HOMEWORK 2, DUE TUESDAY FEBRUARY 9TH

Let X be a locally compact Hausdorff Space. Consider the unital abelian C^*-algebra $A = C_b(X)$ consisting of all bounded continuous functions from X to $\mathbb{C}$.

Let βX be the spectrum (i.e., the set of multiplicative linear functionals with the weak*-topology) of A so that the Gelfand transform gives an isomorphism $\Gamma : A \to C(\beta X)$.

1. Show that there is a natural inclusion map from X into a dense subset of βX such that under this identification βX is the Stone-Čech compactification of X, i.e., if K is any compact Hausdorff space and $f : X \to K$ is continuous then there exists a unique continuous extension of f to βX.

Hint: you may want to use the fact that any compact Hausdorff space is isomorphic to a closed subspace of a cube.

An ultrafilter ω on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ which has the following properties.

(a). $\emptyset \notin \omega$.

(b). If $A \subset B \subset \mathbb{N}$ and $A \in \omega$ then $B \in \omega$.

(c). If $A, B \in \omega$ then $A \cap B \in \omega$.

(d). If $A \subset \mathbb{N}$ then either $A \in \omega$ or $\mathbb{N} \setminus A \in \omega$.

The ultrafilter ω is a principle ultrafilter if there is an element $k \in \mathbb{N}$ such that $\omega = \{A \subset \mathbb{N} \mid k \in A\}$. Otherwise the ultrafilter is said to be free.

Given $a \in \mathbb{C}$ and an element $\{a_n\}_n \in \ell^\infty\mathbb{N}$ we say that $\{a_n\}_n$ converges to a along the ultrafilter ω (and write$\lim_{n\to\omega} a_n = a$) if for each $\varepsilon > 0$ we have $\{n \in \mathbb{N} \mid |a_n - a| < \varepsilon\} \in \omega$.

2. Given an ultrafilter ω show that every bounded sequence $\{a_n\}_n \in \ell^\infty\mathbb{N}$ has a unique limit.

3. Prove that there is a natural correspondence between ultrafilters on $\mathbb{N}$ and multiplicative linear functionals on $\ell^\infty\mathbb{N}$. In particular, show that $\beta\mathbb{N}$ can be identified as the set of ultrafilters on $\mathbb{N}$ in such a way that $\mathbb{N} \subset \beta\mathbb{N}$ are the principle ultrafilters and $\beta\mathbb{N} \setminus \mathbb{N}$ are the free ultrafilters.