Math 208 - Exam 2, March 17, 2009 Name:_________________Key____________________

Problem 1. Find the general solution on $\mathbb{R}$ to each of the following differential equations:

(a). $y^{(4)} + 2y'' + y = 0$.

(b). $y^{(4)} - 8y'' + 16y = \sin t + e^{2t}$.

(c). $y^{(4)} + 2y = t$

(d). $y^{(3)} + y'' - y = te^{-2t}$

Solution 1. (a). The characteristic polynomial for this equation is $r^4 + 2r^2 + 1 = (r^2 + 1)^2$ which has double roots at $r = \pm i$, thus we obtain four linearly independent solutions $\sin t$, $\cos t$, $t \sin t$ and $t \cos t$. Our general solution is then

$$\boxed{y = c_1 \sin t + c_2 \cos t + c_3 t \sin t + c_4 t \cos t,}$$

where c_1, c_2, c_3, and c_4 are constants.

(b). The characteristic polynomial of the corresponding homogeneous equation is $r^4 - 8r^2 + 16 = (r^2 - 4)^2 = (r-2)^2(r+2)^2$ which has double roots at $r = \pm 2$. The general solution to the homogeneous equation is then

$$y_h = c_1 e^{2t} + c_2 te^{-2t} + c_3 te^{2t} + c_4 te^{-2t},$$

where c_1 and c_2 are contants.

Since $\sin t$ is not a solution to the homogeneous equation we use the method of undetermined coefficients to look for a solution to the differential equation $y^{(4)} - 8y'' + 16y = \sin t$ which is of the form $y_1 = A \sin t + B \cos t$. In this case we have $y_1'' = -A \sin t - B \cos t$ and $y_1^{(4)} = y_1 = A \sin t + B \cos t$. Therefore

$$\sin t = y_1^{(4)} - 8y_1'' + 16y_1$$

$$= A \sin t + B \cos t + 8A \sin t + 8B \cos t + 16A \sin t + 16B \cos t = 25A \sin t + 25B \cos t.$$

Since sin and cos are linearly independent we must have that $A = \frac{1}{25}$ and $B = 0$, hence $y_1 = \frac{1}{33} \sin t$.

Both e^{2t} and te^{2t} are solutions to the homogeneous equation, and so using the method of undetermined coefficients we look for a solution to the differential equation $y^{(4)} - 8y'' + 16y = e^{2t}$ of the form $y_2 = At^2e^{2t}$. In this case we have $y_2'' = 4At^2e^{2t} + 8Ate^{2t} + 2Ae^{2t}$ and $y_2^{(4)} = 16At^2e^{2t} + 64Ate^{2t} + 48Ae^{2t}$. Therefore

$$e^{2t} = y_2^{(4)} - 8y_2'' + 16y_2$$

$$= 16At^2e^{2t} + 64Ate^{2t} + 48Ae^{2t} - 8(4At^2e^{2t} + 8Ate^{2t} + 2Ae^{2t}) + 16At^2e^{2t} = 32Ae^{2t}$$

and so we must have that $A = \frac{1}{32}$ hence $y_2 = \frac{1}{32}t^2e^{2t}$.

Combining the above calculations we find the general solution to the non-homogeneous equation as

$$\boxed{y = c_1 e^{2t} + c_2 te^{-2t} + c_3 te^{2t} + c_4 te^{-2t} + \frac{1}{25} \sin t + \frac{1}{32} t^2 e^{2t},}$$

where c_1, c_2, c_3, and c_4 are constants.

(c). The characteristic polynomial of the corresponding homogeneous equation is r^4+2. Using Euler's formula we may write -2 in polar form $-2 = 2e^{(2k+1)\pi i}$ where $k \in \mathbb{Z}$. We then have $(-2)^{1/4} = (2e^{(2k+1)\pi i})^{1/4} = 2^{1/4}e^{(\frac{k}{2}+\frac{1}{4})\pi i}$. Thus we find 4 distinct complex roots of the above equation given as

$$r_1 = 2^{1/4}e^{\frac{\pi}{4}i} = 2^{1/4}(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}) \quad r_2 = 2^{1/4}e^{\frac{3\pi}{4}i} = 2^{1/4}(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}),$$

$$r_3 = 2^{1/4}e^{\frac{5\pi}{4}i} = 2^{1/4}(-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}), \quad r_4 = 2^{1/4}e^{\frac{7\pi}{4}i} = 2^{1/4}(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}).$$

To simplify notation let $a = \frac{8^{1/4}}{2}$, then the general solution to the homogeneous equation is given by

$$y_h = c_1e^{at}\cos(at) + c_2e^{at}\sin(at) + c_3e^{-at}\cos(at) + c_4e^{-at}\sin(at),$$

where c_1, c_2, c_3, and c_4 are constants.

Note that t is not a solution to the homogeneous equation and so we use the method of undetermined coefficients to look for a solution to the non-homogeneous differential equation which is of the form $y_1 = At+B$. In this case we have that $y_1^{(4})=0$ and so

$$t = y_1^{(4)} + 2y = 0 + 2At + 2B.$$

Therefore $B = 0$ and $A = \frac{1}{2}$, hence $y_1 = \frac{1}{2}t$. Combining this with the solution to the homogeneous equation above we get the general solution to the non-homogeneous equation as

$$\boxed{y = c_1e^{at}\cos(at) + c_2e^{at}\sin(at) + c_3e^{-at}\cos(at) + c_4e^{-at}\sin(at) + \frac{1}{2}t,}$$

where c_1, c_2, c_3, and c_4 are constants and $a = \frac{8^{1/4}}{2}$.

(d). The characteristic polynomial of the corresponding homogeneous equation is r^3+r^2-1. The roots of this equation are not so evident, but what is evident is that there is one real root call it r_1 and two complex roots which must be conjugate, call them $x \pm iy$. The general solution to the homogeneous equation is then

$$y_h = c_1e^{r_1t} + c_2e^{xt}\sin(yt) + c_3e^{xt}\cos(yt),$$

where c_1, c_2, and c_3 are constants.

We also see that te^{-2t} is not a solution to the homogeneous equation and hence we use the method of undetermined coefficients to find a solution of the form $y_1 = Ate^{-2t} + Be^{-2t}$. In this case we have $y_1' = -2Ate^{-2t} + (A-2B)e^{-2t}$, $y_1'' = 4Ate^{-2t} + (-4A+4B)e^{-2t}$, and $y_1''' = -8Ate^{-2t} + (12A-8B)e^{-2t}$. Hence

$$te^{-2t} = y_1^{(3)} + y_1'' - y_1$$

$$= (-8Ate^{-2t} + (12A-8B)e^{-2t}) + (4Ate^{-2t} + (-4A+4B)e^{-2t}) - (Ate^{-2t} + Be^{-2t}) = -5Ate^{-2t} + (8A-5B)e^{-2t},$$

and so $A = -\frac{1}{5}$ and $B = -\frac{8}{25}$, hence $y_1 = -\frac{1}{5}te^{-2t} - \frac{8}{25}e^{-2t}$.

Combining this with the above we obtain the general solution to the non-homogeneous equation as

$$\boxed{y = c_1e^{r_1t} + c_2e^{xt}\sin(yt) + c_3e^{xt}\cos(yt) - \frac{1}{5}te^{-2t} - \frac{8}{25}e^{-2t},}$$

where c_1, c_2, and c_3 are constants.

Solving the cubic equation to find r_1, x, and y is probably uninformative, however using a computer one can give the estimates $r_1 \sim .755$, $x \sim -.877$, and $y \sim .745$.

Problem 2. Consider the differential equation $(2-t)y''' + (2t-3)y'' - ty' + y = 0$ for $t < 2$. Notice that $y_1(t) = e^t$ is a solution to this differential equation. Find a solution y which satisfies the initial value conditions $y(0) = -1$, $y'(0) = 1$, and $y''(0) = 1$.

Solution 2. Suppose $y = v(t)e^t$, then $y' = (v' + v)e^t$, $y'' = (v'' + 2v' + v)e^t$, and $y''' = (v''' + 3v'' + 3v' + v)e^t$. Substituting this into the above equation we find

$$(2-t)(v''' + 3v'' + 3v' + v)e^t + (2t-3)(v'' + 2v' + v)e^t - t(v' + v)e^t + ve^t = 0.$$

Collecting the derivatives of v and dividing by e^t gives us

$$(2-t)v''' + (3-t)v'' = 0.$$

Thus we get the first order equation $(2-t)w' + (3-t)w = 0$ where $w = v''$. This equation is separable and so we may find a solution by separating and integrating (note $t > 2$ so that $\ln(t-2)$ is well defined):

$$\ln w = \int \frac{1}{w} dw = \int \frac{3-t}{t-2} dt = \int (-1 + \frac{1}{t-2}) dt = -t + \ln(t-2) + C,$$

hence $v'' = w = C_0 e^{-t}(t-2)$.

Integration by parts then gives

$$v' = C_0 \int e^{-t}(t-2)dt = C_0(-(t-2)e^{-t} + \int e^{-t}dt) = C_0((1-t)e^{-t} + C_1).$$

Another use of integration by parts gives

$$v = C_0(\int ((1-t)e^{-t} + C_1)dt) = C_0((t-1)e^{-t} - \int e^{-t}dt + C_1 t) = C_0(te^{-t} + C_1 t + C_2).$$

Therefore $y = ve^t = C_0(t + C_1 te^t + C_2 e^t)$. The conditions $y(0) = -1$, $y'(0) = 1$, and $y''(0) = 1$ give rise to the equations $-1 = C_0 C_2$, $1 = (1 + C_1 + C_2)$, and $1 = 2C_1 + C_2$. The second two equations tell us that $C_1 = 1$, and $C_2 = -1$, and then the first equation tells us that $C_0 = 1$ hence we find our particular solution as

$$\boxed{y = t + te^t - e^t.}$$

Problem 3. (a). Prove that the set of functions $\{\sin nt, \cos nt\}_{n\in\mathbb{N}}$ is linearlly independent, i.e. there is no linear combination with non-zero coefficients which equals 0.

(b). Determine whether or not the set $\{\sin t \sin \frac{3t}{2}, \sin t \cos \frac{3t}{2}, \sin \frac{t}{2}, \cos \frac{t}{2}\}$ is linearly dependent or independent. Note that each function is a solution to the differential equation $16y^{(4)} - 104y'' + 25y = 0$.

(c). Determine whether or not the set $\{\sin t \sin \frac{3t}{2}, \cos t \cos \frac{3t}{2}, \sin \frac{t}{2}, \cos \frac{t}{2}\}$ is linearly dependent. Note that each function is a solution to the differential equation $16y^{(4)} - 104y'' + 25y = 0$.

Solution 3. (a). To show that the above infinite set of vectors are linearly independent it is enough to show that the sets $\beta_k = \{\sin nt, \cos nt\}_{n=1}^{k}$ are linearly independent for each $k \in \mathbb{N}$. Notice that if we fix $k \in \mathbb{N}$ then β_k has $2k$ functions and moreover each function is a solution to the $2k$-order homogeneous linear differential equation $(\frac{d^2}{dx^2} + 1)(\frac{d^2}{dx^2} + 4)\cdots(\frac{d^2}{dx^2} + k^2)(y) = 0$ since the characteristic polynomial of this equation is $(r^2+1)(r^2+4)\cdots(r^2+k^2)$.

Thus β_k will be linearly independent if and only if the Wronskian of β_k is non-zero at all points. Calculating the Wronskian at $t = 0$ is fairly straightforward, we have

$$W(\beta_k)(0) = \begin{vmatrix} 0 & 0 & & 0 & 1 & 1 & & 1 \\ 1 & 2 & \cdots & k & 0 & 0 & \cdots & 0 \\ 0 & 0 & & 0 & -1 & -4 & & -k^2 \\ & & & & \vdots & & & \\ 0 & 0 & \cdots & 0 & (-1)^{k+1} & (-1)^{k+1}2^{2k-2} & \cdots & (-1)^{k+1}k^{2k-2} \\ (-1)^{k+1} & (-1)^{k+1}2^{2k-1} & & (-1)^{k+1}k^{2k-1} & 0 & 0 & & 0 \end{vmatrix}.$$

Since rearranging rows and multiplying a row by a non-zero constant does not change whether or not the determinant is zero it is enough to check that the following two determinants are non-zero:

$$\begin{vmatrix} 1 & 2 & \cdots & k \\ 1 & 8 & \cdots & k^3 \\ & & \vdots & \\ 1 & 2^{2k-1} & \cdots & k^{2k-1} \end{vmatrix}, \quad \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1 & 4 & \cdots & k^2 \\ & & \vdots & \\ 1 & 2^{2k-2} & \cdots & k^{2k-2} \end{vmatrix}.$$

Also note that the first matrix above can be column reduced to the second matrix so it is enough to check that the second determinant is non-zero. However the second matrix is a Vandermonde matrix which has determinant $\Pi_{1\le m<n\le k}(m^2 - n^2)$. Since $m^2 - n^2 \neq 0$, $\forall 1 \le m < n \le k$ the product is non-zero also hence the functions above are linearly independent.

Another way to show that the above functions are linearly independent is to show that they are actually orthogonal, this is the basis of Fourier series.

(b). Since the set contains 4 functions and each function is a solution to a $4th$ order homogeneous linear differential equation to check for linear independence we just need to compute the Wronskian at a single point. Note that

$$(\sin t \sin \frac{3t}{2})' = \frac{3}{2}\sin t \cos \frac{3t}{2} + \cos t \sin \frac{3t}{2},$$

$$(\sin t \sin \frac{3t}{2})'' = -\frac{13}{4}\sin t \sin \frac{3t}{2} + 3\cos t \cos \frac{3t}{2},$$

and

$$(\sin t \sin \frac{3t}{2})''' = (-\frac{13}{4} - \frac{9}{2})\cos t \sin \frac{3t}{2} + (\frac{39}{8} - 3)\sin t \cos \frac{3t}{2}.$$

Also

$$(\sin t\cos\frac{3t}{2})' = \cos t\cos\frac{3t}{2} - \frac{3}{2}\sin t\sin\frac{3t}{2},$$

$$(\sin t\cos\frac{3t}{2})'' = -\frac{13}{4}\sin t\cos\frac{3t}{2} - 3\cos t\sin\frac{3t}{2},$$

and

$$(\sin t\cos\frac{3t}{2})''' = (-\frac{13}{4} - \frac{9}{2})\cos t\cos\frac{3t}{2} + (\frac{39}{8} + 3)\sin t\sin\frac{3t}{2}$$

Hence the Wronskian of $\sin t\sin\frac{3t}{2}$, $\sin t\cos\frac{3t}{2}$, $\sin\frac{t}{2}$, and $\cos\frac{t}{2}$ evaluated at 0 is

$$\begin{vmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & \frac{1}{2} & 0 \\ 3 & 0 & 0 & -\frac{1}{4} \\ 0 & \frac{-31}{4} & -\frac{1}{8} & 0 \end{vmatrix} = \frac{45}{4} \neq 0.$$

Thus the functions are linearly independent.

(c). This is very similar to part (b). Because we have 4 solutions to a $4th$ order homogeneous linear differential equation it is enough to check the Wronskian. Notice that

$$(\cos t\cos\frac{3t}{2})' = -\sin t\cos\frac{3t}{2} - \frac{3}{2}\cos t\sin\frac{3t}{2},$$

$$(\cos t\cos\frac{3t}{2})'' = -\frac{13}{4}\cos t\cos\frac{3t}{2} + 3\sin t\sin\frac{3t}{2},$$

and

$$(\cos t\cos\frac{3t}{2})''' = (\frac{39}{8} + 3)\cos t\sin\frac{3t}{2} + (\frac{13}{4} + \frac{9}{2})\sin t\cos\frac{3t}{2}.$$

Hence the Wronskian of $\sin t\sin\frac{3t}{2}$, $\cos t\cos\frac{3t}{2}$, $\sin\frac{t}{2}$, and $\cos\frac{t}{2}$ evaluated at 0 is

$$\begin{vmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{2} & 0 \\ 3 & -\frac{13}{4} & 0 & -\frac{1}{4} \\ 0 & 0 & -\frac{1}{8} & 0 \end{vmatrix} = 0.$$

Thus the functions are linearly dependent.

In fact by the trigonometric sum identities we have $\cos t\cos\frac{3t}{2} + \sin t\sin\frac{3t}{2} = \cos\frac{t}{2}$

Problem 4. Let $a, b \in \mathbb{R}$ with $b > 0$ and consider the matrix $A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$. Given $n \in \mathbb{N}$, determine closed expressions (only depending on a, b, and n) for the coefficients of the A^n. For the specific case of $a = -1$ and $b = 6$ use this to give an explicit expression for $e^A = \Sigma_{n=0}^{\infty} \frac{1}{n!} A^n = I + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots$.

Hint: You may want to first consider the case when $a = b = 1$ to get a feeling for the situation.

Solution 4. Note that in the case $a = b = 1$ we may calculate the first few expressions of A^n as:

$$A^1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, A^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, A^3 = \begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}, A^4 = \begin{pmatrix} 5 & 3 \\ 3 & 2 \end{pmatrix}, A^5 = \begin{pmatrix} 8 & 5 \\ 5 & 3 \end{pmatrix}.$$

By this point we should see that there is a pattern emerging. Specifically, the Fibonacci numbers are appearing and so we expect that some sort of recurrence relation is going on. Because of the existence of $(1, 0)$ in the bottom row of A we see for $n \in \mathbb{N}$ that the bottom row of A^{n+1} is exactly equal to the top row of A^n. If we let $(a_0, b_0) = (1, 0)$ and (a_n, b_n) be the top row of A^n we then have that

$$A^n = \begin{pmatrix} a_n & b_n \\ a_{n-1} & b_{n-1} \end{pmatrix}, \forall n \in \mathbb{N}.$$

We therefore have

$$\begin{pmatrix} a_{n+1} & b_{n+1} \\ a_n & b_n \end{pmatrix} = A^{n+1} = AA^n = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_n & b_n \\ a_{n-1} & b_{n-1} \end{pmatrix} = \begin{pmatrix} aa_n + ba_{n-1} & ab_n + bb_{n-1} \\ a_n & b_n \end{pmatrix}.$$

Hence a_n and b_n both satisfy the same difference equation $x_{n+1} - ax_n - bx_{n-1} = 0$, moreover we have initial conditions $a_0 = 1, a_1 = a$, $b_0 = 0, b_1 = b$.

Consider a solution x_n to the difference equation above and let $y(t) = \Sigma_{n=0}^{\infty} \frac{x_n t^n}{n!}$ Note that by the ratio test the radius of convergence of y is ∞. Thus we may differentiage y term by term:

$$y'(t) = \Sigma_{n=0}^{\infty} \frac{x_n t^{n-1}}{(n-1)!} = \Sigma_{n=0}^{\infty} \frac{x_{n+1} t^n}{n!},$$

and

$$y''(t) = \Sigma_{n=0}^{\infty} \frac{x_n t^{n-2}}{(n-2)!} = \Sigma_{n=0}^{\infty} \frac{x_{n+2} t^n}{n!}.$$

Therefore we have that

$$y''(t) - ay'(t) - by(t) = \Sigma_{n=0}^{\infty} (\frac{x_{n+2} t^n}{n!} - a\frac{x_{n+1} t^n}{n!} - b\frac{x_n t^n}{n!})$$

$$= \Sigma_{n=0}^{\infty} \frac{t^n}{n!} (x_{n+2} - ax_{n+1} - bx_n) = 0,$$

i.e. y satisfies the second order homogeneous linear differential equation $y'' - ay - b = 0$.

The characteristic polynomial to this equation is $r^2 - ar - b$ which by the binomial formula has roots $\frac{a \pm \sqrt{a^2+4b}}{2}$. Since $b > 0$ we have that $a^2 + 4b > 0$ and hence this polynomial has two distinct real roots. Therefore any solution to the above differential equation must be of the form

$$y(t) = c_1 e^{\frac{a+\sqrt{a^2+4b}}{2} t} + c_2 e^{\frac{a-\sqrt{a^2+4b}}{2} t}.$$

Furthermore since e^x has the expansion $\Sigma_{n=0}^{\infty} \frac{x^n}{n!}$ we have that x_n which is the n^{th} coefficient of the power expansion of y must be

$$x_n = c_1 (\frac{a + \sqrt{a^2 + 4b}}{2})^n + c_2 (\frac{a - \sqrt{a^2 + 4b}}{2})^n.$$

When $x_n = a_n$ the initial conditions $y(0) = a_0 = 1$ and $y'(0) = a_1 = a$ give rise to the system of linear equations

$$c_1 + c_2 = 1,$$
$$c_1 \frac{a + \sqrt{a^2+4b}}{2} + c_2 \frac{a - \sqrt{a^2+4b}}{2} = a,$$

from which we deduce that $c_1 = \frac{1}{2}(1 + \frac{a\sqrt{a^2+4b}}{a^2+4b})$ and $c_2 = \frac{1}{2}(1 - \frac{a\sqrt{a^2+4b}}{a^2+4b})$.

When $x_n = b_n$ the initial conditions $y(0) = b_0 = 0$ and $y'(0) = b_1 = b$ give rise to the system of linear equations

$$c_1 + c_2 = 0,$$
$$c_1 \frac{a + \sqrt{a^2+4b}}{2} + c_2 \frac{a - \sqrt{a^2+4b}}{2} = b,$$

from which we deduce that $c_1 = \frac{b\sqrt{a^2+4b}}{a^2+4b}$ and $c_2 = -\frac{b\sqrt{a^2+4b}}{a^2+4b}$.

Therefore $A^n = \begin{pmatrix} a_n & b_n \\ a_{n-1} & b_{n-1} \end{pmatrix}$, where

$$\boxed{a_n = \frac{1}{2}(1 + \frac{a\sqrt{a^2+4b}}{a^2+4b})(\frac{a + \sqrt{a^2+4b}}{2})^n + \frac{1}{2}(1 - \frac{a\sqrt{a^2+4b}}{a^2+4b})(\frac{a - \sqrt{a^2+4b}}{2})^n,}$$

and

$$\boxed{b_n = \frac{b\sqrt{a^2+4b}}{a^2+4b}(\frac{a + \sqrt{a^2+4b}}{2})^n - \frac{b\sqrt{a^2+4b}}{a^2+4b}(\frac{a - \sqrt{a^2+4b}}{2})^n,}$$

for all $n \in \mathbb{N}$.

For the case when $a = -1$ and $b = 6$ the above formulas simplify to

$$a_n = \frac{2}{5}2^n + \frac{3}{5}(-3)^n$$

and

$$b_n = \frac{6}{5}2^n - \frac{6}{5}(-3)^n.$$

Therefore $\Sigma_{n=0}^{\infty} \frac{a_n}{n!} = \frac{2}{5}e^2 + \frac{3}{5}e^{-3}$ and $\Sigma_{n=1}^{\infty} \frac{a_{n-1}}{n!} = \frac{1}{5}(e^2 - 1) - \frac{1}{5}(e^{-3} - 1)$, also $\Sigma_{n=0}^{\infty} \frac{b_n}{n!} = \frac{6}{5}e^2 - \frac{6}{5}e^{-3}$ and $1 + \Sigma_{n=1}^{\infty} \frac{b_{n-1}}{n!} = 1 + \frac{3}{5}(e^2 - 1) + \frac{2}{5}(e^{-3} - 1)$. These are exactly the coefficients of $e^A = \Sigma_{n=0}^{\infty} \frac{1}{n!} A^n$ and so we conclude that

$$e^A = \begin{pmatrix} \frac{2}{5}e^2 + \frac{3}{5}e^{-3} & \frac{6}{5}e^2 - \frac{6}{5}e^{-3} \\ \frac{1}{5}(e^2 - 1) - \frac{1}{5}(e^{-3} - 1) & 1 + \frac{3}{5}(e^2 - 1) + \frac{2}{5}(e^{-3} - 1) \end{pmatrix}$$

$$\boxed{= \frac{e^2}{5}\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} + \frac{e^{-3}}{5}\begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}.}$$

Note that if we set $A_1 = \frac{1}{5}\begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix}$ and $A_2 = \frac{1}{5}\begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$ then we have that $A = 2A_1 - 3A_2$, $A_1A_1 = A_1$, $A_2A_2 = A_2$, and $A_1A_2 = A_2A_1 = 0$. From this it follows that if we take any function f which has a power expansion $f(t) = \Sigma_{n=0}^{\infty} c_n t^n$ with radius of convergence greater than 3, then $f(A) = \Sigma_{n=0}^{\infty} c_n A^n$ is well defined and actually equals $f(2)A_1 + f(-3)A_2$. What we did above was just the case $f(t) = e^t$. Also note that the eigenvalues of A are 2 and -3.

Second solution using linear algebra:

The matrix $A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$ has trace a and determinant $-b$, thus the characteristic polynomial is $\chi_A(\lambda) = \lambda^2 - a\lambda - b$ which has roots $\lambda = (a \pm \sqrt{a^2+4b})/2$.

Since $b > 0$ we know that A has two distinct real eigenvalues and hence we know that A can be diagonalized. To find the diagonalization we must first find eigenvectors associated to the eigenvalues.

For $\lambda = (a + \sqrt{a^2+4b})/2$, the eigenspace is the kernel of the matrix $\begin{pmatrix} \frac{a-\sqrt{a^2+4b}}{2} & b \\ 1 & \frac{-a-\sqrt{a^2+4b}}{2} \end{pmatrix}$. By performing Gaussian elimination we reduce this to row reduce echelon form as $\begin{pmatrix} 1 & \frac{-a-\sqrt{a^2+4b}}{2} \\ 0 & 0 \end{pmatrix}$, thus a basis vector for our eigenspace is given by $\begin{pmatrix} \frac{a+\sqrt{a^2+4b}}{2} \\ 1 \end{pmatrix}$.

Similarly, for $\lambda = (a - \sqrt{a^2+4b})/2$ we find a basis vector for our eigenspace given by $\begin{pmatrix} \frac{a-\sqrt{a^2+4b}}{2} \\ 1 \end{pmatrix}$.

Therefore if $P = \begin{pmatrix} \frac{a+\sqrt{a^2+4b}}{2} & \frac{a-\sqrt{a^2+4b}}{2} \\ 1 & 1 \end{pmatrix}$, and $D = \begin{pmatrix} \frac{a+\sqrt{a^2+4b}}{2} & 0 \\ 0 & \frac{a-\sqrt{a^2+4b}}{2} \end{pmatrix}$ then we have that $P^{-1} = \frac{\sqrt{a^2+4b}}{a^2+4b} \begin{pmatrix} 1 & \frac{-a+\sqrt{a^2+4b}}{2} \\ -1 & \frac{a+\sqrt{a^2+4b}}{2} \end{pmatrix}$, and $A = PDP^{-1}$.

In particular since D is a diagonal matrix we have $D^n = \begin{pmatrix} (\frac{a+\sqrt{a^2+4b}}{2})^n & 0 \\ 0 & (\frac{a-\sqrt{a^2+4b}}{2})^n \end{pmatrix}$ and so$A^n = (PDP^{-1})^n = PD^nP^{-1}$. Finding the entries of A^n now just amounts to the easy exercise of multiplying three matrices together.

For the case when $a = -1$ and $b = 6$ we have $A^n = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2^n & 0 \\ 0 & (-3)^n \end{pmatrix} \frac{1}{5} \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix}$, and in general for any analytic function f which is well defined at 2 and -3 we will have that $f(A) = \begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} f(2) & 0 \\ 0 & f(-3) \end{pmatrix} \frac{1}{5} \begin{pmatrix} 1 & 3 \\ -1 & 2 \end{pmatrix} = \frac{f(2)}{5} \begin{pmatrix} 2 & 6 \\ 1 & 3 \end{pmatrix} + \frac{f(-3)}{5} \begin{pmatrix} 3 & -6 \\ -1 & 2 \end{pmatrix}$. When $f(x) = e^x$ we obtain the answer above.

Problem 5. Fix $n \in \mathbb{N}$ and consider the differential equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$.

(a). Determine two linearly independent solutions in powers of x for $|x| < 1$.

(b). Show that there is a polynomial solution P_n such that $P_n(1) = 1$, calculate P_0, P_1, P_2, and P_3.

(c). Show that $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$, for all $n \in \mathbb{N}$.

(d). Calculate $\int_{-1}^{1} P_n(x)P_m(x)dx$, for all $n, m \in \mathbb{N}$.

(e). Show that $\{P_n\}_{n\in\mathbb{N}}$ forms a basis for the space of polynomials, i.e. show that they are linearly independent and that every polynomial is a linear combination of P_n's.

Solution 5. (a). We want to find solutions of the form $y = \Sigma_{k=0}^{\infty} a_k x^k$. If such a solution exists then we can differentiate to get $y' = \Sigma_{k=0}^{\infty} k a_k x^{k-1}$ and $y'' = \Sigma_{k=0}^{\infty} k(k-1)a_k x^{k-2} = \Sigma_{k=0}^{\infty}(k+2)(k+1)a_{k+2}x^k$.

Since y satisfies the differential equation we have that

$$0 = (1-x^2)\Sigma_{k=0}^{\infty} k(k-1)a_n x^{k-2} - 2x(\Sigma_{k=0}^{\infty} k a_k x^{k-1}) + n(n+1)(\Sigma_{k=0}^{\infty} a_k x^k)$$

$$= \Sigma_{k=0}^{\infty}((k+2)(k+1)a_{k+2} - k(k-1)a_k - 2ka_k + n(n+1)a_k)x^k.$$

Hence we must have that each of the coefficients above equals 0, i.e.

$$(k+2)(k+1)a_{k+2} + (-k(k+1) + n(n+1))a_k = 0, \ \forall k \geq 0.$$

Thus a_0 and a_1 are undetermined while

$$a_2 = -\frac{1}{2}n(n+1)a_0,$$

$$a_3 = \frac{1}{3\cdot 2}(2 - n(n+1))a_1,$$

$$a_4 = \frac{1}{4\cdot 3}(2\cdot 3 - n(n+1))a_2 = \frac{1}{4\cdot 3\cdot 2}(2\cdot 3 - n(n+1))(0 - n(n+1))a_0,$$

$$a_5 = \frac{1}{5\cdot 4}(3\cdot 4 - n(n+1))a_3 = \frac{1}{5\cdot 4\cdot 3\cdot 2}(3\cdot 4 - n(n+1))(2 - n(n+1))a_1.$$

In general we can see that we have

$$a_{2k} = a_0\frac{1}{(2k)!}\Pi_{j=1}^{k}((2j-2)(2j-1) - n(n+1)), \ k \geq 0$$

and

$$a_{2k+1} = a_1\frac{1}{(2k+1)!}\Pi_{j=1}^{k}((2j-1)(2j) - n(n+1)), \ k \geq 0.$$

Thus we get two solutions

$$\boxed{y_1 = a_0\Sigma_{k=0}^{\infty}\frac{1}{(2k)!}[\Pi_{j=1}^{k}((2j-2)(2j-1) - n(n+1))]x^{2k},}$$

and

$$\boxed{y_2 = a_1\Sigma_{k=0}^{\infty}\frac{1}{(2k+1)!}[\Pi_{j=1}^{k}((2j-1)(2j) - n(n+1))]x^{2k+1}.}$$

To check the radius of convergence for these solutions we use the ratio test. For y_1, if none of the coefficients are 0 then we have

$$\left(\frac{1}{(2k+2)!}[\Pi_{j=1}^{k+1}((2j-2)(2j-1)-n(n+1))]\right)/\left(\frac{1}{(2k)!}[\Pi_{j=1}^{k}((2j-2)(2j-1)-n(n+1))]\right)$$

$$= ((2k)(2k+1)-n(n+1))/(2k+2)(2k+1) \to_{k\to\infty} 1.$$

Thus the radius of convergence is $1/1 = 1$. Similarly, for y_2, if none of the coefficients are 0 then we have

$$\left(\frac{1}{(2k+3)!}[\Pi_{j=1}^{k+1}((2j-1)(2j)-n(n+1))]\right)/\left(\frac{1}{(2k+1)!}[\Pi_{j=1}^{k}((2j-1)(2j)-n(n+1))]\right)$$

$$= ((2k+2)(2k+1)-n(n+1))/(2k+3)(2k+2) \to_{k\to\infty} 1.$$

Thus the radius of convergence is $1/1 = 1$.

Note that in the above formulas if some coefficient is 0 then all coefficients following it will also be 0, and so in this case our solution is a polynomial and hence has infinite radius of convergence.

(b). If n is odd then there will be some j such that $n = 2j-1$ thus the j^{th} coefficient of y_2 will be 0 and so will all coefficients following it. Hence y_2 will be a polynomial of degree at most n. If n is even then there will be some j such that $n = 2j$ in which case the j^{th} coefficient of y_1 will be 0 and so will all coefficients following it. Hence in this case y_1 will be a polynomial of degree at most n. Thus there is always a polynomial solution P_n which we may normalize and assume that $P_n(1) = 1$.

From the above formulas, we can see that the first few polynomial solutions are given by

$$\boxed{P_0 = 1, \quad P_1 = x, \quad P_2 = \frac{-1}{2} + \frac{3}{2}x^2, \quad P_3 = \frac{-3}{2}x + \frac{5}{2}x^3.}$$

(c). From part (b) we know that there is only one polynomial solution up to normalization, thus we need to check that $\frac{d^n}{dx^n}(x^2-1)^n$ satisfies the differential equation, then we know that $\frac{d^n}{dx^n}(x^2-1)^n$ is a multiple of P_n and it is easy to check the normalization $\frac{1}{2^n n!}$

If we set $v(x) = (x^2-1)^n$ then $v'(x) = 2nx(x^2-1)^{n-1}$ and so we have $(x^2-1)v' - 2nxv = 0$.

If we differentiate this equation then we have

$$0 = ((x^2-1)v'' + 2xv') - 2n(xv' + v),$$

differentiating this again yields

$$0 = ((x^2-1)v^{(3)} + 4xv'' + 2v') - 2n(xv'' + 2v'),$$

differentiating a third time yields

$$0 = ((x^2-1)v^{(4)} + 6xv^{(3)} + 2(1+2)v'') - 2n(xv^{(3)} + 3v'').$$

After differentiating $n+1$ times we obtain

$$0 = ((x^2-1)v^{(n+2)} + 2(n+1)xv^{(n+1)} + n(n+1)v^{(n)}) - 2n(xv^{(n+1)} + (n+1)v^{(n)})$$

$$= (x^2-1)v^{(n+2)} + 2xv^{(n+1)} - n(n+1)v^{(n)}.$$

Thus $v^{(n)} = \frac{d^n}{dx^n}(x^2-1)^n$ satisfies the differential equation.

(d). Assume $n \leq m$, from part (c) we have

$$\int_{-1}^{1} P_n P_m dx = \frac{1}{2^{n+m}n!m!}\int_{-1}^{1} (\frac{d^n}{dx^n}(x^2-1)^n)(\frac{d^m}{dx^m}(x^2-1)^m)dx.$$

Notice that if $k \geq 0$ then we have $\frac{d^{m-k-1}}{dx^{m-k-1}}(x^2-1)^m$ is a multiple of (x^2-1), and hence when we plug in ± 1 we get 0. With this in mind we may integrate by parts n times to obtain

$$\int_{-1}^{1} P_n P_m dx = \frac{1}{2^{n+m}n!m!}\int_{-1}^{1} (\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n)(x^2-1)^m dx.$$

If $n < m$ then $\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n = 0$ and hence the above integral is 0. If $n = m$ then $\frac{d^{m+n}}{dx^{m+n}}(x^2-1)^n = (2n)!$ and so

$$\int_{-1}^{1} P_n^2 dx = \frac{(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1} (x^2-1)^n dx.$$

This integral is a type of problem that one would conquer in a combinatorics class. Using the binomial theorem it is possible to show that the above integral gives

$$\int_{-1}^{1} P_n^2 dx = 2/(2n+1).$$

(e). Let V_n be the vector space of all polynomials of degree at most n. Then a standard basis for V_n is given by $\{1, x, x^2, \ldots, x^n\}$ and in particular we see that V_n is $n+1$ dimensional. $\{P_0, P_1, \ldots, P_n\}$ is a set of $n+1$ functions in V_n and hence this set is a basis for V_n if and only if it is linearly independent. If we can show this for all $n \geq 0$ then we have the result.

Let $a_0, a_1, \ldots, a_n$ be real numbers and suppose that $\Sigma_{k=0}^n a_k P_k = 0$. Then for $0 \leq j \leq n$ we have

$$\begin{aligned} 0 &= \int_{-1}^{1} P_j(\Sigma_{k=0}^n a_k P_k)dx \\ &= \Sigma_{k=0}^n a_k(\int_{-1}^{1} P_j P_k dx) \\ &= a_j \int_{-1}^{1} P_j^2 dx = a_j 2/(2n+1), \end{aligned}$$

where the last two equalities follow from part (d).

Therefore $a_j = 0$, for $0 \leq j \leq n$ which shows that $\{P_0, P_1, \ldots, P_n\}$ is linearly independent and hence is a basis.