

Problem 1. A large tank is half full with a mixture of salt dissolved in water. Pure water is pumped into the tank at the rate of 3 gallons per minute, and water is let out of the tank at the rate of 1 gallon per minute, thus overall the tank is filling up. Assuming the mixture in the tank is kept uniform by stirring, what percentage of the original salt will still be in the tank when it is full?

Simplify your answer as much as possible.

Solution 1. Let v be the volume of mixture which starts in the tank (in gallons), r_i and c_i (resp. r_o, c_o) the rate and concentration of salt for the water coming into the tank (resp. out of the tank), we know that the change in the amount of salt in the tank during an interval Δt is going to be the amount of salt that comes in $r_i c_i \Delta t$ minus the amount of salt that goes out $r_o c_o \Delta t$. The concentration of salt in the tank at time t is $c_o = c_o(t)$ and is given by the amount of salt in the tank at time t divided by the amount of mixture in the tank at time t , this gives rise to the differential equation:

$$\frac{dX}{dt} = r_i c_i - \frac{r_o X}{v + (r_i - r_o)t}.$$

Once we plug in the known constants $r_i = 3$, $r_o = 1$, and $c_i = 0$ we have

$$\frac{dX}{dt} = -\frac{X}{v + 2t}.$$

By separation of variables and integration we obtain

$$\ln |X| = \int \frac{1}{X} \frac{dX}{dt} dt = - \int \frac{1}{v + 2t} dt = -\frac{1}{2} \ln |v + 2t| + C.$$

Note X , and $v + t$ are always positive quantities and so by exponentiating both sides of the equation we have

$$X = \frac{C_1}{\sqrt{v + 2t}},$$

where $C_1 = e^C$.

We are looking for the percentage of the salt which is still in the tank once the tank is full. The volume of the tank is $2v$ and the tank is filling at a rate of 2 gallon per minute, thus the tank will be full after $v/2$ minutes, hence the desired quantity we are looking for is

$$X(v/2)/X(0) = \left(\frac{C_1}{\sqrt{v + 2(v/2)}}\right) / \left(\frac{C_1}{\sqrt{v}}\right) = \frac{1}{\sqrt{2}} = \sqrt{2}/2.$$

Thus $\sqrt{2}/2$ or approximately 71% of the salt will be left.