compact.*

Definition 2.2.21. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$. A function $f \in L^2(X, \mathcal{B}, \mu)$ is **almost periodic** if the Γ orbit $\{\sigma_\gamma(f) \mid \gamma \in \Gamma\}$ is pre-compact in the $\|\cdot\|_2$-topology.

Proposition 2.2.22. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a discrete group Γ on a standard probability space $(X, \mathcal{B}, \mu)$. Let $\mathcal{A} \subset \mathcal{B}$ be the σ-algebra generated by all almost periodic functions in $L^2(X, \mathcal{B}, \mu)$. Then $\mathcal{A}$ is Γ-invariant, and $f \in L^2(X, \mathcal{B}, \mu)$ is almost periodic if and only if f is $\mathcal{A}$ measurable.*

Proof. For each $f \in L^2(X, \mathcal{B}, \mu)$ denote by S_f the Γ-orbit of f, and consider the set $AP(L^2(X, \mathcal{B}, \mu))$ of all almost periodic functions. Note that it is obvious that $AP(L^2(X, \mathcal{B}, \mu))$ is Γ-invariant, contains the scalars, and is closed under scalar multiplication.

Since addition is continuous with respect to $\|\cdot\|_2$, if $f, g \in AP(L^2(X, \mathcal{B}, \mu))$ we have that $\overline{S_f} + \overline{S_g}$ is compact being the continuous image of the compact set $\overline{S_f} \times \overline{S_g}$. We therefore have that $S_{f+g} \subset \overline{S_f} + \overline{S_g}$ is precompact, hence $AP(L^2(X, \mathcal{B}, \mu))$ is closed under addition. Similarly, if $f, g \in AP(L^2(X, \mathcal{B}, \mu))$ then we have that

$$|f|, \overline{f}, \max\{|f|, |g|\}, \min\{|f|, |g|\} \in AP(L^2(X, \mathcal{B}, \mu)).$$

Also, if we also have that $g \in L^\infty(X, \mathcal{B}, \mu)$ then $fg \in AP(L^\infty(X, \mathcal{B}, \mu))$.

If $f_n \in AP(L^2(X, \mathcal{B}, \mu))$ and $f \in L^2(X, \mathcal{B}, \mu)$ such that $\|f_n - f\|_2 \to 0$, then fix $\varepsilon > 0$, and take $n \in \mathbb{N}$ such that $\|f_n - f\|_2 < \varepsilon/2$. Since f_n is almost periodic we have that S_{f_n} is totally bounded, hence there exists a finite set $\subset L^2(X, \mathcal{B}, \mu)$ such that $\inf_{h \in \mathcal{F}} \|\sigma_\gamma(f_n) - h\|_2 < \varepsilon/2$ for all $\gamma \in \Gamma$. By the triangle inequality we then have $\inf_{h \in \mathcal{F}} \|\sigma_\gamma(f) - h\|_2 < \varepsilon$. This shows that S_f is totally bounded and hence $f \in AP(L^2(X, \mathcal{B}, \mu))$.

The operations above then generate all $\mathcal{A}$-measurable functions and so we obtain the result. □

Definition 2.2.23. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a discrete group Γ on a standard probability space $(X, \mathcal{B}, \mu)$. Suppose $\mathcal{A} \subset \mathcal{B}$ is a Γ-invariant σ-algebra, then we say that the action $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$ is a **factor** of the action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$.

Note that the Koopman representation of a factor $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$ is a subrepresentation of the Koopman representation of $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$. Hence, Proposition 2.2.11 together with Proposition 2.2.22 gives the following.

Proposition 2.2.24. *Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be a measure preserving action of a discrete group Γ on a standard probability space $(X, \mathcal{B}, \mu)$. Then there exists a unique maximal factor $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$ which is compact. Moreover, $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is weak mixing if and only if $\mathcal{A}$ is trivial, i.e., when $\mathcal{A}$ consists only of null or co-null sets.*

2.3 A remark about measure spaces

So far we have been considering actions of a countable group on a general probability space $(X, \mathcal{B}, \mu)$. For most aspects of ergodic theory this is the proper setting. However, occasionally this level of generality can be problematic. For example, consider the action of $\mathbb{Z}$ by rotation on the circle $\mathbb{T}$. We can consider $\mathbb{T}$ equipped with either the Borel σ-algebra, or the Lebesgue σ-algebra. If we are not concerned with sets of measure 0, then both of these systems contain the same information, however the identity map from the Borel σ-algebra to the Lebesgue σ-algebra is not measurable, and these two systems are not isomorphic under our notion of isomorphism.

For another example, consider the case when $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ are measure preserving actions. Then the diagonal action $\Gamma \curvearrowright (X \times Y, \mathcal{B} \otimes \mathcal{A}, \mu \times \nu)$ has the factor $\Gamma \curvearrowright (X \times Y, \mathcal{B} \otimes \{Y, \emptyset\}, \mu \times \nu)$, which we would like to identify with the action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$. However, a problem arises because the projection map from $X \times Y$ to X is not almost everywhere 1-1 and hence is not an isomorphism of actions.

One way to overcome this problem is to restrict ourselves to only consider actions on nice measure spaces. Specifically, one considers the class of **standard probability spaces**. A standard probability space $(X, \mathcal{B}, \mu)$ is a probability space such that the underlying σ-algebra space $(X, \mathcal{B})$ is isomorphic (as σ-algebra spaces) to a Polish[2] topological space with its Borel σ-algebra. In this setting we do not allow actions on $\mathbb{T}$ with the Lebesgue measure, or actions on the space $(X \times Y, \mathcal{B} \otimes \{Y, \emptyset\}, \mu \times \nu)$ since these are not standard probability spaces, and so the problems above do not arise.

An alternate approach, which we will take here, is to continue to allow general probability spaces, but instead generalize our notion of equivalence so that the spaces above are equivalent under this more general notion. Thus, from now on we will say that two probability spaces $(X, \mathcal{B}, \mu)$ and $(Y, \mathcal{A}, \nu)$ are isomorphic if there is an integral preserving unital $*$-isomorphism from $L^\infty(Y, \mathcal{A}, \nu)$ to $L^\infty(X, \mathcal{B}, \mu)$. We will also say that two measure preserving actions $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ are isomorphic (or conjugate) if there exists a $*$-isomorphism $\hat{\theta}$ from $L^\infty(Y, \mathcal{A}, \nu)$ to $L^\infty(X, \mathcal{B}, \mu)$ such that $\hat{\theta} \circ \sigma_\gamma = \sigma_\gamma \circ \hat{\theta}$ for all $\gamma \in \Gamma$. Note that if $\theta : X \to Y$ is a bijection such that θ and θ^{-1} is measure preserving such that $\theta \circ \gamma = \gamma \circ \theta$ for all $\gamma \in \Gamma$, then $\hat{\theta} : L^\infty(Y, \mathcal{A}, \nu) \to L^\infty(X, \mathcal{B}, \mu)$ given by $\hat{\theta}(f) = f \circ \theta^{-1}$ implements an isomorphism between the two actions.

We also have a process (essentially the GNS-construction) which takes us from a general action on a separable measure space to an isomorphic action on a standard Borel space.

Proposition 2.3.1. *Let A be a unital $*$-algebra with a state[3] ϕ such that for all $a \in A$ there exists $K > 0$ such that $\phi(a^*abb^*) \leq K\phi(bb^*)$ for all $b \in A$. Let*

[2] A topological space is Polish if it is separable and has a complete metric which induces the topology

[3] A state on a unital $*$-algebra is a linear map $\phi : A \to \mathbb{C}$ such that $\phi(1) = 1$, and $\phi(a^*a) \geq 0$ for all $a \in A$.

Γ be a countable group and suppose $\sigma : \Gamma \to \mathrm{Aut}(A)$ is an action of Γ on A by unital $*$-homomorphisms such that $\phi(\sigma_\gamma(a)) = \phi(a)$ for all $a \in A$, and $\gamma \in \Gamma$.

Then there exists a compact Hausdorff space X, a continuous action $\Gamma \curvearrowright X$, a Γ-invariant Radon probability measure μ on X, and a unital $*$-homomorphism $\pi : A \to L^\infty(X, \mu)$ such that $\pi \circ \sigma_\gamma = \sigma_\gamma \circ \pi$ for all $\gamma \in \Gamma$, and $\int \pi(a)\, d\mu = \phi(a)$, for all $a \in A$.

Moreover, if A is countable generated as an algebra, then X is separable (and hence a Polish space).

Proof. We endow A with the inner-product $\langle a, b\rangle_\phi = \phi(b^* a)$, by quotienting out by the kernel of this inner-product and then taking the completion, we obtain a Hilbert space $L^2(A, \phi)$. Given $a \in A$ we denote by $\hat{a}$ the equivalence class of a in $L^2(A, \phi)$. If $a, b \in A$, then we may consider a acting on $L^2(A, \phi)$ by left multiplication. The formula

$$\|\widehat{ab}\|_\phi^2 = \phi(a^* a b b^*) \leq K \|\hat{b}\|_\phi^2$$

shows that left multiplication is well defined and bounded, we therefore obtain a $*$-homomorphism $\pi_0 : A \to \mathcal{B}(L^2(A, \phi))$.

We denote by $C^*(A)$ the abelian C^*-algebra generated by $\pi_0(A)$ in $\mathcal{B}(L^2(A, \phi))$, and we denote by X the Gelfand spectrum of $C^*(A)$. By Gelfand's Theorem we have that X is a compact Hausdorff space and we obtain a $*$-isomorphism from $C^*(A)$ to $C(X)$. We therefore obtain a $*$-homomorphism $\pi : A \to C(X)$ by applying the Gelfand transform to the image $\pi_0(A)$.

On $C(X) \cong C^*(A)$ we may consider the state $\hat{\phi}$ given by $\hat{\phi}(x) = \langle x\hat{1}, \hat{1}\rangle_\phi$. By the Riesz Representation Theorem the state $\hat{\phi}$ corresponds to a Radon measure μ on X such that $\hat{\phi}(x) = \int x\, d\mu$ for all $x \in C(X)$. We therefore have that for all $a \in A$

$$\int \pi(a)\, d\mu = \hat{\phi}(\pi(a))$$
$$= \langle \pi_0(a)\hat{1}, \hat{1}\rangle_\phi = \langle \hat{a}, \hat{1}\rangle_\phi = \phi(a).$$

Since $\sigma : \Gamma \to \mathrm{Aut}(A)$ preserves the state ϕ, we have that for all $\gamma \in \Gamma$, and $a, b \in A$

$$\|\widehat{\sigma_\gamma(a)b}\|_\phi^2 = \phi(\sigma_\gamma(a^* a) b b^*) = \phi(a^* a \sigma_{\gamma^{-1}}(b b^*)) = \|\widehat{a\sigma_{\gamma^{-1}}(b)}\|_\phi^2.$$

Hence, if we define σ_γ on $\pi_0(A)$ by $\sigma_\gamma(\pi_0(a)) = \pi_0(\sigma_\gamma(a))$ then this is well defined and preserves the operator norm, hence extends to a map (which we still denote by σ) from Γ to $\mathrm{Aut}(C^*(A))$. The Gelfand transform then gives a continuous action of Γ on X, such that $\sigma_\gamma(x) = x \circ \gamma^{-1}$ for all $x \in C(X)$.

An easy calculation then shows that $\pi \circ \sigma_\gamma = \sigma_\gamma \circ \pi$ for all $\gamma \in \Gamma$, and we have that for all $\gamma \in \Gamma$, and $x \in C(X)$

$$\int x\, d\gamma_* \mu = \int x \circ \gamma^{-1}\, d\mu = \hat{\phi}(\sigma_\gamma(x)) = \hat{\phi}(x) = \int x\, d\mu.$$

Hence μ is Γ-invariant.

Finally, if A is countably generated as an algebra then $C^*(A)$ is a separable C^*-algebra and hence the Gelfand spectrum X is separable. ∎

Corollary 2.3.2. *Suppose $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is a measure preserving action of a countable group Γ on a probability space $(X, \mathcal{B}, \mu)$ such that $L^2(X, \mathcal{B}, \mu)$ is separable. Then the action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is isomorphic to a measure preserving action of Γ on a standard probability space $(Y, \mathcal{A}, \nu)$.*

Proof. Since $L^2(X, \mathcal{B}, \mu)$ is separable there exists a countably generated $*$-subalgebra $A \subset L^\infty(X, \mathcal{B}, \mu) \subset L^2(X, \mathcal{B}, \mu)$ which is dense in $L^2(X, \mathcal{B}, \mu)$. Since Γ is countable we may also assume that A is Γ-invariant. By Proposition 2.3.1 there exists a measure preserving action of Γ on a standard probability space $(Y, \mathcal{A}, \nu)$ and homomorphism $\pi : A \to L^\infty(Y, \mathcal{A}, \nu)$ which is Γ-equivariant and preserves the integrals.

Since π preserves the integrals we may extend it to a unitary $U : L^2(X, \mathcal{B}, \mu) \to L^2(Y, \mathcal{A}, \nu)$ which also preserves the integrals, and is again Γ-equivariant. The map $f \mapsto U^* f U$ then gives a $*$-isomorphism between $L^\infty(Y, \mathcal{A}, \nu)$ and $L^\infty(X, \mathcal{B}, \nu)$ which preserves the integrals and is again Γ-equivariant. □

2.4 Gaussian actions

Let $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation of a countable group Γ. The aim of this section, which is taken from [PS], is to describe the construction of a measure-preserving action of Γ on a non-atomic standard probability space (X, μ) such that $\mathcal{H}$ is realized as a subspace of $L^2_{\mathbb{R}}(X, \mu)$ and π is contained in the Koopman representation $\Gamma \curvearrowright L^2_0(X, \mu)$. The action $\Gamma \curvearrowright (X, \mu)$ is referred to as the **Gaussian action** associated to π.

Given a Hilbert space $\mathcal{H}$, the n-symmetric tensor $\mathcal{H}^{\odot n}$ is the subspace of $\mathcal{H}^{\otimes n}$ fixed by the action of the symmetric group S_n by permuting the indices. For $\xi_1, \ldots, \xi_n \in \mathcal{H}$, we define their symmetric tensor product $\xi_1 \odot \cdots \odot \xi_n \in \mathcal{H}^{\odot n}$ to be $\frac{1}{n!} \sum_{\sigma \in S_n} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)}$. Denote

$$\mathfrak{S}(\mathcal{H}) = (\mathbb{R}\Omega \oplus \bigoplus_{n=1}^{\infty} \mathcal{H}^{\odot n}) \otimes_{\mathbb{R}} \mathbb{C},$$

with renormalized inner product such that $\|\xi\|^2_{\mathfrak{S}(\mathcal{H})} = n! \|\xi\|^2$, for $\xi \in \mathcal{H}^{\odot n}$.

For $\xi \in \mathcal{H}$ let x_ξ be the **symmetric creation operator**,

$$x_\xi(\Omega) = \xi, \;\; x_\xi(\eta_1 \odot \cdots \odot \eta_k) = \xi \odot \eta_1 \odot \cdots \odot \eta_k,$$

and its adjoint,

$$x^*_\xi(\Omega) = 0, \;\; x^*_\xi(\eta_1 \odot \cdots \odot \eta_k) = \sum_{i=1}^{k} \langle \eta_i, \xi \rangle \eta_1 \odot \cdots \odot \widehat{\eta_i} \odot \cdots \odot \eta_k.$$

Let

$$s(\xi) = \frac{1}{2}(x_\xi + x^*_\xi),$$

and note that it is an unbounded, self-adjoint operator on $\mathfrak{S}(\mathcal{H})$.

The moment generating function $M(t)$ for the Gaussian distribution is defined to be

$$M(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(tx)\exp(-x^2/2)dx = \exp(t^2/2).$$

It is easy to check that if $\|\xi\| = 1$ then

$$\langle s(\xi)^n \, \Omega, \Omega\rangle = M^{(n)}(0) = \frac{(2k)!}{2^k k!},$$

if $n = 2k$ and 0 if n is odd. Hence, $s(\xi)$ may be regarded as a Gaussian random variable. Note that if $\xi, \eta \in \mathcal{H}$ then $s(\xi)$ and $s(\eta)$ commute, moreover, if $\xi \perp \eta$, then

$$\langle s(\xi)^m s(\eta)^n \Omega, \Omega\rangle = \langle s(\xi)^m \Omega, \Omega\rangle \langle s(\eta)^n \Omega, \Omega\rangle,$$

for all $m, n \in \mathbb{N}$; thus, $s(\xi)$ and $s(\eta)$ are independent random variables.

From now on we will use the convention $\xi_1\xi_2 \cdots \xi_k$ to denote the symmetric tensor $\xi_1 \odot \xi_2 \odot \cdots \odot \xi_k$. Let Ξ be a basis for $\mathcal{H}$ and

$$\mathcal{S}(\Xi) = \{\Omega\} \cup \{s(\xi_1)s(\xi_2)\cdots s(\xi_k)\Omega \mid \xi_1, \xi_2, \ldots, \xi_k \in \Xi\}.$$

Lemma 2.4.1. *The set $\mathcal{S}(\Xi)$ is a (non-orthonormal) basis of $\mathfrak{S}(\mathcal{H})$.*

Proof. We will show that $\xi_1 \cdots \xi_k \in \operatorname{sp}(\mathcal{S}(\Xi))$, for all $\xi_1, \ldots, \xi_k \in \mathcal{H}$. We have $\Omega \in \operatorname{sp}(\mathcal{S}(\Xi))$. Also, since $s(\xi)\Omega = \xi$, $\mathcal{H} \subset \operatorname{sp}(\mathcal{S}(\Xi))$. Now as $s(\xi_1)\cdots s(\xi_k)\Omega = P(\xi_1, \ldots, \xi_k)$ is a polynomial in $\xi_1, \ldots, \xi_k$ of degree k with top term $\xi_1 \cdots \xi_k$, the result follows by induction on k. □

Let $u(\xi_1, \ldots, \xi_k) = \exp(\pi i s(\xi_1)\cdots s(\xi_k))$ and $u(\xi_1, \ldots, \xi_k)^t = \exp(\pi i t s(\xi_1)\cdots s(\xi_k))$. Denote by A the von Neumann algebra generated by all such $u(\xi_1, \ldots, \xi_k)$, which is the same as the von Neumann algebra generated by the spectral projections of the unbounded operators $s(\xi_1)\cdots s(\xi_k)$.

Theorem 2.4.2. *We have that $L^2(A, \tau) \cong \mathfrak{S}(\mathcal{H})$, and A is a maximal abelian $*$-subalgebra of $\mathcal{B}(\mathfrak{S}(\mathcal{H}))$ with faithful state $\tau = \langle \cdot\, \Omega, \Omega\rangle$.*

Proof. By Lemma 2.4.1, $A \mapsto A\Omega$ is an embedding of A into $\mathfrak{S}(\mathcal{H})$. By Stone's Theorem

$$\lim_{t\to 0} \frac{u(\xi_1, \ldots, \xi_k)^t - 1}{\pi i t}\Omega = s(\xi_1)\cdots s(\xi_k)\Omega;$$

hence, $A\Omega$ is dense in $\mathfrak{S}(\mathcal{H})$. This implies that A is maximal abelian in $B(\mathfrak{S}(\mathcal{H}))$. □

There is a natural strong operator topology continuous embedding $\mathcal{O}(\mathcal{H}) \hookrightarrow \mathcal{U}(\mathfrak{S}(\mathcal{H}))$ given by

$$T \mapsto T^{\mathfrak{S}} = 1 \oplus \bigoplus_{n=1}^{\infty} T^{\odot n}.$$

It follows that there is an embedding $\mathcal{O}(\mathcal{H}) \hookrightarrow \mathrm{Aut}(A, \tau)$, $T \mapsto \sigma_T$, which can be identified on the unitaries $u(\xi_1, \ldots, \xi_k)$ by

$$\sigma_T(u(\xi_1, \ldots, \xi_k)) = \mathrm{Ad}(T^{\mathfrak{S}})(u(\xi_1, \ldots, \xi_k)) = u(T(\xi_1), \ldots, T(\xi_k)).$$

Thus for an orthogonal representation $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$, there is a natural action $\sigma^\pi : \Gamma \to \mathrm{Aut}(A, \tau)$ given by $\sigma^\pi_\gamma(u(\xi_1, \ldots, \xi_k)) = u(\pi_\gamma(\xi_1), \ldots, \pi_\gamma(\xi_k)) = \mathrm{Ad}(\pi^{\mathfrak{S}}_\gamma)(u(\xi_1, \ldots, \xi_k))$. Applying Proposition 2.3.1 we then obtain a measure preserving probability space $(X, \mathcal{B}, \mu)$ and an action of Γ on this space so that we can identify A with $L^\infty(X, \mu)$ in a state preserving Γ-equivariant manor. The action of Γ on $(X, \mathcal{B}, \mu)$ is the Gaussian action associated to π.

We have an explicit description of the Koopman representation of $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ given by $\pi^{\mathfrak{S}}(\gamma) = (\pi(\gamma))^{\mathfrak{S}} \ominus 1$. Hence, we have that ergodic properties which remain stable with respect to tensor products transfer from π to σ^π.

Proposition 2.4.3. *Let $\pi : \Gamma \to \mathcal{O}(\mathcal{H})$ be an orthogonal representation of a countable group Γ. Then the Gaussian action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is ergodic, if and only if it is weak mixing, if and only if π is weak mixing.*

Proof. If π is not weak mixing, then there exists $\xi \in \mathcal{H}^{\otimes 2}$ such that for all $\gamma \in \Gamma$, $\pi^{\otimes 2}(\gamma)(\xi) = \xi$. Viewing ξ as a Hilbert-Schmidt operator on $\mathcal{H}$, let $|\xi| = (\xi\xi^*)^{1/2}$. Since the map $\xi \otimes \eta \mapsto \eta \otimes \xi$ is the same as taking the adjoint of the corresponding Hilbert-Schmidt operator, we have that $|\xi| \in \mathcal{H}^{\odot 2}$ and $\pi^{\odot 2}(\gamma)(|\xi|) = |\xi|$. Since $\pi^{\odot 2}$ embeds into the Koopman representation of the Gaussian action, it follows from Lemma 2.2.5 that $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is not ergodic.

Conversely, if $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is not ergodic then neither is the Koopman representation $\pi^{\mathfrak{S}}$. But $\pi^{\mathfrak{S}}$ is a sub-representation of $\bigoplus_{n=1}^\infty \pi^{\otimes n} = \pi \otimes (1 \oplus \bigoplus_{n=1}^\infty \pi^{\otimes n})$. Hence if $\pi^{\mathfrak{S}}$ has non-trivial invariant vectors then π is not weak mixing by Proposition 1.5.6. □

2.5 Ergodic theorems

Using the Koopman representation, von Neumann's Ergodic Theorem for Hilbert spaces can be rephrased for actions.

Theorem 2.5.1 (Von Neumann's Ergodic Theorem [vN32])**.** *Let $\Gamma \curvearrowright (X, \mu)$ be a measure preserving action of a countable amenable group Γ on a probability space $(X, \mathcal{B}, \mu)$. Let $E_{\mathcal{I}} \in \mathcal{B}(L^2(X, \mathcal{B}, \mu))$ be the projection onto the subspace of Γ-invariant functions. Let $F_n \subset \Gamma$ be a Følner sequence, then for each $f \in L^2(X, \mathcal{B}, \mu)$ we have that*

$$\| \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} \sigma_\gamma(f) - E_{\mathcal{I}}(f) \|_2 \to 0.$$

To state Birkoff's Ergodic Theorem we first need to discuss conditional expectations. Note that if $(X, \mathcal{B}, \mu)$ is a probability space, then we may consider a function $f \in L^\infty(X, \mathcal{B}, \mu)$ as a bounded operator on $L^2(X, \mathcal{B}, \mu)$ by multiplication, i.e., $g \mapsto fg$. To avoid confusion in the next lemmas, when we consider a function $f \in L^\infty(X, \mathcal{B}, \mu)$ as an element in $L^2(X, \mathcal{B}, \mu)$ we will continue to use the notation f, however, when we consider f as a bounded operator we will denote this operator by M_f.

Lemma 2.5.2. *Let $(X, \mathcal{B}, \mu)$ be a probability space, then $L^\infty(X, \mathcal{B}, \mu) \subset \mathcal{B}(L^2(X, \mathcal{B}, \mu))$ is a maximal abelian self-adjoint subalgebra.*

Proof. Suppose $T \in \mathcal{B}(L^2(X, \mathcal{B}, \mu))$ commutes with $L^\infty(X, \mathcal{B}, \mu)$, we want to show that there exists some $f \in L^2(X, \mathcal{B}, \mu)$ such that $T(g) = fg$ for all $g \in L^2(X, \mathcal{B}, \mu)$.

Let $f = T(1) \in L^2(X, \mathcal{B}, \mu)$, if $g \in L^\infty(X, \mathcal{B}, \mu) \subset L^2(X, \mathcal{B}, \mu)$ then we have

$$T(g) = TM_g(1) = M_g T(1) = M_g(f) = fg.$$

Hence, if we consider the functions $f_t = 1_{[0,t)}(|f|)f \in L^\infty(X, \mathcal{B}, \mu)$, then we have

$$\begin{aligned} \|f_t\|_\infty &= \sup_{g \in L^2(X,\mathcal{B},\mu), \|g\|_2 = 1} \|f_t g\|_2 \\ &= \sup_{g \in L^2(X,\mathcal{B},\mu), \|g\|_2 = 1} \|T(1_{[0,t)}(|f|)g)\|_2 \\ &\leq \sup_{g \in L^2(X,\mathcal{B},\mu), \|g\|_2 = 1} \|T\| \|1_{[0,t)}(|f|)g\|_2 \leq \|T\|. \end{aligned}$$

Therefore, $\|f\|_\infty = \lim_{t \to \infty} \|f_t\|_\infty \leq \|T\|$, and so $f \in L^\infty(X, \mathcal{B}, \mu)$. Moreover, since $L^\infty(X, \mathcal{B}, \mu) \subset L^2(X, \mathcal{B}, \mu)$ is dense, we have that $T(g) = fg$ for all $g \in L^2(X, \mathcal{B}, \mu)$. ☐

If $\mathcal{A} \subset \mathcal{B}$ is σ-subalgebra we will denote by $E_\mathcal{A} \in \mathcal{B}(L^2(X, \mathcal{B}, \mu))$ the orthogonal projection onto $L^2(X, \mathcal{A}, \mu)$. Note that if $g \in L^\infty(X, \mathcal{A}, \mu)$ then we can think of M_g as a bounded operator on $L^2(X, \mathcal{A}, \mu)$ or on $L^2(X, \mathcal{B}, \mu)$, and since the product of two $\mathcal{A}$-measurable functions is again $\mathcal{A}$-measurable it follows that

$$M_g E_\mathcal{A} = E_\mathcal{A} M_g.$$

Then for each $f \in L^\infty(X, \mathcal{B}, \mu)$ and $g \in L^\infty(X, \mathcal{A}, \mu)$ we have that

$$(E_\mathcal{A} M_f E_\mathcal{A}) M_g = E_\mathcal{A} M_f M_g E_\mathcal{A} = M_g (E_\mathcal{A} M_f E_\mathcal{A}).$$

By Lemma 2.5.2 there exists a function $f_0 \in L^\infty(X, \mathcal{B}, \mu)$ such that $E_\mathcal{A} M_f E_\mathcal{A}^* = M_{f_0}$. It also follows from the proof of Lemma 2.5.2 that actually $f_0 = E_\mathcal{A}(f)$ and so we see that $E_\mathcal{A}(L^\infty(X, \mathcal{B}, \mu)) \subset L^\infty(X, \mathcal{A}, \mu)$, and $\|E_\mathcal{A}(f)\|_\infty \leq \|f\|_\infty$ for all $f \in L^\infty(X, \mathcal{B}, \mu)$.

The map $E_\mathcal{A} : L^\infty(X, \mathcal{B}, \mu) \to L^\infty(X, \mathcal{A}, \mu)$ is called the **conditional expectation** with respect to $\mathcal{A}$.

Lemma 2.5.3. *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $\mathcal{A} \subset \mathcal{B}$ be a σ-subalgebra, then the conditional expectation $E_\mathcal{A} : L^\infty(X, \mathcal{B}, \mu) \to L^\infty(X, \mathcal{A}, \mu)$ satisfies the following properties:*

(1). *$E_{\mathcal{A}}$ preserves the integral, i.e., $\int E_{\mathcal{A}}(f)\,d\mu = \int f\,d\mu$, for all $f \in L^\infty(X,\mathcal{B},\mu)$.*

(2). *$E_{\mathcal{A}}$ preserves positivity and for each $f \in L^\infty(X,\mathcal{B},\mu)$ we have $|E_{\mathcal{A}}(f)| \leq E_{\mathcal{A}}(|f|)$.*

(3). *$E_{\mathcal{A}}$ is $L^\infty(X,\mathcal{A},\mu)$-bimodular, i.e., if $f \in L^\infty(X,\mathcal{B},\mu)$ and $g \in L^\infty(X,\mathcal{A},\mu)$ then $E_{\mathcal{A}}(fg) = E_{\mathcal{A}}(f)g$.*

(4). *$E_{\mathcal{A}}$ extends continuously to a contraction on $L^p(X,\mathcal{B},\mu)$ for all $1 \leq p \leq \infty$.*

Proof. (1) follows because if $f \in L^\infty(X,\mathcal{B},\mu)$ then we have

$$\int E_{\mathcal{A}}(f)\,d\mu = \langle E_{\mathcal{A}}(f), 1\rangle_{L^2(X,\mathcal{B},\mu)} = \langle f, E_{\mathcal{A}}(1)\rangle_{L^2(X,\mathcal{B},\mu)} = \int f\,d\mu.$$

We have (2) because for all $f \in L^\infty(X,\mathcal{B},\mu)$ and $g \in L^2(X,\mathcal{A},\mu)$ we have

$$\langle E_{\mathcal{A}} M_{|f|} E_{\mathcal{A}} g, g\rangle = \langle M_{|f|} E_{\mathcal{A}} g, E_{\mathcal{A}} g\rangle \geq 0,$$

and

$$\langle ((M_{|E_{\mathcal{A}}(f)|})^2 - (E_{\mathcal{A}} M_{|f|} E_{\mathcal{A}})^2) g, g\rangle = \langle (E_{\mathcal{A}} M_{|f|}(1 - E_{\mathcal{A}}) M_{|f|} E_{\mathcal{A}}) g, g\rangle$$

$$= \langle (1 - E_{\mathcal{A}})|f|g, |f|g\rangle \geq 0.$$

Hence $E_{\mathcal{A}}(|f|) \geq 0$, and $|E_{\mathcal{A}}(f)|^2 \geq E_{\mathcal{A}}(|f|)^2$.

(3) follows because as noted above the product of two $\mathcal{A}$-measurable functions is again $\mathcal{A}$-measurable.

Finally, we have (4) because if $1 \leq p < \infty$, then for every $f \in L^p(X,\mathcal{B},\mu)$ and $g \in L^\infty(X,\mathcal{A},\mu)$ we have

$$\int E_{\mathcal{A}}(f)g\,d\mu = \int E_{\mathcal{A}}(fg)\,d\mu = \int fg\,d\mu,$$

hence if $1 < q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$ then

$$\|E_{\mathcal{A}}(f)\|_p = \sup_{g \in L^\infty(X,\mathcal{A},\mu), \|g\|_q \leq 1} \left|\int E_{\mathcal{A}}(f)g\,d\mu\right|$$

$$\leq \sup_{g \in L^\infty(X,\mathcal{B},\mu), \|g\|_q \leq 1} \left|\int fg\,d\mu\right| = \|f\|_p.$$

□

Note that because $E_{\mathcal{A}}$ extends continuously to a contraction on $L^p(X,\mathcal{B},\mu)$ for any $1 \leq p \leq \infty$ we will use the same notation $E_{\mathcal{A}}$ for any of these extensions.

Exercise 2.5.4. If $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is a measure preserving action, and $\mathcal{A}$ is Γ-invariant, then show that

$$\sigma_\gamma(E_{\mathcal{A}}(f)) = E_{\mathcal{A}}(\sigma_\gamma(f)),$$

for all $\gamma \in \Gamma$, and $f \in L^p(X, \mathcal{B}, \mu)$, $1 \leq p \leq \infty$.

Theorem 2.5.5 (Birkoff's Ergodic Theorem [Bir31])**.** *Let $\mathbb{Z} \curvearrowright^T (X, \mathcal{B}, \mu)$ be a measure preserving action of $\mathbb{Z}$ on a probability space $(X, \mathcal{B}, \mu)$. Let $\mathcal{I} \subset \mathcal{B}$ denote the σ-subalgebra of Γ-invariant sets. Then for each $f \in L^1(X, \mathcal{B}, \mu)$ we have $\| \cdot \|_1$, and almost everywhere convergence*

$$\frac{1}{N} \Sigma_{n=0}^{N-1} \sigma_n(f) \to E_{\mathcal{I}}(f).$$

We will model the proof of this theorem by the proof of von Neumann's Ergodic Theorem (this is the approach taken in [PY98], that is to say we will prove the theorem for functions of the form $g - \sigma_n(g)$ which are easier to handle, and we will also show that we can approximate an arbitrary function f with $E_{\mathcal{I}}(f) = 0$ by some linear combination of functions of the above type.

Lemma 2.5.6. *Let $\mathbb{Z} \curvearrowright^T (X, \mathcal{B}, \mu)$ be a measure preserving action of $\mathbb{Z}$ on a probability space $(X, \mathcal{B}, \mu)$. Let $\mathcal{I} \subset \mathcal{B}$ denote the σ-subalgebra of Γ-invariant sets. Then for each $\varepsilon > 0$, and $f \in L^1(X, \mathcal{B}, \mu)$ with $E_{\mathcal{I}}(f) = 0$, there exists $g \in L^\infty(X, \mathcal{B}, \mu)$ such that $\|f - (\sigma_1(g) - g)\|_1 < \varepsilon$.*

Proof. Let B be the $\| \cdot \|_1$ closure of the subspace $\{\sigma_1(g) - g \mid g \in L^\infty(X, \mathcal{B}, \mu)\}$. If $h \in L^\infty(X, \mathcal{B}, \mu)$ such that $\int (\sigma_1(g) - g) h \, d\mu = 0$ for all $g \in L^\infty(X, \mathcal{B}, \mu)$ then we have

$$\int \sigma_1(g)(h - \sigma_1(h)) \, d\mu = \int (\sigma_1(g) - g) h \, d\mu = 0.$$

Thus $\sigma_1(h) = h$ and hence $\sigma_n(h) = h$ for all $n \in \mathbb{Z}$. We therefore have that $h \in L^\infty(X, \mathcal{I}, \mu)$ and hence

$$\int f h \, d\mu = \int E_{\mathcal{I}}(fh) \, d\mu = \int E_{\mathcal{I}}(f) h \, d\mu = 0.$$

Since $L^\infty(X, \mathcal{B}, \mu) = (L^1(X, \mathcal{B}, \mu))^*$ by the Hahn-Banach Theorem this shows that $f \in B$. □

For any $f \in L^1(X, \mathcal{B}, \mu)$ and $\varepsilon > 0$, we will denote

$$E_\varepsilon(f) = \{x \in X \mid \limsup_{N \to \infty} \frac{1}{N} |\Sigma_{n=0}^{N-1} \sigma_n(f)(x)| \geq \varepsilon\}.$$

Lemma 2.5.7. *Using the notation above, for each $f \in L^1(X, \mathcal{B}, \mu)$ we have $\mu(E_{2\varepsilon}(f)) \leq \frac{1}{\varepsilon} \int |f| \, d\mu$.*

Proof. Suppose $f \geq 0$, and for each $M \geq 1$ consider

$$E_\varepsilon^M(f) = \{x \in X \mid \sup_{1 \leq N \leq M} \frac{1}{N} \Sigma_{n=0}^{N-1} \sigma_n(f)(x) \geq \varepsilon\}.$$

We will first show that if $P > M$ then we have the inequality

$$\Sigma_{n=0}^{P-1} \sigma_n(f)(x) \geq \varepsilon \Sigma_{n=0}^{P-M-1} \sigma_n(1_{E_\varepsilon^M(f)})(x).$$

Indeed, if $x \in X$, then we may consider the first $0 \leq n_0 \leq P - M$ such that $x \in T^{n_0}(E_\varepsilon^M(f))$, we then have that there exists some $n_0 + 1 \leq N_0 \leq n_0 + M - 1$, such that $\Sigma_{n=n_0}^{N_0} \sigma_n(f)(x) \geq \varepsilon(N_0 - n_0)$. Hence, we have that

$$\Sigma_{n=0}^{P-1} \sigma_n(f)(x) \geq \sigma_{n_0}(f) + \Sigma_{n=N_0}^{P-1} \sigma_n(f)(x)$$

$$= (\varepsilon \Sigma_{n=0}^{n_0-1} \sigma_n(1_{E_\varepsilon^M(f)})(x)) + \sigma_{n_0}(f)(x) + \Sigma_{n=N_0}^{P-1} \sigma_n(f)(x)$$

$$\geq (\varepsilon \Sigma_{n=0}^{N_0-1} \sigma_n(1_{E_\varepsilon^M(f)})(x)) + \Sigma_{n=N_0}^{P-1} \sigma_n(f)(x).$$

We may now consider the first $N_0 \leq n_1 \leq P - M$ such that $x \in T^{n_1}(E_\varepsilon^M(f))$ and the corresponding $n_1 + 1 \leq N_1 \leq n_1 + M - 1$, the same argument then gives

$$\Sigma_{n=0}^{P-1} \sigma_n(f)(x) \geq (\varepsilon \Sigma_{n=0}^{N_1-1} \sigma_n(1_{E_\varepsilon^M(f)})(x)) + \Sigma_{n=N_1}^{P-1} \sigma_n(f).$$

Continuing in this manor we obtain the inequality

$$\Sigma_{n=0}^{P-1} \sigma_n(f)(x) \geq \varepsilon \Sigma_{n=0}^{P-M-1} \sigma_n(1_{E_\varepsilon^M(f)})(x),$$

for all $x \in X$, and $P > M$.

Integrating this inequality we obtain

$$P \int f \, d\mu = \int \Sigma_{n=0}^{P-1} \sigma_n(f)(x) \, d\mu$$

$$\geq \varepsilon \int \Sigma_{n=0}^{P-M-1} \sigma_n(1_{E_\varepsilon^M(f)})(x) \, d\mu = \varepsilon(P - M)\mu(E_\varepsilon^M(f)).$$

Dividing by P and taking the limit as $P \to \infty$ we have that

$$\int f \, d\mu \geq \varepsilon \mu(E_\varepsilon^M(f)).$$

for all $M \geq 1$.

Since $E_\varepsilon^M(f)$ are increasing as M increases, and since $E_\varepsilon(f) \subset \cup_{M=1}^\infty E_\varepsilon^M(f)$, we have that

$$\int f \, d\mu \geq \varepsilon \mu(E_\varepsilon(f)).$$

For general $f \in L^1(X, \mathcal{B}, \mu)$ we may consider $f = f_+ - f_-$ where $f_+, f_- \geq 0$, and $f_+ f_- = 0$. We then have

$$\int |f| \, d\mu \geq \varepsilon(\mu(E_\varepsilon(f_+)) + \mu(E_\varepsilon(f_-))) \geq \varepsilon \mu(E_{2\varepsilon}(f)).$$

$\square$

Proof of Birkoff's Ergodic Theorem. Let $f \in L^1(X, \mathcal{B}, \mu)$ be given, and note that by subtracting $E_{\mathcal{I}}(f)$ we may assume that $E_{\mathcal{I}}(f) = 0$.

Let $\delta, \varepsilon > 0$, and consider $E_\varepsilon(f)$ as defined above. By Lemma 2.5.6 there exists $g \in L^\infty(X, \mathcal{B}, \mu)$ such that $\|f - (\sigma_1(g) - g)\|_1 < \delta\varepsilon/4 < \delta$. For each $x \in X$ we have that

$$\limsup_{N\to\infty} \frac{1}{N} |\Sigma_{n=0}^{N-1} \sigma_n(\sigma_1(g) - g)(x)| \leq \limsup_{N\to\infty} \frac{1}{N} |(\sigma_N(g) - g)(x)|$$

$$\leq \limsup_{N\to\infty} \frac{2\|g\|_\infty}{N} = 0.$$

Hence $E_{\varepsilon/2}(\sigma_1(g) - g) = \emptyset$.

By Lemma 2.5.7 we therefore have that

$$\mu(E_\varepsilon(f)) \leq \mu(E_{\varepsilon/2}(\sigma_1(g) - g)) + \mu(E_{\varepsilon/2}(f - (\sigma_1(g) - g)))$$

$$\leq \frac{4}{\varepsilon}\|f - (\sigma_1(g) - g)\|_1 < \delta,$$

and also

$$\limsup_{N\to\infty} \|\frac{1}{N}\Sigma_{n=1}^{N-1}\sigma_n(f)\|_1$$

$$\leq \limsup_{N\to\infty} \|\frac{1}{N}\Sigma_{n=1}^{N-1}(\sigma_1(g) - g)\|_1 + \|f - (\sigma_1(g) - g)\|_1 < \delta.$$

Since δ and ε are arbitrary this shows that $\frac{1}{N}\Sigma_{n=1}^{N-1}\sigma_n(f) \to 0$ almost everywhere and in $\|\cdot\|_1$. □

2.6 Recurrence theorems

Theorem 2.6.1 (Poincaré's Recurrence Theorem [Poi90])**.** *Let $\mathbb{Z} \curvearrowright^T (X, \mathcal{B}, \mu)$ be a measure preserving action on a probability space $(X, \mathcal{B}, \mu)$. If $A \subset X$ is measurable, such that $\mu(A) > 0$ then for almost every point $x \in A$, the orbit $\mathbb{Z}x$ returns to A infinitely often.*

Proof. Let $F \subset A$ be the set of points x such that $T^n(x) \notin A$, for all $n > 0$. Then $F = A \setminus (\cup_{n\in\mathbb{N}} T^n(A))$ is measurable and if $m > n$ then we have $T^m(F) \cap T^n(F) = \emptyset$. Indeed, if $T^{-m}(x) \in F \subset A$ then by the definition of F we have that $T^{-n}(x) = T^{m-n}(T^{-m}(x)) \notin A$. We therefore have that for any $N \in \mathbb{N}$

$$\mu(F) = \frac{1}{N}\Sigma_{n=1}^N \mu(T^m(F)) = \frac{1}{N}\mu(\cup_{n=1}^N T^m(F)) \leq \frac{1}{N}.$$

Hence $\mu(F) = 0$ and the theorem follows easily. □

If $\mathbb{Z} \curvearrowright^T (X, \mathcal{B}, \mu)$ is a measure preserving action on a probability space $(X, \mathcal{B}, \mu)$, and $A \subset X$ is measurable, such that $\mu(A) > 0$, then for each $x \in A$ we let $n_A(x) \in \mathbb{N} \cup \{\infty\}$ be the smallest natural number such that $T^{n_A(x)}(x) \in A$. It is easy to see that n_A is measurable and by Poincaré's Recurrence Theorem, n_A is almost everywhere finite.

For $n \in \mathbb{N}$ we let $A_n \subset A$ be the set of points $x \in A$ such that $T^n(x) \in A$ but $T^k(x) \notin A$ for all $0 < k < n$, so that $n_A = \Sigma_{n\in\mathbb{N}} n 1_{A_n}$. If $0 \leq k_1 < n_1 < \infty$, and $0 \leq k_2 < n_2 < \infty$, then we have that $T^{k_1}(A_{n_1}) \cap T^{k_2}(A_{n_2}) = \emptyset$ unless $n_1 = n_2$ and $k_1 = k_2$. Indeed, if $k_1 < k_2$ and $x \in T^{k_1}(A_{n_1}) \cap T^{k_2}(A_{n_2})$, then $T^{-k_2}(x) \in A_{n_2}$ and $T^{k_2-k_1}(T^{-k_2}(x)) = T^{-k_1}(x) \in A_{n_1} \subset A$, hence $k_2 \geq k_2 - k_1 \geq n_2 > k_2$ which cannot happen. On the other hand, if $k_1 = k_2$, and $n_1 \neq n_2$ then $T^{k_1}(A_{n_1}) \cap T^{k_2}(A_{n_2}) = T^{k_1}(A_{n_1} \cap A_{n_2}) = T^{k_1}(\emptyset) = \emptyset$.

There is a nice picture to go along with this disjoint decomposition known as the Kakutani tower, however I am not TEX savvy enough to draw it, so instead I will refer to page 45 in [Pet83].

Theorem 2.6.2 ([Kac47])**.** *Let $\mathbb{Z} \curvearrowright^T (X, \mathcal{B}, \mu)$ be a measure preserving action on a probability space $(X, \mathcal{B}, \mu)$. If $A \subset X$ is measurable, such that $\mu(A) > 0$, then*

$$\int_A n_A \, d\mu \leq 1,$$

with equality when the action is ergodic.

Proof. Using the notation above, we have that

$$\int_A n_A \, d\mu = \Sigma_{n\in\mathbb{N}} n\mu(A_n)$$

$$= \Sigma_{n\in\mathbb{N}} \Sigma k = 0^{n-1} \mu(T^k(A_n)) = \mu(\cup_{n\in\mathbb{N}} \cup_{k=0}^{n-1} T^k(A_n)) \leq 1.$$

The set $\cup_{n\in\mathbb{N}} \cup_{k=0}^{n-1} T^k(A_n)$ is a $\mathbb{Z}$-invariant set which contains A, hence if the action is ergodic then we have

$$\int_A n_A \, d\mu = \mu(\cup_{n\in\mathbb{N}} \cup_{k=0}^{n-1} T^k(A_n)) = 1.$$

□

Theorem 2.6.3 ([Roh48])**.** *Let $\mathbb{Z} \curvearrowright^T (X, \mathcal{B}, \mu)$ be an ergodic measure preserving action on a probability space $(X, \mathcal{B}, \mu)$. Then for any $N \in \mathbb{N}$, and $\varepsilon > 0$, there exists $A \subset X$ measurable, such that $A, T(A), \ldots, T^{N-1}(A)$ are pairwise disjoint, and $\mu(\cup_{j=0}^{N-1} T^j(A)) > 1 - \varepsilon$, and $\mu(A \Delta T^N(A)) < \varepsilon$.*

Proof. This theorem is an easy exercise if $(X, \mathcal{B}, \mu)$ is not a diffuse subspace, hence we will assume that it is diffuse. Given $B \subset X$ with $\mu(B) > 0$ then we can consider the Kakutani tower $\{T^k(B_n)\}_{n\in\mathbb{N}, 0\leq k<n}$ associated to B. If we then consider $A = \cup_{n\geq N} \cup_{0\leq k<n/N} T^{kN}(B_n)$ then we have that $A, T(A), \ldots, T^{N-1}(A)$ are pairwise disjoint. Since the action is ergodic, we also have that

$$\mu(\cup_{j=0}^{N-1} T^j(A)) \geq 1 - \mu(\cup_{n\in\mathbb{N}} \cup_{\max\{n-N,0\}\leq k<n} T^k(B_n))$$

$$\geq 1 - N\Sigma_{n\in\mathbb{N}} \mu(B_n) = 1 - N\mu(B),$$

and

$$\mu(A\Delta T^N(A)) \leq 2\Sigma_{n\geq N}\mu(B_n) \leq 2\mu(B).$$

Hence, if we choose $B \subset X$ measurable, such that $0 < \mu(B) < \varepsilon/N$ (which is possible since $(X, \mathcal{B}, \mu)$ is diffuse), then we obtain a set A which satisfies the desired properties. □

Theorem 2.6.4 ([Khi35]). *Let $\Gamma \curvearrowright (X, \mu)$ be a measure preserving action of a countable amenable group Γ on a probability space $(X, \mathcal{B}, \mu)$. Suppose $f \in L^2(X, \mu)$, $f > 0$. Then for each $\varepsilon > 0$, there exists $F \subset \Gamma$ finite such that for all $\gamma_0 \in \Gamma$ we have that*

$$\gamma_0 F \cap \{\gamma \in \Gamma \mid \int \sigma_\gamma(f) f \,\mathrm{d}\mu \geq \|f\|_1^2 - \varepsilon\} \neq \emptyset.$$

Proof. Let $F_n \subset \Gamma$ be a Følner sequence for Γ. By von Neumann's Ergodic Theorem there exists $n \in \mathbb{N}$, such that

$$\|\frac{1}{|F_n|}\Sigma_{\gamma\in F_n^{-1}}\sigma_\gamma(f) - E_{\mathcal{I}}(f)\|_2 < \varepsilon/(1 + \|f\|_2).$$

Therefore, we have that

$$|\langle \frac{1}{|F_n|}\Sigma_{\gamma\in\gamma_0 F_n^{-1}}\sigma_\gamma(f) - E_{\mathcal{I}}(f), f\rangle| = |\langle \frac{1}{|F_n|}\Sigma_{\gamma\in F_n^{-1}}\sigma_\gamma(f) - E_{\mathcal{I}}(f), \sigma_{\gamma_0^{-1}}(f)\rangle|$$

$$\leq \|\frac{1}{|F_n|}\Sigma_{\gamma\in F_n^{-1}}\sigma_\gamma(f) - E_{\mathcal{I}}(f)\|_2\|f\|_2 < \varepsilon.$$

Hence,

$$\langle \frac{1}{|F_n|}\Sigma_{\gamma\in\gamma_0 F_n^{-1}}\sigma_\gamma(f), f\rangle \geq \langle E_{\mathcal{I}}(f), f\rangle - \varepsilon$$

$$= \|E_{\mathcal{I}}(f)\|^2 - \varepsilon \geq |\langle E_{\mathcal{I}}(f), 1\rangle|^2 - \varepsilon = \|f\|_1^2 - \varepsilon.$$

□

Corollary 2.6.5. *Let $\mathbb{Z} \curvearrowright^T (X, \mathcal{B}, \mu)$ be a measure preserving action on a probability space $(X, \mathcal{B}, \mu)$, then for each $A \subset X$ measurable, $\mu(A) > 0$, and $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $j \in \mathbb{N}$ we have*

$$\{j, j+1, \ldots, j+K\} \cap \{k \in \mathbb{N} \mid \mu(T^k(E) \cap E) \geq \mu(E)^2 - \varepsilon\} \neq \emptyset.$$

Proof. This follows by applying Theorem 2.6.4 to the Følner sequence $F_n = \{-1, -2, \ldots, -n\}$ for $\mathbb{Z}$, and the function $f = \chi_E$. □

Chapter 3

Uniform multiple recurrence

Furstenberg's Recurrence Theorem [Fur77] states that if $\mathbb{Z}\curvearrowright^{T}(X,\mathcal{B},\mu)$ is a measure preserving action on a probability space $(X,\mathcal{B},\mu)$ and $A\subset X$ is a measurable subset such that $\mu(A)>0$, then for each $k\geq 1$ there exists $n\geq 1$ such that

$$\mu(A\cap T^{n}(A)\cap\cdots\cap T^{kn}(A))>0.$$

This was then generalized by Furstenberg and Katznelson [FK78] where it is shown that if $k\in\mathbb{N}$, and $\mathbb{Z}\curvearrowright^{T}(X,\mathcal{B},\mu)$ is a measure preserving action on a probability space $(X,\mathcal{B},\mu)$, then for each measurable set $A\subset X$, such that $\mu(A)>0$, we actually have uniform multiple recurrence

$$\liminf_{N\to\infty}\frac{1}{N}\Sigma_{n=0}^{N-1}\mu(T^{n}(A)\cap T^{2n}(A)\cap\cdots\cap T^{kn}(A))>0.$$

A further generalization of these theorems which is conjectured by Bergelson is the following.

Conjecture 3.0.6 ([Ber96])**.** Let Γ be a countable amenable group with a Følner sequence $F_n\subset\Gamma$. Suppose $k\in\mathbb{N}$ and $\Gamma\curvearrowright^{\alpha^{j}}(X,\mathcal{B},\mu)$ are measure preserving actions of Γ on a probability space $(X,\mathcal{B},\mu)$, for $0\leq j<k$. Assume moreover that the actions pairwise commute, i.e., $\alpha^{i}_{\gamma}\circ\alpha^{j}_{\lambda}=\alpha^{j}_{\lambda}\circ\alpha^{i}_{\gamma}$, for all $i\neq j$, and $\gamma,\lambda\in\Gamma$. Then for each measurable subset $A\subset X$ with $\mu(A)>0$ we have

$$\liminf_{n\to\infty}\frac{1}{|F_n|}\Sigma_{\gamma\in F_n^{-1}}\mu(\alpha^{0}_{\gamma}(A)\cap\alpha^{0}_{\gamma}\alpha^{1}_{\gamma}(A)\cap\cdots\cap\alpha^{0}_{\gamma}\cdots\alpha^{k-1}_{\gamma}(A))>0.$$

The purpose of the next two sections is to establish this conjecture in the extreme cases when either all the actions involved are compact or they are all weak mixing.

3.1 Multiple recurrence for compact actions

We will first introduce a bit of notation. If $(X, \mathcal{B}, \mu)$ is a probability space, we will denote by $X^k = \Pi_{j=0}^{k-1} X$, $\mathcal{B}^{\otimes k} = \otimes_{j=0}^{k-1} \mathcal{B}$, and $\mu^k = \Pi_{j=0}^{k-1} \mu$. Given $f_0, f_1, \ldots, f_{k-1} \in L^\infty(X, \mathcal{B}, \mu)$ we denote by $f_0 \otimes f_1 \otimes \cdots \otimes f_{k-1}$ the function on X^k given by

$$(f_0 \otimes \cdots \otimes f_{k-1})(x_0, \ldots, x_{k-1}) = f_0(x_0) f_1(x_1) \cdots f_{k-1}(x_{k-1}).$$

We also denote by $L^\infty(X, \mathcal{B}, \mu)^{\otimes_{\mathrm{alg}} k} \subset L^\infty(X^k, \mathcal{B}^{\otimes k}, \mu^k)$ the algebra generated by all functions of the form $f_0 \otimes \cdots \otimes f_{k-1}$. Note that this extends to a unitary between $L^2(X, \mathcal{B}, \mu)^{\overline{\otimes} k}$ and $L^2(X^k, \mathcal{B}^{\otimes k}, \mu^k)$, hence we will identify these two spaces.

Theorem 3.1.1. *Let Γ be a countable amenable group with a Følner sequence $F_n \subset \Gamma$. Suppose $k \in \mathbb{N}$ and $\Gamma \curvearrowright^{\alpha^j} (X, \mathcal{B}, \mu)$ are compact, measure preserving actions of Γ on a probability space $(X, \mathcal{B}, \mu)$, for $0 \leq j < k$. Suppose $f \in L^\infty(X, \mathcal{B}, \mu)$, $f \neq 0$. Then we have*

$$\liminf_{n \to \infty} \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} \| \sigma_\gamma^0(f) \sigma_\gamma^1(f) \cdots \sigma_\gamma^{k-1}(f) \|_2^2 > 0.$$

Proof. Suppose $f \in L^\infty(X, \mathcal{B}, \mu)$ as above, note that we may assume that $f \geq 0$, and $\|f\|_2 = 1$. Let $\tilde{f} = f \otimes f \otimes \cdots \otimes f \in L^\infty(X^k, \mathcal{B}^{\otimes k}, \mu^k)$, and fix $\delta > 0$ to be chosen later. Since each action α^j is compact, we have that $\alpha^j(\Gamma) \subset \mathrm{Aut}(X, \mathcal{B}, \mu)$ is precompact in the weak topology. Thus the diagonal action $\Gamma \curvearrowright^{\tilde{\alpha}} (X^k, \mathcal{B}^{\otimes k}, \mu^k)$ is also compact since $\tilde{\alpha}(\Gamma) \subset \mathrm{Aut}(X^k, \mathcal{B}^{\otimes k}, \mu^k)$ is contained in a product of compact groups. Hence by Lemma 1.7.10 there exists a finite set $E_\delta \subset \Gamma$ such that $\inf_{\gamma_0 \in E_\delta} \|\sigma_\gamma(\tilde{f}) - \sigma_{\gamma_0}(\tilde{f})\|_2 < \delta$, for all $\gamma \in \Gamma$.

Therefore by the pigeon hole principle, for each finite set $F \subset \Gamma$ there exists $\gamma_0 \in E_\delta$ such that $\Phi(F, \gamma_0) = \{\gamma \in F \mid \|\sigma_\gamma(\tilde{f}) - \sigma_{g_0}(\tilde{f})\|_2 < \delta\}$ satisfies

$$|\Phi(F, \gamma_0)| \geq |F| / |E_\delta|.$$

Note that if $\gamma \in \gamma_0^{-1} \Phi(F, \gamma_0)$ then for each $0 \leq i < k$ we have

$$2\langle \sigma_\gamma^i(f), f \rangle \geq 2\Pi_{j=0}^{k-1} \langle \sigma_\gamma^j(f), f \rangle$$

$$= 2 - \|\sigma_\gamma(\tilde{f}) - \tilde{f}\|_2^2 \geq 2 - \delta^2,$$

and hence

$$\|f - \sigma_\gamma^i(f)\|_2^2 = 2 - 2\langle \sigma_\gamma(f), f \rangle \leq \delta^2.$$

We therefore have

$$\|\sigma_\gamma^0(f) \sigma_\gamma^1(f) \cdots \sigma_\gamma^{k-1}(f)\|_2$$

$$\geq \|f^k\|_2 - \Sigma_{j=0}^{k-1} \|f\|_\infty^{k-1} \|\sigma_\gamma^j(f) - f\|_2$$

$$\geq \|f^k\|_2 - k\|f\|_\infty^{k-1} \delta.$$

Thus, if we set $\delta = \|f^k\|_2/2k\|f\|_\infty^{k-1}$ then we have

$$\liminf_{n\to\infty} \frac{1}{|F_n|}\Sigma_{\gamma\in F_n^{-1}}\|\sigma_\gamma^0(f)\cdots\sigma_\gamma^{k-1}(f)\|_2^2$$

$$\geq \liminf_{n\to\infty} \sup_{\gamma_0\in E_\delta} \frac{|\gamma_0^{-1}\Phi(F,\gamma_0)\cap F_n|}{|F_n|}\|f^k\|_2^2/4$$

$$= \liminf_{n\to\infty} \sup_{\gamma_0\in E_\delta} \frac{|\gamma_0^{-1}\Phi(F,\gamma_0)\cap \gamma_0^{-1}F_n|}{|F_n|}\|f^k\|_2^2/4$$

$$\geq \|f^k\|_2^2/4|E_\delta| > 0.$$

□

Note that if α^i is compact for all $0 \leq i < k$ and the actions pairwise commute, then an easy exercise shows that the actions $\alpha^0\alpha^1\cdots\alpha^i$ are also compact, hence by considering $f = 1_A$, the above theorem verifies Conjecture 3.0.6 in this case.

3.2 Multiple recurrence for weak mixing actions

Definition 3.2.1. Suppose $\Gamma\curvearrowright(X,\mathcal{B},\mu)$ is a measure preserving action of a countable amenable group Γ on a probability space $(X,\mathcal{B},\mu)$. Let $F_n \subset \Gamma$ be a Følner sequence, and suppose $A \subset L^\infty(X,\mathcal{B},\mu)$ is a Γ-invariant unital, self-adjoint subalgebra. A state φ on A is said to be **generic** with respect to F_n, and μ if

$$\lim_{n\to\infty} \frac{1}{|F_n|}\Sigma_{\gamma\in F_n^{-1}}\varphi(\sigma_\gamma(f)) = \int f\,d\mu,$$

for all $f \in A$.

The example to keep in mind is when $\mathbb{Z}\curvearrowright(\mathbb{T},\mathcal{B}_{\rm orel},\mu)$ is the rotation action of $\mathbb{Z}$ on the circle by an irrational (modulo 2π) angle. Where A is the algebra of continuous functions on $\mathbb{T}$, and $\varphi \in A^*$ is given by evaluation at some point $z_0 \in \mathbb{T}$. Then it follows from Birkoff's Ergodic Theorem that for almost all choices of z_0, this state is generic. This justifies the terminology.

Lemma 3.2.2. *Suppose $\Gamma\curvearrowright(X,\mathcal{B},\mu)$ is an ergodic, measure preserving action of a countable amenable group Γ on a probability space $(X,\mathcal{B},\mu)$. Let $F_n \subset \Gamma$ be a Følner sequence, and suppose $A \subset L^\infty(X,\mathcal{B},\mu)$ is a Γ-invariant unital, self-adjoint subalgebra. If φ is a state on A which is generic with respect to F_n, and μ then*

$$\lim_{n\to\infty} \frac{1}{|F_n|}\Sigma_{\gamma\in F_n^{-1}}\sigma_\gamma(f) = \int f\,d\mu,$$

for all $f \in A$, where the convergence is in $L^2(A,\varphi)$.

Proof. Note, that when φ is given by $\varphi(f) = \int f\,d\mu$ then this is just a restatement of von Neumann's Ergodic Theorem.

Let $\varepsilon > 0$ be given, and suppose that $f \in A$. Note that by subtracting the integral we may assume that $\int f\,d\mu = 0$. Since the action is ergodic it follows from von Neumann's Ergodic Theorem that there exists $n_0 \in \mathbb{N}$ such that

$$\|\frac{1}{|F_{n_0}|}\Sigma_{\gamma_0 \in F_{n_0}^{-1}}\sigma_{\gamma_0}(f)\|^2_{L^2(X,\mathcal{B},\mu)} < \varepsilon.$$

Using the triangle, followed by the Cauchy-Schwartz inequality $((|\frac{1}{N}\Sigma_{n=0}^{N-1}a_n|)^2 \leq (\frac{1}{N}\Sigma_{n=0}^{N-1}|a_n|)^2 \leq \frac{1}{N}\Sigma_{n=0}^{N-1}|a_n|^2)$, and then using the fact that φ is generic we then have

$$\limsup_{n\to\infty}\varphi(|\frac{1}{|F_n|}\Sigma_{\gamma\in F_n^{-1}}\frac{1}{|F_{n_0}|}\Sigma_{\gamma_0\in F_{n_0}^{-1}}\sigma_{\gamma\gamma_0}(f)|^2)$$
$$\leq \limsup_{n\to\infty}\frac{1}{|F_n|}\Sigma_{\gamma\in F_n^{-1}}\varphi(\sigma_\gamma(|\frac{1}{|F_{n_0}|}\Sigma_{\gamma_0\in F_{n_0}^{-1}}\sigma_{\gamma_0}(f)|^2))$$
$$= \|\frac{1}{|F_{n_0}|}\Sigma_{\gamma_0\in F_{n_0}^{-1}}\sigma_{\gamma_0}(f)\|^2_{L^2(X,\mathcal{B},\mu)} < \varepsilon.$$

Also, for each $\gamma_0 \in F_{n_0}$ we have that

$$\limsup_{n\to\infty}\|\frac{1}{|F_n|}\Sigma_{\gamma\in F_n^{-1}}(\sigma_\gamma(f) - \sigma_{\gamma\gamma_0}(g))\|_\infty \leq \limsup_{n\to\infty}\frac{|F_n\Delta\gamma_0^{-1}F_n|}{|F_n|} = 0.$$

Hence, combining this with the above inequality we have

$$\limsup_{n\to\infty}\varphi(|\frac{1}{|F_n|}\Sigma_{\gamma\in F_n^{-1}}\sigma_\gamma(f)|^2) < \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary this shows that

$$\limsup_{n\to\infty}\|\frac{1}{|F_n|}\Sigma_{\gamma\in F_n^{-1}}\sigma_\gamma(f)\|^2_{L^2(A,\varphi)} = 0.$$

□

Exercise 3.2.3. Generalize the above lemma to arbitrary measure preserving actions. That is to say, suppose $\Gamma \curvearrowright (X,\mathcal{B},\mu)$ is a measure preserving action of a countable amenable group Γ on a probability space $(X,\mathcal{B},\mu)$. Let $F_n \subset \Gamma$ be a Følner sequence, and suppose $A \subset L^\infty(X,\mathcal{B},\mu)$ is a Γ-invariant unital, self-adjoint subalgebra such that if $L^\infty(X,\mathcal{I},\mu)$ is the algebra of bounded Γ-invariant functions then $L^\infty(X,\mathcal{I},\mu) \cap A$ is dense in $L^2(X,\mathcal{I},\mu)$. Suppose φ is a state on A such that for all $f \in A$ we have

$$\lim_{n\to\infty}\frac{1}{|F_n|}\Sigma_{\gamma\in F_n^{-1}}\varphi(\sigma_\gamma(f)) = \int f\,d\mu.$$

Then show that for all $f \in A$ we have

$$\lim_{n\to\infty}\frac{1}{|F_n|}\Sigma_{\gamma\in F_n^{-1}}\sigma_\gamma(f) = E_{\mathcal{I}}(f),$$

where the convergence is in $L^2(A,\varphi)$.

Theorem 3.2.4. *Let Γ be a countable amenable group with Følner sequence $F_n \subset \Gamma$. Suppose $k \in \mathbb{N}$ and $\Gamma \curvearrowright^{\alpha^j} (X, \mathcal{B}, \mu)$ are weak mixing, measure preserving actions of Γ on a probability space $(X, \mathcal{B}, \mu)$, for $0 \leq j < k$. Assume moreover that the actions pairwise commute, i.e., $\alpha^i_\gamma \circ \alpha^j_\lambda = \alpha^j_\lambda \circ \alpha^i_\gamma$, for all $i \neq j$, and $\gamma, \lambda \in \Gamma$, and that the actions $\gamma \mapsto \alpha^i_\gamma \alpha^{i+1}_\gamma \cdots \alpha^j_\gamma$ are weak mixing for all $0 \leq i \leq j < k$. Suppose $f_0, \ldots, f_{k-1} \in L^\infty(X, \mathcal{B}, \mu)$. Then we have*

$$\lim_{n \to \infty} \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} \sigma^0_\gamma(f_0)(\sigma^0_\gamma \sigma^1_\gamma(f_1)) \cdots (\sigma^0_\gamma \cdots \sigma^{k-1}_\gamma(f_{k-1})) = \Pi_{j=0}^{k-1}(\int f_j \, d\mu),$$

where the convergence is in $L^2(X, \mathcal{B}, \mu)$.

Proof. Consider the action $\Gamma \curvearrowright (X^k, \mathcal{B}^{\otimes k}, \mu^k)$ given by $\gamma(x_0, x_1, \ldots, x_{k-1}) = (\alpha^0_\gamma x_0, \alpha^0_\gamma \alpha^1_\gamma x_1, \ldots, \alpha^0_\gamma \cdots \alpha^{k-1}_\gamma x_{k-1})$.

Denote by ν the diagonal measure on X^k given by $\nu(A) = \mu(\{x \in X \mid (x, x, \ldots, x) \in A\})$. Then we obtain a well defined state φ on $A = L^\infty(X, \mathcal{B}, \mu)^{\otimes_{\rm alg} k}$ by

$$\varphi(\tilde{f}) = \int \tilde{f} \, d\nu,$$

for all $\tilde{f} \in A$.

By Lemma 3.2.2, in order to prove the theorem it is enough to show that φ is generic with respect to F_n, and μ^k. We will do by induction on k.

Note that $k = 1$ is trivial. If this holds for $k \geq 1$, and $f_0, f_1, \ldots, f_k \in L^\infty(X, \mathcal{B}, \mu)$, then by Lemma 3.2.2 we have that

$$\lim_{n \to \infty} \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} \sigma^1_\gamma(f_1) \cdots (\sigma^1_\gamma \cdots \sigma^k_\gamma(f_k)) = \Pi_{j=1}^{k}(\int f_j \, d\mu),$$

in $L^2(X, \mathcal{B}, \mu)$.

Multiplying this by f_0 and integrating we obtain

$$\lim_{n \to \infty} \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} \int f_0 \sigma^1_\gamma(f_1) \cdots (\sigma^1_\gamma \cdots \sigma^k_\gamma(f_k)) \, d\mu = \Pi_{j=0}^{k}(\int f_j \, d\mu).$$

Applying σ^0_γ then gives the result. □

Corollary 3.2.5. *Let $\mathbb{Z} \curvearrowright^T (X, \mathcal{B}, \mu)$ be a weak mixing, measure preserving action on a probability space $(X, \mathcal{B}, \mu)$. Then for each $k \in \mathbb{N}$, and $f_0, f_1, \ldots, f_{k-1} \in L^\infty(X, \mathcal{B}, \mu)$ we have*

$$\lim_{N \to \infty} \frac{1}{N} \Sigma_{n=0}^{N-1} \sigma_n(f_0) \sigma_{2n}(f_1) \cdots \sigma_{kn}(f_{k-1}) = \Pi_{j=0}^{k-1}(\int f_j \, d\mu),$$

where the convergence is in $L^2(X, \mathcal{B}, \mu)$.

Proof. If we consider the action $\alpha^i : \mathbb{Z} \to \mathrm{Aut}(X, \mathcal{B}, \mu)$ which takes the generator of $\mathbb{Z}$ to T^i, then it follows from Corollary 1.7.7 that these actions satisfy the hypotheses of the above theorem. □

3.3 Relatively independent joinings and the basic construction

Having established Furstenberg's multiple recurrence for the cases of compact and weak mixing actions of $\mathbb{Z}$, it follows from Proposition **??** that every probability measure preserving action $\mathbb{Z}\curvearrowright(X, \mathcal{B}, \mu)$ has a non-trivial factor $\mathbb{Z}\curvearrowright(X, \mathcal{A}, \mu)$ for which Furstenberg's multiple recurrence holds. To establish this principle in the general case we must show how to extend this result to larger factors. The strategy for doing this is to replace $\mathbb{C}$ with $L^\infty(X, \mathcal{A}, \mu)$ and to mimic the previous constructions and proofs.

3.3.1 Joinings

Let $\Gamma\curvearrowright(X, \mathcal{B}, \mu)$, and $\Gamma\curvearrowright(Y, \mathcal{A}, \nu)$ be two measure preserving action of a countable group Γ on probability spaces $(X, \mathcal{B}, \mu)$, and $(Y, \mathcal{A}, \nu)$. For the moment let us denote by $B = L^\infty(X, \mathcal{B}, \mu)$ and $A = L^\infty(Y, \mathcal{A}, \nu)$. Suppose $\phi : B \to A$ is a map which is positive ($\phi(f) \geq 0$, whenever $f \geq 0$), unital ($\phi(1) = 1$), preserves the integrals ($\int \phi(f)\, d\nu = \int f\, d\mu$, for all $f \in B$), and is Γ-equivariant ($\phi(\sigma_\gamma(f)) = \sigma_\gamma(\phi(f))$, for all $\gamma \in \Gamma$, and $f \in B$).

Then on the algebra $B \otimes_{\rm alg} A$ we obtain a state τ such that for $\Sigma_{j=1}^n b_j \otimes a_j \in B \otimes_{\rm alg} A$ we have

$$\tau(\Sigma_{j=1}^n b_j \otimes a_j) = \int \Sigma_{j=1}^n \phi(b_j) a_j\, d\mu.$$

Since ϕ is Γ-equivariant, it follows from Proposition 2.3.1 that there exists a probability space $(Z, \mathcal{C}, \eta)$ with a measure preserving action of Γ, and there exists a $*$-homomorphism $\pi : B \otimes_{\rm alg} A \to L^\infty(Z, \mathcal{C}, \eta)$ such that $\pi \circ \sigma_\gamma = \sigma_\gamma \circ \pi$, for all $\gamma \in \Gamma$, and $\int \pi(x)\, d\eta = \tau(x)$ for all $a \in B \otimes_{\rm alg} A$.

Because ϕ is unital, and preserves the integral, it follows that $\pi_{|B\otimes 1}$ and $\pi_{|1\otimes A}$ injective and integral preserving. Thus, $L^\infty(Z, \mathcal{C}, \eta)$ contains isomorphic copies of $L^\infty(X, \mathcal{B}, \mu)$ and $L^\infty(Y, \mathcal{A}, \nu)$. Specifically, if we denote by $\mathcal{C}_X \subset \mathcal{C}$ (resp. $\mathcal{C}_Y \subset \mathcal{C}$) the σ-subalgebra generated by the image of $\pi_{|B\otimes 1}$ (resp. $\pi_{|1\otimes A}$) then $\mathcal{C}_X$, and $\mathcal{C}_Y$ are Γ-invariant, and we have that $\pi_{|B\otimes 1}$, and $\pi_{|1\otimes A}$ give Γ-eqivariant isomrophism of $L^\infty(X, \mathcal{B}, \mu)$ onto $L^\infty(Z, \mathcal{C}_X, \eta)$, and $L^\infty(Y, \mathcal{A}, \nu)$ onto $L^\infty(Z, \mathcal{C}_Y, \eta)$.

Moreover, after this identification, we recover the map ϕ by the formula $\phi(f) = E_{\mathcal{C}_Y}(f) \in L^\infty(Z, \mathcal{C}_Y, \eta)$, for all $f \in L^\infty(Z, \mathcal{C}_X, \eta)$. Indeed, this follows since if $b \in B$, and $a \in A$, then we have

$$\int \phi(b) a\, d\nu = \tau(b \otimes a)$$

$$= \int \pi(b)\pi(a)\, d\eta = \int E_{\mathcal{C}_Y}(\pi(b))\pi(a)\, d\eta.$$

Given $b \in B$, and $a \in A$, we will often abuse notation and denote by $b \otimes a$ the function $\pi(b \otimes a) \in L^\infty(Z, \mathcal{C}, \eta)$. One should be careful however with this abuse of notation because π need not be faithful in general.

Definition 3.3.1. Let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ be two measure preserving action of a countable group Γ on probability spaces $(X, \mathcal{B}, \mu)$, and $(Y, \mathcal{A}, \nu)$. A **joining** of these two actions is a measure preserving action $\Gamma \curvearrowright (Z, \mathcal{C}, \eta)$ together with Γ-equivariant, integral preserving embeddings of $L^\infty(X, \mathcal{B}, \mu)$ and $L^\infty(Y, \mathcal{A}, \nu)$ in to $L^\infty(Z, \mathcal{C}, \eta)$, such that $\mathcal{C}$ is the σ-algebra generated by these embeddings.

If $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ is the same action as $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ then we say that a joining is a self joining of the action $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$.

From the discussion above, joinings are in 1-1 correspondence with Γ-equivariant, unital, integral preserving, positive maps $\phi : L^\infty(X, \mathcal{B}, \mu) \to L^\infty(Y, \mathcal{A}, \nu)$.

If $\Gamma \curvearrowright (Z_0, \mathcal{C}, \eta)$ is a probability measure preserving action, and we have two probability measure preserving actions $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$, and Γ-equivariant embeddings $\alpha : L^\infty(Z_0, \mathcal{C}, \eta) \to L^\infty(X, \mathcal{B}, \mu)$ and $\beta : L^\infty(Z_0, \mathcal{C}, \eta) \to L^\infty(Y, \mathcal{A}, \nu)$, then we obtain a Γ-equivariant, positive, unital, integral preserving map from $L^\infty(X, \mathcal{B}, \mu)$ to $L^\infty(Y, \mathcal{A}, \nu)$ by first taking the conditional expection from $L^\infty(X, \mathcal{B}, \mu)$ to $\alpha(L^\infty(Z_0, \mathcal{C}, \eta))$ and then applying the isomorphism $\beta \circ \alpha^{-1}$. The joining corresponding to this map is called the **relatively independent joining** over $(Z_0, \mathcal{C}, \eta)$. We denote the new space on which Γ acts by $X \times_{\alpha(\mathcal{C}) = \beta(\mathcal{C})} Y$, or simply by $X \times_{\mathcal{C}} Y$ if the embeddings α and β are clear from the context. We also denote by $L^\infty(X, \mathcal{B}, \mu) \otimes^{\mathrm{alg}}_{\alpha(\mathcal{C}) = \beta(\mathcal{C})} L^\infty(X, \mathcal{B}, \mu)$ the vector space generated by functions of the type $b \otimes a$ where $b \in B$, and $a \in A$.

A special case to consider is when $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ contains a Γ-invariant σ-subalgebra $\mathcal{A}$, and we have $(Z_0, \mathcal{C}, \eta) = (X, \mathcal{A}, \mu)$, then we may consider the relatively independent self joining over $(X, \mathcal{A}, \mu)$. Note, however that a relatively independent joinings consists not only of invariant σ-subalgebras, but also the ways in which we are including these subalgebras into the larger algebras. For instance, if $\alpha \in \mathrm{Aut}(L^\infty(X, \mathcal{A}, \mu))$ is a Γ-equivariant automorphism, then we obtain a new relatively independent joining by considering the alternate embedding $\alpha : L^\infty(X, \mathcal{A}, \mu) \to L^\infty(X, \mathcal{A}, \mu) \subset L^\infty(X, \mathcal{B}, \mu)$. In general, the actions $\Gamma \curvearrowright X \times_{\mathcal{A}} X$ and $\Gamma \curvearrowright X \times_{\mathcal{A} = \alpha(\mathcal{A})} X$ need not be isomorphic.

3.3.2 The basic construction

Relatively independent joinings over the trivial σ-subalgebra corresponds to taking a diagonal action on a product space. We have seen previously that a useful tool in analyzing the structure of product actions $\Gamma \curvearrowright X \times X$ was the identification between $L^2(X \times X)$ and the Hilbert-Schmidt operators on $L^2(X)$. This allowed us to use tools such as functional calculus. There is an analog of the Hilbert-Schmidt operators in the setting of relatively independent joinings which we will now describe.

Suppose $(X, \mathcal{B}, \mu)$ is a probability space and $\mathcal{A} \subset \mathcal{B}$ is a σ-subalgebra. The **basic construction** associated to the inclusion $L^\infty(X, \mathcal{A}, \mu) \subset L^\infty(X, \mathcal{B}, \mu)$ is the algebra $L^\infty(X, \mathcal{A}, \mu)' \subset \mathcal{B}(L^2(X, \mathcal{B}, \mu))$, of operators in $\mathcal{B}(L^2(X, \mathcal{B}, \mu))$ which commute with $L^\infty(X, \mathcal{A}, \mu)$. We will denote the basic construction by

$\langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$, where $e_\mathcal{A} \in \mathcal{B}(L^2(X, \mathcal{B}, \mu))$ is the orthogonal projection onto $L^2(X, \mathcal{A}, \mu)$.

Note that we have

$$L^\infty(X, \mathcal{B}, \mu)e_\mathcal{A}L^\infty(X, \mathcal{B}, \mu) = \mathrm{sp}\{\mathrm{f}e_\mathcal{A}\mathrm{g} \mid \mathrm{f}, \mathrm{g} \in \mathrm{L}^\infty(\mathrm{X}, \mathcal{B}, \mu)\} \subset \langle \mathrm{L}^\infty(\mathrm{X}, \mathcal{B}, \mu), \mathrm{e}_\mathcal{A}\rangle,$$

and that this is an algebra since if $f \in L^\infty(X, \mathcal{B}, \mu)$ we have

$$e_\mathcal{A} f e_\mathcal{A} = E_\mathcal{A}(f)e_\mathcal{A}.$$

Note that we distinguish here the the projection $e_\mathcal{A}$, which is an operator on $L^2(X, \mathcal{A}, \mu)$, from the conditional expectation $E_\mathcal{A}$, which is an operator on $L^\infty(X, \mathcal{A}, \mu)$.

Exercise 3.3.2. Suppose $S \in \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$, has polar decomposition $S = V|S|$. Show that $V \in \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$.

Lemma 3.3.3. *Suppose $(X, \mathcal{B}, \mu)$ is a probability space and $\mathcal{A} \subset \mathcal{B}$ is a σ-subalgebra. For each $S \in \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$ there exists a unique $\phi(S) \in L^\infty(X, \mathcal{A}, \mu)$ such that*

$$e_\mathcal{A} S e_\mathcal{A} = \phi(S)e_\mathcal{A}.$$

Moreover, the map $S \mapsto \phi(S)$ is a unital, positivity preserving extension of $E_\mathcal{A}$, which is $L^\infty(X, \mathcal{A}, \mu)$-bimodular, and continuous with respect to the weak operator topology.

Proof. By Lemma 2.5.2 $L^\infty(X, \mathcal{A}, \mu)$ is a maximal abelian subalgebra of $\mathcal{B}(L^2(X, \mathcal{A}, \mu))$. Thus, since $e_\mathcal{A} S e_\mathcal{A}$ restricted to $L^2(X, \mathcal{A}, \mu)$ commutes $L^\infty(X, \mathcal{A}, \mu)$, there exists a unique element $\phi(S) \in L^\infty(X, \mathcal{A}, \mu)$ such that $e_\mathcal{A} S e_\mathcal{A} f = \phi(S)f = \phi(S)e_\mathcal{A} f$, for all $f \in L^2(X, \mathcal{A}, \mu)$. If $f \in L^2(X, \mathcal{A}, \mu)^\perp \subset L^2(X, \mathcal{B}, \mu)$ then we have $e_\mathcal{A} S e_\mathcal{A} f = 0 = \phi(S)e_\mathcal{A}$.

That $x \mapsto \phi(x)$ is unital, positivity preserving, and weak operator topology continuous, follows from the fact that it is the composition of the map $x \mapsto e_\mathcal{A} x e_\mathcal{A}$ and the $*$-homomorphism $a e_\mathcal{A} \mapsto a$. □

Exercise 3.3.4 (Generalized Cauchy-Schwartz inequality)**.** Prove that for all $x, y \in \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$ we have

$$|\phi_\mathcal{A}(y^*x)|^2 \leq \phi_\mathcal{A}(y^*y)\phi_\mathcal{A}(x^*x).$$

In the case where the σ-algebra $\mathcal{A}$ is trivial we have that $e_\mathcal{A}$ is the rank 1 projection on to the subspace $\mathbb{C}1 \subset L^2(X, \mathcal{B}, \mu)$, and thus operators of the form $f e_\mathcal{A} g$ were rank 1 projections. Rather than working with a Hilbert space basis $\{\xi_i\} \subset L^2(X, \mathcal{B}, \mu)$ as before, we could have just as easily worked with the family of partial isometries from $\mathbb{C}1$ to $\mathbb{C}\xi_i$. This motivates the following lemma.

Lemma 3.3.5. *Suppose $(X, \mathcal{B}, \mu)$ is a probability space and $\mathcal{A} \subset \mathcal{B}$ is a σ-subalgebra. There exists a family of partial isometries $\{v_i\}_{i \in I} \subset \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$ such that*

$$(a). \quad v_i v_i^* \leq e_\mathcal{A}, \quad \text{for all } i \in I;$$

$$(b). \quad v_i v_j^* = 0, \quad \text{for all } i, j \in I, i \neq j;$$

$$(c). \quad \Sigma_{i\in I} v_i^* e_{\mathcal{A}} v_i = 1.$$

Proof. A simple argument with Zorn's Lemma shows that there is a maximal (with respect to inclusion) family of partial isometries $\{v_i\}_{i\in I}$ satisfying conditions (a) and (b) above.

Let $P = \Sigma_{i\in I} v_i^* e_{\mathcal{A}} v_i$, and consider $S \in (1-P)\cdot\langle L^\infty(X,\mathcal{B},\mu), e_{\mathcal{A}}\rangle \cdot e_{\mathcal{A}}$. By considering the polar decomposition $S = V|S|$, we have that $V^*V = \mathrm{Proj}_{\overline{\mathrm{Range}(S)}} \leq e_{\mathcal{A}}$, $VV^* = \mathrm{Proj}_{\overline{\mathrm{Range}(S^*)}} \leq (1-P)$, and $V \in \langle L^\infty(X,\mathcal{B},\mu), e_{\mathcal{A}}\rangle$. Thus by maximality of the family $\{v_i\}_{i\in I}$ we must have that $V = 0$, and hence $S = 0$.

Thus $\{0\} = (1-P)\cdot\langle L^\infty(X,\mathcal{B},\mu), e_{\mathcal{A}}\rangle\cdot e_{\mathcal{A}}\cdot L^2(X,\mathcal{B},\mu) \supset (1-P)\cdot L^\infty(X,\mathcal{B},\mu)\cdot 1$. Since $L^\infty(X,\mathcal{B},\mu)\cdot 1$ is dense in $L^2(X,\mathcal{B},\mu)$ this shows that $P = 1$. □

A family of partial isometries which satisfy the conditions above will be called an **operator basis** for $\langle L^\infty(X,\mathcal{B},\mu), e_{\mathcal{A}}\rangle$. Note that since $v_i v_i^* \leq e_{\mathcal{A}}$ we have that $v_i v_i^* = \phi_{\mathcal{A}}(v_i v_i^*) \in L^\infty(X,\mathcal{A},\mu)$ for each $i \in I$.

Definition 3.3.6. Suppose $(X,\mathcal{B},\mu)$ is a probability space and $\mathcal{A} \subset \mathcal{B}$ is a σ-subalgebra. Let $\{v_i\}_{i\in I}$ be an operator basis for $\langle L^\infty(X,\mathcal{B},\mu), e_{\mathcal{A}}\rangle$. An operator $S \in \langle L^\infty(X,\mathcal{B},\mu), e_{\mathcal{A}}\rangle$ is **Hilbert-Schmidt class** with Hilbert-Schmidt norm if

$$\|S\|_{\mathrm{HS}}^2 = \Sigma_{i\in I}\int \phi_{\mathcal{A}}(v_i S^* S v_i^*)\, d\mu < \infty.$$

The quantity $\|S\|_{\mathrm{HS}}$ is the Hilbert-Schmidt norm of S.

This definition does not depend on the operator basis $\{v_i\}_{i\in I}$, this can be seen from the following analogue of Perseval's identity. If $\{w_j\}_{j\in J}$ is another operator basis, then we have

$$\Sigma_{i\in I}\int \phi_{\mathcal{A}}(v_i S^* S v_i^*)\, d\mu = \Sigma_{i\in I}\Sigma_{j\in J}\int \phi_{\mathcal{A}}(v_i S^*(w_j^* e_{\mathcal{A}} w_j) S v_i^*)\, d\mu$$

$$= \Sigma_{i\in I}\Sigma_{j\in J}\int \phi_{\mathcal{A}}(v_i S^* w_j^*)\phi_{\mathcal{A}}(w_j S v_i^*)\, d\mu$$

$$= \Sigma_{j\in J}\Sigma_{i\in I}\int \phi_{\mathcal{A}}(v_j S v_i)\phi_{\mathcal{A}}(v_i S^* w_j^*)\, d\mu$$

$$= \Sigma_{j\in J}\int \phi_{\mathcal{A}}(w_j S^* S w_j^*).$$

Exercise 3.3.7. Show that $\phi_{\mathcal{A}}$ is a contraction from the Hilbert-Schmidt norm to $L^2(X,\mathcal{A},\mu)$.

Note that it follows from above that we may approximate S in the Hilbert-Schmidt norm with finite sums of the form $\Sigma_{i,j} w_j^*(w_j S v_i^*) v_i$. Also, if $\varepsilon > 0$, and we consider $g_j, h_i \in L^\infty(X,\mathcal{B},\mu)$ such that $\|g_j - w_j^*(1)\|_2, \|v_i v_i^* - h_i v_i^*(1)\|_2 < \varepsilon$, then we have

$$= \|\phi_{\mathcal{A}}((w_j S v_i^*) - (w_j g_j \phi_{\mathcal{A}}(w_j S v_i^*) e_{\mathcal{A}} h_i v_i^*))\|_2$$

$$= \|\phi_{\mathcal{A}}(w_j S v_i^*) - \phi_{\mathcal{A}}(w_j g_j)\phi_{\mathcal{A}}(w_j S v_i^*)\phi_{\mathcal{A}}(h_i v_i^*)\|_2$$
$$\leq \|\phi_{\mathcal{A}}(w_j S v_i^*)\|_\infty (\|w_j w_j^* - \phi_{\mathcal{A}}(w_j g_j)\|_2 + \|v_i v_i^* - \phi_{\mathcal{A}}(h_i v_i^*)\|_2)$$
$$\leq 2\|S\|\varepsilon.$$

It also follows that

$$\Sigma_{(k,l)\in I\times J,(k,l)\neq(i,j)} \|\phi_{\mathcal{A}}(w_l g_j)\|_2^2 \|\phi_{\mathcal{A}}(h_i v_k^*)\|_2^2 < 2\varepsilon^2 + \varepsilon^4,$$

and hence

$$\|w_j^*(w_j S v_i^*) v_i - g_j \phi_{\mathcal{A}}(w_j S v_i^*) e_{\mathcal{A}} h_i\|_{\text{HS}}^2 \leq 2(\|S\| + 1)\varepsilon^2 + \varepsilon^4.$$

Since $\varepsilon > 0$ was arbitrary we may then use the triangle inequality to conclude that the $*$-algebra $L^\infty(X, \mathcal{B}, \mu) e_{\mathcal{A}} L^\infty(X, \mathcal{B}, \mu)$ is dense in the Hilbert-Schmidt norm.

If $T \in \langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}}\rangle$ then $T^*T \leq \|T\|^2$, and since $\phi_{\mathcal{A}}$ is positivity preserving it follows that

$$\|TS\|_{\text{HS}}^2 = \Sigma_{i\in I} \int \phi_{\mathcal{A}}(v_i S^* T^* T S v_i^*)\, d\mu \leq \|T\|^2 \|S\|_{\text{HS}}^2.$$

Also, it follows from the argument above that the adjoint operator $S \mapsto S^*$ is an anti-linear isometry, and hence we also have

$$\|ST\|_{\text{HS}}^2 \leq \|T\|^2 \|S\|_{\text{HS}}^2.$$

In particular, we see that the Hilbert-Schmidt class in $\langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}}\rangle$ is a two sided ideal.

Exercise 3.3.8. Given a Hilbert-Schmidt class operator $S \in \langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}}\rangle$ show that $\|S\|_{\text{HS}} = 0$ if and only if $S = 0$.

The Hilbert-Schmidt norm has an associated inner product

$$\langle S, T\rangle_{\text{HS}} = \Sigma_{i\in I} \int \phi_{\mathcal{A}}(v_i T^* S v_i^*)\, d\mu,$$

which is well defined by the generalized Cauchy-Schwartz inequality, and does not depend on the operator basis from the arguments above.

Thus, the class of Hilbert-Schmidt operators is an inner-product space. This space is not complete in general[1], we denote the Hilbert space completion by $L^2\langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}}\rangle$.

Even though the class of Hilbert-Schmidt operators is not a complete space in general we do have that it is complete when we restrict to the Hilbert-Schmidt operators whose uniform norm is bounded by some fixed constant.

[1]Consider the case when $\mathcal{A} = \mathcal{B}$, then it is easy to see that the class of Hilbert-Schmidt operators coincides with $L^\infty(X, \mathcal{B}, \mu)$, and the inner-product structure is the usual inner-product on $L^2(X, \mathcal{B}, \mu)$.

Proposition 3.3.9. *Suppose $(X, \mathcal{B}, \mu)$ is a probability space and $\mathcal{A} \subset \mathcal{B}$ is a σ-subalgebra. Suppose $K > 0$ and consider the convex set*

$$B_K = \{S \in \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle \mid \|S\| \leq K\}.$$

Then B_K is complete in the Hilbert-Schmidt norm.

Proof. Fix an operator basis $\{v_i\}_{i\in I}$ for $\langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$. Since $\phi_\mathcal{A}$ is a contraction from the Hilbert-Schmidt norm to $L^2(X, \mathcal{A}, \mu)$, if $S_n \in B_K$ is Cauchy in the Hilbert-Schmidt norm, then for all $i, j \in I$, $\phi(v_j S_n v_i^*) \in L^\infty(X, \mathcal{A}, \mu)$ is Cauchy in $L^2(X, \mathcal{A}, \mu)$ and we also have that $\|\phi(v_j S_n v_i^*)\|_\infty \leq K$. Hence, there exists $g_{i,j} \in L^\infty(X, \mathcal{A}, \mu)$ such that $\|g_{i,j}\|_\infty \leq K$, and

$$\lim_{n\to\infty} \|v_j^*(v_j S_n v_i^*)v_i - v_j^* g_{i,j} v_i\|_{\mathrm{HS}} = \lim_{n\to\infty} \|v_j^* \phi(v_j S_n v_i^*) e_\mathcal{A} v_i - v_j^* g_{i,j} v_i\|_{\mathrm{HS}}$$

$$\leq \lim_{n\to\infty} \|\phi(v_j S_n v_i^*) e_\mathcal{A} - g_{i,j} e_\mathcal{A}\|_{\mathrm{HS}} = 0.$$

Since $\mathrm{Range}(v_j^*)$ are pairwise orthogonal subspaces we may then consider the sum

$$S = \Sigma_{i,j\in I} v_j^* g_{i,j} v_i \in \mathcal{B}(L^2(X, \mathcal{B}, \mu)).$$

Then $\|S\| \leq K$ and it is easy to see that $S \in \langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$. More over for any finite set $I_0 \subset I$ it follows from the triangle inequality that

$$\lim_{n\to\infty} \|\Sigma_{i,j\in I_0} v_j^*(v_j S_n v_i^*)v_i - v_j^*(v_j S v_i^*)v_i\|_{\mathrm{HS}} = 0.$$

Since S_n is Cauchy in the Hilbert-Schmidt norm this implies that S is Hilbert-Schmidt class and $\|S_n - S\|_{\mathrm{HS}} \to 0$. ☐

If $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is a measure preserving action of a countable group Γ, such that $\mathcal{A}$ is Γ-invariant, then we may consider the relatively independent self joining $\Gamma \curvearrowright X \times_\mathcal{A} X$ defined above. We may then define a map $\Xi : L^\infty(X, \mathcal{B}, \mu) \otimes_\mathcal{A} L^\infty(X, \mathcal{B}, \mu) \to L^\infty(X, \mathcal{B}, \mu) e_\mathcal{A} L^\infty(X, \mathcal{B}, \mu)$ by linearly extending the formula

$$\Xi(b \otimes a) = b e_\mathcal{A} a,$$

for all $a, b \in L^\infty(X, \mathcal{B}, \mu)$.

If $\{v_i\}_{i\in I}$ is an operator basis for $\langle L^\infty(X, \mathcal{B}, \mu), e_\mathcal{A}\rangle$, we then have that for all $\Sigma_{k=1}^n b_k \otimes a_k \in L^\infty(X, \mathcal{B}, \mu) \otimes_\mathcal{A} L^\infty(X, \mathcal{B}, \mu)$

$$\|\Sigma_{k=1}^n b_k \otimes a_k\|_{L^2(X\times_\mathcal{A} X)}^2 = \int \Sigma_{k,l=1}^n E_\mathcal{A}(b_k^* b_l) a_l a_k^* \, d\mu$$

$$= \int \Sigma_{k,l=1}^n \phi_\mathcal{A}(b_k^* b_l) \phi_\mathcal{A}(a_l a_k^*) \, d\mu$$

$$= \Sigma_{i,j\in I} \int \Sigma_{k,l=1}^n \phi_\mathcal{A}(b_k^* v_i^* e_\mathcal{A} v_i b_l) \phi_\mathcal{A}(a_l v_j^* e_\mathcal{A} v_j a_k^*) \, d\mu$$

$$= \Sigma_{i,j\in I} \int \Sigma_{k,l=1}^n \phi_\mathcal{A}(b_k^* v_i^*) \phi_\mathcal{A}(v_i b_l) \phi_\mathcal{A}(a_l v_j^*) \phi_\mathcal{A}(v_j a_k^*) \, d\mu$$

$$= \Sigma_{i,j\in I} \int \Sigma_{k,l=1}^{n} \phi_{\mathcal{A}}(v_i b_l e_{\mathcal{A}} a_l v_j^*) \phi_{\mathcal{A}}(v_j a_k^* e_{\mathcal{A}} b_k^* v_i^*)\, d\mu$$

$$= \|\Xi(\Sigma_{k=1}^{n} b_k \otimes a_k)\|_{\mathrm{HS}}^2.$$

Hence Ξ is well defined and extends to a unitary operator (which we will also denote by Ξ) from $L^2(X \times_{\mathcal{A}} X)$ to $L^2\langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}}\rangle$.

Moreover, this unitary implements an equivalence between the Koopman representation of $\Gamma \curvearrowright X \times_{\mathcal{A}} X$ and the representation of Γ on $L^2\langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}}\rangle$, given by $S \mapsto \sigma_\gamma S \sigma_{\gamma^{-1}}$.

Suppose $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$, and $\Gamma \curvearrowright (Y, \mathcal{A}, \nu)$ are two probability measure preserving actions and we have Γ-equivariant embeddings $\alpha : L^\infty(Z_0, \mathcal{C}, \eta) \to L^\infty(X, \mathcal{B}, \mu)$ and $\beta : L^\infty(Z_0, \mathcal{C}, \eta) \to L^\infty(Y, \mathcal{A}, \nu)$. We may consider the class of operators $S \in \mathcal{B}(L^2(X, \mathcal{B}, \mu), L^2(Y, \mathcal{A}, \mu))$ such that $S\alpha(g) = \beta(g)S$ and $|S^*S|^{1/2}$ is in the Hilbert-Schmidt class of $\langle L^\infty(X, \mathcal{B}, \mu), e_{\alpha(\mathcal{C})}\rangle$. In this case we can consider the norm given by $\|S\|_{\mathrm{HS}} = \||S^*S|^{1/2}\|_{\mathrm{HS}}$, and consider the completion under this norm.

It then follows that this is a Hilbert space, and we may consider the map Ξ which linearly extends the formula $\Xi(b \otimes a) = b e_{\beta(\mathcal{C})=\alpha(\mathcal{C})} a$ (here we view $e_{\beta(\mathcal{C})=\alpha(\mathcal{C})}$ as an operator from $L^2(X, \mathcal{B}, \mu)$ to $L^2(Y, \mathcal{A}, \nu)$). Then just as above Ξ extends to a unitary operator which implements an isomorphism between the Koopman representation $\Gamma \curvearrowright Y \times_{\beta(\mathcal{C})=\alpha(\mathcal{C})} X$ and the representation given by $S \mapsto \sigma_\gamma^{\mathcal{A}} S \sigma_{\gamma^{-1}}^{\mathcal{B}}$.

Exercise 3.3.10. Fill in the details to the previous paragraphs.

3.4 Ergodic, weak mixing, and compact extensions

Definition 3.4.1. Let Γ be a countable group. An extension $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ of a probability measure preserving action $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$ is an **ergodic extension** if the σ-algebra $\mathcal{I}$ of Γ-invariant sets is contained in $\mathcal{A}$.

Definition 3.4.2. Let Γ be a countable group. An extension $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ of a probability measure preserving action $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$ is a **weak mixing extension** if for any $\varepsilon > 0$ and any finite set $\mathcal{F} \subset L^2(X, \mathcal{B}, \mu)$ such that $E_{\mathcal{A}}(f) = 0$ for all $f \in \mathcal{F}$, there exists $\gamma \in \Gamma$ such that for all $f, g \in \mathcal{F}$ we have

$$\|E_{\mathcal{A}}(\sigma_\gamma(f)\overline{g})\|_1 < \varepsilon.$$

Definition 3.4.3. Let Γ be a countable group and let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be an extension of a probability measure preserving action $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$. A function $f \in L^2(X, \mathcal{B}, \mu)$ is **almost periodic** relative to $L^\infty(X, \mathcal{A}, \mu)$ if for all $\varepsilon > 0$ there exist $g_1, \ldots, g_n \in L^\infty(X, \mathcal{B}, \mu)$ such that for all $\gamma \in \Gamma$ we have

$$\sigma_\gamma(f) \in_\varepsilon L^2(X, \mathcal{A}, \mu)g_1 + \cdots + L^2(X, \mathcal{A}, \mu)g_n.$$

That is to say, for each $\gamma \in \Gamma$ there exist $k_1^\gamma, \ldots, k_n^\gamma \in L^2(X, \mathcal{A}, \mu)$ such that

$$\|\sigma_\gamma(f) - \Sigma_{j=1}^n k_j^\gamma g_j\|_2 < \varepsilon.$$

The proof of the following proposition is a straight forward generalization of the proof of Proposition 2.2.22 and we leave the details to the reader.

Proposition 3.4.4. *Let Γ be a countable group and let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be an extension of a probability measure preserving action $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$. Let $\mathcal{C} \subset \mathcal{B}$ be the σ-algebra generated by all functions which are almost periodic relative to $L^\infty(X, \mathcal{A}, \mu)$. Then $\mathcal{A} \subset \mathcal{C}$, $\mathcal{C}$ is Γ-invariant, and $f \in L^2(X, \mathcal{B}, \mu)$ is almost periodic relative to $L^\infty(X, \mathcal{A}, \mu)$ if and only if f is $\mathcal{C}$ measurable.*

Definition 3.4.5. Let Γ be a countable group. An extension $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ of a probability measure preserving action $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$ is a **compact extension** if every $f \in L^2(X, \mathcal{B}, \mu)$ is almost periodic relative to $L^\infty(X, \mathcal{A}, \mu)$.

Note that Proposition 3.4.4 shows that there is a unique maximal compact extension of $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$ which is a factor of $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$.

We can now generalize the relationship between ergodic, weak mixing, and compact actions that we have seen before.

Theorem 3.4.6. *Let Γ be a countable group and let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be an extension of a probability measure preserving action $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$. The following conditions are equivalent:*

(1). *The extension is weak mixing.*

(2). *There are no non-trivial functions $f \in L^2(X, \mathcal{B}, \mu)$ which are almost periodic relative to $L^\infty(X, \mathcal{A}, \mu)$.*

(3). *The relatively independent self joining $\Gamma \curvearrowright X \times_{\mathcal{A}} X$ is an ergodic extension of $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$.*

(4). *For any ergodic extension $\Gamma \curvearrowright (Y, \mathcal{C}, \nu)$ of $\Gamma \curvearrowright (X, \mathcal{A}, \nu)$, the relatively independent self joining $\Gamma \curvearrowright X \times_{\mathcal{A}} Y$ is an ergodic extension of $\Gamma \curvearrowright (X, \mathcal{A}, \nu)$.*

(5). *The relatively independent self joining $\Gamma \curvearrowright X \times_{\mathcal{A}} X$ is a weak mixing extension of $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$.*

(6). *For any weak mixing extension $\Gamma \curvearrowright (Y, \mathcal{C}, \nu)$ of $\Gamma \curvearrowright (X, \mathcal{A}, \nu)$, the relatively independent self joining $\Gamma \curvearrowright X \times_{\mathcal{A}} Y$ is a weak mixing extension of $\Gamma \curvearrowright (X, \mathcal{A}, \nu)$.*

Proof. For (1) $\implies$ (2) suppose that $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is a weak mixing extension and $f \in L^\infty(X, \mathcal{B}, \mu)$ is almost periodic relative to $L^\infty(X, \mathcal{A}, \mu)$, such that $E_{\mathcal{A}}(f) = 0$. Consider $\varepsilon > 0$, then there exists $g_1, \ldots, g_n \in L^\infty(X, \mathcal{B}, \mu)$ such that for every $\gamma \in \Gamma$, there are $k_1^\gamma, \ldots, k_n^\gamma \in L^2(X, \mathcal{A}, \mu)$ such that

$$\|\sigma_\gamma(f) - \Sigma_{j=1}^n k_j^\gamma g_j\|_2 < \varepsilon. \tag{3.1}$$

Note that since $\sup_{\gamma \in \Gamma} \|\sigma_\gamma(f)\|_\infty = \|f\|_\infty$ we may assume that

$$K = \sup_{1 \leq j \leq n,\ \gamma \in \Gamma} \|k_j^\gamma\|_\infty < \infty.$$

Also, by replacing g_j with $g_j - E_{\mathcal{A}}(g_j)$ we may assume that $E_{\mathcal{A}}(g_j) = 0$ for $1 \leq j \leq n$.

Since $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is a weak mixing extension there exists $\gamma \in \Gamma$ such that for all $1 \leq j \leq n$ we have

$$\|E_{\mathcal{A}}(\sigma_\gamma(f)\overline{g_j})\|_1 < \varepsilon.$$

Hence, for all $1 \leq j \leq n$ we also have

$$|\langle \sigma_\gamma(f), k_j^\gamma g_j \rangle| \leq \|E_{\mathcal{A}}(\sigma_\gamma(f)\overline{g_j k_j^\gamma})\|_1 = \|E_{\mathcal{A}}(\sigma_\gamma(f)\overline{g_j})\overline{k_j^\gamma}\|_1 \leq K\varepsilon. \tag{3.2}$$

Combining (3.1) and (3.2) we then have

$$\|f\|_2^2 \leq \|\sigma_\gamma(f)\|_2^2 + \|\Sigma_{j=1}^n k_j^\gamma g_j\|_2^2$$

$$= \|\sigma_\gamma(f) - \Sigma_{j=1}^n k_j^\gamma g_j\|_2^2 - 2\Re\langle \sigma_\gamma(f), \Sigma_{j=1}^n k_j^\gamma g_j \rangle \leq \varepsilon^2 + 2Kn\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary this shows that $f = 0$.

For (2) $\implies$ (3) suppose that the relatively independent self joining $\Gamma \curvearrowright X \times_{\mathcal{A}} X$ is not an ergodic extension. Hence, there exists a non-trivial Γ-invariant function $f \in L^2(X \times_{\mathcal{A}} X)$ such that $E_{\mathcal{A}}(f) = 0$. Using the identification from Section **??** we then produce a non-trivial element $S \in L^2\langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}} \rangle$ such that $\phi_{\mathcal{A}}(S) = 0$, and $\mathrm{Ad}(\sigma_\gamma)(\mathrm{S}) = \mathrm{S}$ for all $\gamma \in \Gamma$. Note that we may assume $\|S\|_{\mathrm{HS}} = 1$, and hence by considering an operator $S_0 \in \langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}} \rangle$ such that $\|S_0\|_{\mathrm{HS}} = 1$, and $\|S_0 - S\|_{\mathrm{HS}} < 1/4$, we have that $\|\mathrm{Ad}(\sigma_\gamma)(\mathrm{S}_0) - \mathrm{S}_0\|_{\mathrm{HS}} < 1/2$, for all $\gamma \in \Gamma$.

If we consider then the closed (in the Hilbert-Schmidt norm) convex hull of $\{\mathrm{Ad}(\sigma_\gamma)(\mathrm{S}_0) \mid \gamma \in \Gamma\}$, then the element T of minimal norm is $\mathrm{Ad}(\sigma_\gamma)$-invariant, satisfies $T \notin L^\infty(X, \mathcal{A}, \mu)$, and is bounded in the uniform norm by Proposition 3.3.9. Consider the unit ball $B_1 = \{f \in L^2(X, \mathcal{B}, \mu) \mid \|f\|_2 \leq 1\}$, then $T(B_1) \subset L^2(X, \mathcal{B}, \mu)$ is Γ-invariant.

If $\varepsilon > 0$ is given and we consider an operator basis $\{v_i\}_{i \in I}$ for $\langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}} \rangle$ then we may find a finite set $I_0 \subset I$ such that

$$\|T - \Sigma_{i,j \in I_0} v_j^* \phi_{\mathcal{A}}(v_j T v_i^*) v_i\|_{\mathrm{HS}} < \varepsilon/2\|T\|.$$

It is then not hard to see that if we consider $f_j \in L^\infty(X, \mathcal{B}, \mu)$ such that $\|f_j - v_j^*(1)\|_2 < \varepsilon/2|I_0|$, then for all $f \in T(B_1)$ we have

$$f \in_\varepsilon \Sigma_{j \in I_0} L^2(X, \mathcal{A}, \mu) f_j.$$

This shows that every function in $T(B_1)$ is almost periodic relative to $L^\infty(X, \mathcal{A}, \mu)$. The same argument also shows the same for every function in $T^*(B_1)$. Since

$T \neq e_{\mathcal{A}} T e_{\mathcal{A}}$ this shows that there exists a function which is not in $L^2(X, \mathcal{A}, \mu)$ which is almost periodic relative to $L^\infty(X, \mathcal{A}, \mu)$.

For (3) $\implies$ (1) if the action is not a weak mixing extension, then there exists $c > 0$ and a finite set $\mathcal{F} \subset L^2(X, \mathcal{B}, \mu)$ such that $E_{\mathcal{A}}(f) = 0$ for all $f \in \mathcal{F}$, and for all $\gamma \in \Gamma$ we have

$$\Sigma_{f,g \in \mathcal{F}} \|E_{\mathcal{A}}(\sigma_\gamma(f)\overline{g})\|_2^2 \geq \Sigma_{f,g \in \mathcal{F}} \|E_{\mathcal{A}}(\sigma_\gamma(f)\overline{g})\|_1^2 \geq c.$$

If we consider the operator $T = \Sigma_{f \in \mathcal{F}} f e_{\mathcal{A}} \overline{f} \in \langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}} \rangle$ then we have that T has finite Hilbert-Schmidt norm, and for each $\gamma \in \Gamma$ we have

$$\begin{aligned} &\langle \mathrm{Ad}(\sigma_\gamma)(\mathrm{T}), \mathrm{T} \rangle_{\mathrm{HS}} \\ &= \Sigma_{f,g \in \mathcal{F}} \langle \sigma_\gamma(f) e_{\mathcal{A}} \sigma_\gamma(\overline{f}), g e_{\mathcal{A}} \overline{g} \rangle_{\mathrm{HS}} \\ &= \Sigma_{f,g \in \mathcal{F}} \|E_{\mathcal{A}}(\sigma_\gamma(f)\overline{g})\|_2^2 \geq c \end{aligned}$$

Therefore, by Proposition 1.5.2 there exists a non-trivial $\mathrm{Ad}(\sigma_\gamma)$-invariant vector in $L^2\langle L^\infty(X, \mathcal{B}, \mu), e_{\mathcal{A}} \rangle$. From Section **??** it then follows that $\Gamma \curvearrowright X \times_{\mathcal{A}} X$ is not an ergodic extension.

The proof that (3) $\Leftrightarrow$ (4) then follows from the remarks at the end of Section **??**. If $\Gamma \curvearrowright X \times_{\mathcal{A}} Y$ is not an ergodic extension then just as above we construct an operator $T \in \mathcal{B}(L^2(Y, \mathcal{C}, \nu), L^2(X, \mathcal{B}, \mu))$ which commutes with the embeddings of $\mathcal{A}$ measurable functions, intertwines the Koopman representations of Γ, and such that $T \neq e_{\mathcal{A}} T e_{\mathcal{A}}$. by considering the operators $(T^*T)^{1/2}$ and $(TT^*)^{1/2}$, the result follows easily.

The equivalence of (1)-(4) shows that if $\Gamma \curvearrowright X \times_{\mathcal{A}} X$ is an ergodic extension then $\Gamma \curvearrowright (X \times_{\mathcal{A}} X) \times_{\mathcal{A}} (X \times_{\mathcal{A}} X)$ is also an ergodic extension. This then shows that (1)-(5) are equivalent, and a similar argument works for (6) as well. □

Corollary 3.4.7. *Let Γ be a countably infinite amenable group with a Følner sequence $F_n \subset \Gamma$, and let $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ be an extension of a probability measure preserving action $\Gamma \curvearrowright (X, \mathcal{A}, \mu)$. Then $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is a weak mixing extension if and only if for all $f, g \in L^2(X, \mathcal{B}, \mu)$ such that $E_{\mathcal{A}}(f) = E_{\mathcal{A}}(g) = 0$ we have that*

$$\lim_{n \to \infty} \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} \int |E_{\mathcal{A}}(\sigma_\gamma(f)\overline{g})|^2 \, d\mu = 0.$$

Proof. It is clear that this condition implies that $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is a weak mixing extension and so we need only prove the converse.

Assume that $\Gamma \curvearrowright (X, \mathcal{B}, \mu)$ is a weak mixing extension then by Theorem 3.4.6 the relatively independent self joining $\Gamma \curvearrowright X \times_{\mathcal{A}} X$ is an ergodic extension, and hence if $f, g \in L^\infty(X, \mathcal{B}, \mu)$ are such that $E_{\mathcal{A}}(f) = E_{\mathcal{A}}(g) = 0$, then we have that $\|E_{\tilde{\mathcal{I}}}(f \otimes g)\|_2 \leq \|E_{\mathcal{A}}(f \otimes g)\|_2 = 0$, where $\tilde{I}$ is the σ-algebra of Γ-invariant sets for $X \times_{\mathcal{A}} X$.

By von Neumann's Ergodic Theorem we then have

$$\lim_{n \to \infty} \frac{1}{|F_n|} \Sigma_{\gamma \in F_n^{-1}} \int |E_{\mathcal{A}}(\sigma_\gamma(f)\overline{g})|^2 \, d\mu$$

$$= \lim_{n\to\infty} \frac{1}{|F_n|} \Sigma_{\gamma\in F_n^{-1}} \langle (\sigma_\gamma \otimes_{\mathcal{A}} \sigma_\gamma)(f \otimes \overline{f}), g \otimes \overline{g} \rangle$$

$$= \langle E_{\tilde{\mathcal{I}}}(f \otimes \overline{f}), g \otimes \overline{g} \rangle = 0.$$

We then obtain the general case when $f, g \in L^2(X, \mathcal{B}, \mu)$ by continuity. □

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