HOMEWORK 3, DUE THURSDAY FEBRUARY 18TH

1. Let A be a C^*-algebra and φ be a state on A. Show that if $x \in A$ is self-adjoint and $\varphi(x^2) = \varphi(x)^2$ then $\varphi(xy) = \varphi(x)\varphi(y)$ and $\varphi(yx) = \varphi(y)\varphi(x)$, for all $y \in A$.

2. Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a representation of a unital C^*-algebra A, and let $\xi \in \mathcal{H}$ be a unit vector such that $\pi(A)\xi$ is dense in $\mathcal{H}$. Show that the GNS-representation corresponding to the vector state $\phi(x) = \langle \pi(x)\xi, \xi \rangle$ is equivalent to the representation π.

3. Let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a representation of a unital C^*-algebra A. Show that π is not irreducible if and only if there exists a projection $P \in \mathcal{B}(\mathcal{H})$ which commutes with A and such that $P \neq 0, 1$.

4. Let A be a unital C^*-algebra and let $S(A)$ denote the state space of A endowed with the weak*-topology in A^*. Show that $S(A)$ is a convex and compact subspace of A^*.

Recall that an extreme point of a convex set is a point which cannot be decomposed as a non-trivial convex combination of other points. Note that it then follows from the previous exercise and the Krein-Milman Theorem that if a C^*-algebra A is unital then $S(A)$ is the closed convex hull of it's extreme points. In particular $S(A)$ has extreme points since we know that it is non-empty. Extreme points in $S(A)$ are called pure states.

5. Let A be a unital C^*-algebra and $\varphi \in S(A)$. Show that φ is a pure state if and only if the GNS-representation π_φ corresponding to φ is irreducible.

6. Let A be a separable C^*-algebra.

(a). Show that there is a faithful representation $\pi : A \to \mathcal{B}(\mathcal{H})$ such that $\mathcal{H}$ is separable.

(b). Show that A has a faithful state, i.e. there is a state $\varphi \in S(A)$ such that for all $x \in A$, if $\varphi(x^*x) = 0$ then $x = 0$.