Math 208 - Exam 3, April 14, 2009 Name:__________________Key______________________

Problem 1 (20 points). Find the general solution to the differential equation $y' + xy = 1 + x$ in powers of x. You may leave your answer in the form of a recurrence relation for the coefficients of y, in any case however you should explicitly compute the first four terms of the power series expansion.

Solution 1. Suppose $y = \Sigma_{n=0}^{\infty} a_n x^n$ is a solution. Then

$$1 + x = y' + xy = \Sigma_{n=1}^{\infty} a_n n x^{n-1} + \Sigma_{n=0}^{\infty} a_n x^{n+1}$$

$$= \Sigma_{n=0}^{\infty} a_{n+1}(n+1)x^n + \Sigma_{n=1}^{\infty} a_{n-1} x^n$$

$$= a_1 + (2a_2 + a_0)x + \Sigma_{n=2}^{\infty}(a_{n+1}(n+1) + a_{n-1})x^n.$$

Thus we have $a_1 = 1$, $2a_2 + a_0 = 1$, and $a_{n+1}(n+1) + a_{n-1} = 0$ for $n \geq 2$.

Our recurrence relation is then given by

$$\boxed{a_{n+1} = -\frac{a_{n-1}}{n+1}, \qquad n \geq 2,}$$

or explicitly as

$$\boxed{a_{2n+1} = (-1)^n \frac{2^n n!}{(2n+1)!}, \quad n \geq 0, \qquad \text{and} \qquad a_{2n} = (-1)^n \frac{(a_0 - 1)}{2^n n!}, \quad n \geq 1.}$$

Note that by the ratio test y will have a positive radius of convergence. Using these formulas we can compute the first few coefficients of y as a_0, $a_1 = 1$, $a_2 = \frac{1-a_0}{2}$, $a_3 = -\frac{1}{3}$, $a_4 = -\frac{a_2}{4} = \frac{a_0-1}{8}$.

Therefore we have

$$\boxed{y = a_0 + x + \frac{1-a_0}{2}x^2 - \frac{1}{3}x^3 + \frac{a_0 - 1}{8}x^4 + \cdots,}$$

or explicitly as

$$\boxed{y = a_0(1 + \Sigma_{n=1}^{\infty}(-1)^n \frac{1}{2^n n!}x^{2n}) - \Sigma_{n=1}^{\infty}(-1)^n \frac{1}{2^n n!}x^{2n} + \Sigma_{n=0}^{\infty}(-1)^n \frac{2^n n!}{(2n+1)!}x^{2n+1}.}$$

Note that the above equation is a first order linear equation and so we can alternately try to solve this equation by multiplying by an integrating factor $\mu(x) = e^{x^2/2}$. This leads to the solution

$$y = a_0 e^{-x^2/2} + e^{-x^2/2} \int_0^x (1+t)e^{t^2/2} dt.$$

Problem 2 (20 points). Consider the differential equation $x^2y''-2xy'+cy = 0$ where c is a real number. Find all values of c such that there exists a solution y of this differential equation which satisfies $\lim_{x\to\infty} y(x) = \infty$.

Solution 2. This is Euler's equation. The associated polynomial to this equation is $r(r-1) - 2r + c = r^2 - 3r + c$. By the binomial formula the roots of this equation are

$$r = \frac{3 \pm \sqrt{9-4c}}{2}.$$

When $9 - 4c \geq 0$ then we have a solution to the differential equation given by $y = x^r$, $x > 0$ where $r = \frac{3+\sqrt{9-4c}}{2} > 0$. Since $r > 0$ we have $\lim_{x\to\infty} x^r = \infty$.

When $9 - 4c < 0$ we can write the roots as $\mu + i\lambda$ where $\mu = \frac{3}{2} > 0$ and $\lambda = \frac{1}{2}\sqrt{4c-9} > 0$. In this case the general solution is given by $y = c_1 x^\mu \sin(\lambda \ln x) + c_2 x^\mu \cos(\lambda \ln x)$. When $\lambda \ln x = 2k\pi$ we have $y(x) = c_2 x^\mu$, when $\lambda \ln x = 2k\pi + \pi$ we have $y(x) = -c_2 x^\mu$, when $\lambda \ln x = 2k\pi + \frac{\pi}{2}$ we have $y(x) = c_1 x^\mu$, and when $\lambda \ln x = 2k\pi + \frac{3\pi}{2}$ we have $y(x) = -c_1 x^\mu$. If we have a non-zero solution then either c_1 or c_2 is non-zero and hence we have that y oscillates between positive and negative values and so in particular $\lim_{x\to\infty} y(x) \neq \infty$.

Thus a solution y exists with $\lim_{x\to\infty} y(x) = \infty$ precisely when $9 - 4c \geq 0$, or

$$\boxed{c \leq 9/4.}$$

Note that $\frac{3-\sqrt{9-4c}}{2} > 0$ for $c > 0$ and hence one can show that for $0 < c \leq 9/4$ we have in fact that every non-zero solution y satisfies either $\lim_{x\to\infty} y(x) = \infty$ or $\lim_{x\to\infty} y(x) = -\infty$.

Problem 3 (20 points). Find a non-zero solution in powers of x to the differential equation $x^2y'' + 3xy' + (1+x)y = 0$. Note that you do not have to find the general solution, any non-zero solution is fine.

Solution 3. Note that $x_0 = 0$ is a regular singular point to the above equation and hence we should look for a solution of the form $\Sigma_{n=0}^{\infty} a_n x^{n+r}$ where $a_0 \neq 0$ and r is a root of the indicial equation $r(r-1)+3r+1 = 0$, i.e. $(r+1)^2 = r^2 + 2r + 1 = 0$ and hence $r = -1$.

Therefore we look for a solution of the form $y = \Sigma_{n=0}^{\infty} a_n x^{n-1} = \Sigma_{n=-1}^{\infty} a_{n+1} x^n$. Plugging this y into the above differential equation we then have $xy' = \Sigma_{n=0}^{\infty} a_n (n-1) x^{n-1} = \Sigma_{n=-1}^{\infty} a_{n+1} n x^n$, and $x^2 y'' = \Sigma_{n=0}^{\infty} a_n (n-1)(n-2) x^{n-1} = \Sigma_{n=-1}^{\infty} a_{n+1} n(n-1) x^n$, hence

$$0 = x^2y'' + 3xy' + (1+x)y = a_0(2 - 3 + 1) + \Sigma_{n=0}^{\infty}(a_{n+1}n(n-1) + a_{n+1}3n + a_{n+1} + a_n)x^n.$$

Therefore we have that $a_{n+1}(n+1)^2 = a_{n+1}(n^2 - n + 3n + 1) = -a_n$, for $n \geq 0$. From this it is easy to calculate a_{n+1} as

$$a_{n+1} = (-1)^{n+1} a_0 / ((n+1)!)^2, \qquad n \geq 0.$$

Therefore if we set $a_0 = 1$ then we obtain a non-zero solution as

$$\boxed{y = \Sigma_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} x^{n-1},}$$

for $x \neq 0$.

Problem 4 (20 points). Find a solution to the initial value problem $y'' + y = g(t)$, $y(0) = 0$, $y'(0) = 1$, where

$$g(t) = \begin{cases} t/2, & 0 \leq t < 6 \\ 3, & t \geq 6 \end{cases}$$

Solution 4. By taking the Laplace transform of the above equation we have

$$(s^2+1)\mathcal{L}\{y\} - 1 = s^2\mathcal{L}\{y\} - sy(0) - y'(0) + \mathcal{L}\{y\} = \mathcal{L}\{g\}.$$

We can rewrite $g(t)$ as $g(t) = \frac{t}{2} + U_6(t)(3 - \frac{t}{2}) = \frac{t}{2} - U_6(t)\frac{(t-6)}{2}$ hence we find the Laplace transform of g as

$$\mathcal{L}\{g(t)\} = \mathcal{L}\{\frac{t}{2}\} - \mathcal{L}\{U_6(t)\frac{(t-6)}{2}\} = \frac{1}{2s^2} - e^{-6s}\frac{1}{2s^2}.$$

Therefore we may solve for $\mathcal{L}\{y\}$ as

$$\mathcal{L}\{y\} = \frac{1}{s^2+1} + \frac{1}{2s^2(s^2+1)} - \frac{e^{-6s}}{2s^2(s^2+1)}.$$

By partial fractions we have

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1},$$

hence we may apply the inverse Laplace transform to the above equation and obtain

$$y = \sin t + \frac{1}{2}(t - \sin t) - U_6(t)\frac{1}{2}((t-6) - \sin(t-6)),$$

or

$$\boxed{y = \frac{1}{2}(t + \sin t - U_6(t)[t - 6 - \sin(t-6)]).}$$

Problem 5 (20 points). Consider the integral equation $y(t) + \int_0^t (t-w)y(w)dw = 1$. Use the Laplace transform to find a solution y to this equation.

Hint: Use the convolution formula.

Solution 5. We recognize the formula $\int_0^t (t-w)y(w)dw = t * y$ hence by applying the Laplace transform to the above equation we have

$$\mathcal{L}\{y\} + \frac{1}{s^2}\mathcal{L}\{y\} = \mathcal{L}\{y\} + \mathcal{L}\{t * y\} = \mathcal{L}\{1\} = \frac{1}{s}.$$

Solving for $\mathcal{L}\{y\}$ gives

$$\mathcal{L}\{y\} = \frac{s}{s^2+1}.$$

Taking the inverse Laplace transform then gives

$$\boxed{y = \cos t.}$$

Problem 6 (Extra Credit - 10 points). Given $\delta > 0$, define the function $g_\delta : [0, \infty) \to \mathbb{R}$ by

$$g_\delta(t) = \begin{cases} 1/\delta, & 0 \le t < \delta \\ 0, & t \ge \delta \end{cases}$$

Show that if $f : [0, \infty) \to \mathbb{R}$ is a continuous function then $\lim_{\delta \to 0}(f * g_\delta)(t) = f(t)$, for all $t > 0$.

Solution 6. Fix $t > 0$ and define the function $F : (-\infty, t) \to \mathbb{R}$ by $F(\delta) = \int_0^\delta f(t - w)dw$, note that since f is continuous F must be differentiable, also note that $F(0) = 0$.

We then have that

$$\begin{aligned} \lim_{\delta \to 0}(f * g_\delta)(t) &= \lim_{\delta \to 0} \int_0^t f(t - w) g_\delta(w) dw \\ &= \lim_{\delta \to 0} \frac{1}{\delta} \int_0^\delta f(t - w) dw \\ &= \lim_{\delta \to 0} \frac{F(\delta) - F(0)}{\delta}. \end{aligned}$$

By the definition of the derivative we have

$$\lim_{\delta \to 0} \frac{F(\delta) - F(0)}{\delta} = F'(0).$$

By the fundamental theorem of calculus we then have

$$\lim_{\delta \to 0}(f * g_\delta)(t) = F'(0) = f(t - 0) = f(t).$$