Math 208 - Exam 1, February 10, 2009 Name:________Key________

Problem 1 (15 points). Find an explicit solution to the differential equation: $y' - (\tan t)y = \frac{1}{\cos t}$ for $\frac{-\pi}{2} < t < \frac{\pi}{2}$, subject to the initial condition $y(0) = -1$.

Hint: It may be useful to remember the formula: $(\ln(\cos t))' = -\tan t$, for $\frac{-\pi}{2} < t < \frac{\pi}{2}$.

Solution 1. This is a linear equation and hence we look for an integrating factor μ which satisfies $\mu' = -(\tan t)\mu$. Since $\mu' = -(\tan t)\mu$ we must have that $\ln \mu = \int \frac{1}{\mu} d\mu = -\int (\tan t) dt = \ln(\cos t)$ and hence $\mu = \cos t$.

Multiplying by μ the above equation becomes $\frac{d}{dt}(y \cos t) = (\cos t)y' - (\sin t)y = 1$. Integrating gives us $y \cos t = t + C$. Where C is a constant.

The initial condition gives $-1 = y(0) = C$ and hence for $\frac{-\pi}{2} < t < \frac{\pi}{2}$ we have

$$y = \frac{t-1}{\cos t}.$$

Problem 2 (15 points). Suppose that a sum S is invested at the annual rate of return 8% compounded continuously.

(a). Find the time T required for the initial amount to double.

(b). How much should we invest today if we wanted to have the sum of $40,000 in 20 years?

Solution 2. S will satisfy the differential equation $\frac{dS}{dt} = rS$ where $r = .08$ is the rate of return. Solving this gives $S = S(0)e^{rt}$ where $S(0)$ is the initial amount invested.

(a). From above we have $2S(0) = S(T) = S(0)e^{rT}$, hence

$$T = \frac{1}{r}\ln 2 = \frac{1}{.08}\ln 2 \ \text{ years.}$$

(b). If $40,000 = S(20) = S(0)e^{20r}$ then

$$S(0) = 40,000e^{-20r} = \$40,000e^{-1.6}.$$

Problem 3 (20 points). A tank initially contains 100 gallons of water with 50 pounds of salt. Pure water is pumped into the tank at the rate of 2 gallons per minute. The mixture is then pumped at the same rate into a second tank which initially contains 100 gallons of pure water. The mixture is then allowed to flow out of the second tank again at the rate of 2 gallons per minute.

(a). Find an expression for the amount of salt in the first tank at time t.

(b). Find the time $t_{\max}$ when the second tank contains the most amount of salt. How much salt does the second tank contain at this time?

Solution 3. In general the change in the amount of salt in a tank will be the rate at which salt is added minus the rate at which salt is subtracted. Thus if $X(t)$ is the amount of salt in the first tank at time t and $Y(t)$ is the amount of salt in the second tank at time t. Then X and Y will satisfy the following differential equations:

$$\frac{dX}{dt} = -\frac{2X}{100},$$
$$\frac{dY}{dt} = \frac{2X}{100} - \frac{2Y}{100}.$$

(a). Solving the above differential equation for X we get $X = C_0 e^{-\frac{t}{50}}$ where C_0 is a constant. The initial condition $X(0) = 50$ then gives us

$$X(t) = 50e^{-\frac{t}{50}}.$$

(b). Substituting the expression for X into the differential equation for Y above gives $\frac{dY}{dt} = e^{-\frac{t}{50}} - \frac{Y}{50}$. Solving this linear differential equation gives $Y = te^{-\frac{t}{50}} + C_1 e^{-\frac{t}{50}}$, where C_1 is a constant. The initial condition $Y(0) = 0$ gives $C_1 = 0$ and so

$$Y = te^{-\frac{t}{50}}.$$

In order to find the maximum of this function we take the derivative to get

$$\frac{dY}{dt} = e^{-\frac{t}{50}} - \frac{te^{-\frac{t}{50}}}{50}.$$

Setting this equal to 0 gives $1 - \frac{t_{\max}}{50} = 0$ and hence we have

$$t_{\max} = 50 \quad \text{minutes},$$

$$Y(t_{\max}) = 50e^{-1} \quad \text{pounds}.$$

Problem 4 (15 points). Find an explicit solution for $x > \frac{1}{2}\sqrt{2}$ to the differential equation: $y' = \frac{4x^2+3y^2}{2xy}$, subject to the initial condition $y(1) = -2$.

Solution 4. This differential equation can be viewed as a homogeneous equation, a Bernoulli equation, or an exact equation depending on our perspective.

Method 1: (homogeneous)

$y' = \frac{4x^2+3y^2}{2xy} = 2\frac{x}{y} + \frac{3}{2}\frac{y}{x}$, hence if we make the substitution $v = y/x$ then $\frac{dy}{dx} = v + x\frac{dv}{dx}$ and hence we get the new differential equation

$$v + x\frac{dv}{dx} = \frac{2}{v} + \frac{3}{2}v.$$

Simplifying gives us $\frac{dv}{dx} = (\frac{4+v^2}{2v})/x$. This is a separable equation and integrating gives us

$$\ln(4 + v^2) = \int \frac{2v}{4 + v^2} dv = \int \frac{1}{x} dx = \ln x + C.$$

Therefore $v = \pm\sqrt{C_0 x - 4}$ and hence $y = xv = \pm x\sqrt{C_0 x - 4}$ for some constant C_0. The initial condition $y(1) = -2$ tells us that $C_0 = 8$ and we are taking the negative branch of the square root, hence

$$y(x) = -2x\sqrt{2x - 1}.$$

Method 2: (Bernoulli)

$y' = \frac{4x^2+3y^2}{2xy} = \frac{3}{2x}y + 2xy^{-1}$, hence if we make the substitution $v = y^2$ then $\frac{dy}{dx} = \frac{1}{2\sqrt{v}}\frac{dv}{dx}$ and hence we get the new differential equation

$$\frac{1}{2\sqrt{v}}\frac{dv}{dx} = \frac{3}{2x}\sqrt{v} + \frac{2x}{\sqrt{v}}.$$

Multiplying both sides of the equation by $2\sqrt{v}$ gives $\frac{dv}{dx} = \frac{3}{x}v + 4x$. This is a linear equation and hence we multiply by an integrating factor μ which satisfies the differential equation $\mu' = -\frac{3}{x}\mu$, i.e. $\mu = \frac{1}{x^3}$. We then have

$$\frac{d}{dx}(\frac{v}{x^3}) = \frac{1}{x^3}\frac{dv}{dx} - \frac{3}{x^4}v = \frac{4}{x^2},$$

and so integrating gives us $v = x^3(-\frac{4}{x} + C)$, i.e.

$$y = \pm\sqrt{x^3 C - 4x^2}.$$

The initial condition $y(1) = -2$ tells us that $C = 8$ and we are taking the negative branch of the square root, hence

$$y(x) = -2x\sqrt{2x - 1}.$$

Method 3: (Exact)

Rewriting the above differential equation gives us $M + N\frac{dy}{dx} = (-4x^2 - 3y^2) + 2xy\frac{dy}{dx} = 0$. This is not an exact equation since $M_y = -6y \neq 2y = N_x$. However since $\frac{-4}{x} = (M_y - N_x)/N$ does not depend on y we know that we may multiply by an integrating factor $\mu(x)$ which satisfies $\mu' = \frac{-4}{x}\mu$ to make the equation become exact.

$\mu = \frac{1}{x^4}$ gives such a function and so now we have the exact equation $(-\frac{4}{x^2} - \frac{3y^2}{x^4}) + \frac{2y}{x^3}\frac{dy}{dx} = 0$. Thus we know that there exists some function $F(x, y)$ such that

$$F_x = -\frac{4}{x^2} - \frac{3y^2}{x^4},$$

$$F_y = \frac{2y}{x^3}.$$

Integrating F_x gives $F = \frac{4}{x} + \frac{y^2}{x^3} + g(y)$. Differentiating this with respect to y gives $\frac{2y}{x^3} = F_y = \frac{2y}{x^3} + g'(y)$, and hence $g = 0$ so we get the implicit solution $\frac{4}{x} + \frac{y^2}{x^3} = C$. Solving for y gives $y = \pm\sqrt{Cx^3 - 4x^2}$. Again, the initial condition $y(1) = -2$ tells us that $C = 8$ and we are taking the negative branch of the square root, hence

$$y(x) = -2x\sqrt{2x - 1}.$$

Problem 5 (20 points). Find an implicit solution to the differential equation: $y' = -\frac{2y+3xy^2+1}{x+2x^2y}$.

Solution 5. Rewriting the above equation we have $M + N\frac{dy}{dx} = (2y + 3xy^2 + 1) + (x + 2x^2y)\frac{dy}{dx} = 0$. This is not exact since $M_y = 2 + 6xy \neq 1 + 4xy = N_x$. However since $\frac{1}{x} = \frac{1+2xy}{x+2x^2y} = (M_y - N_x)/N$ does not depend on y we know that we may multiply by an integrating factor $\mu(x)$ which satisfies $\mu' = \frac{1}{x}\mu$ to make the equation become exact.

$\mu = x$ gives such a function and so we now have the exact equation $(2yx+3x^2y^2+x)+(x^2+2x^3y)\frac{dy}{dx} = 0$. Thus we know that there exists some function $F(x, y)$ such that

$$F_x = 2yx + 3x^2y^2 + x,$$

$$F_y = x^2 + 2x^3y.$$

Integrating F_x gives $F = yx^2+x^3y^2+\frac{x^2}{2}+g(y)$. Differentiating this with respect to y gives $x^2+2x^3y = F_y = x^2 + 2x^3y + g'(y)$, and hence $g = 0$ so we get the implicit solution

$$yx^2 + x^3y^2 + \frac{x^2}{2} = C.$$

Problem 6 (15 points). Find an explicit solution on $\mathbb{R}$ to the differential equation: $y'' + y' - 6y = 0$, subject to the initial conditions $y(0) = 1$, $y'(0) = -1$.

Solution 6. This is a homogeneous linear equation with constant coefficients. The characteristic polynomial to this equation is $r^2 + r - 6$ which has two distinct real roots 2 and -3. Hence a solution to the above differential equation must be of the form:

$$C_1 e^{2t} + C_2 e^{-3t},$$

where C_1 and C_2 and constants.

The initial conditions give us $1 = y(0) = C_1 + C_2$ and $-1 = y'(0) = 2C_1 - 3C_2$. Solving this system of linear equations gives $C_1 = \frac{2}{5}$ and $C_2 = \frac{3}{5}$. Hence

$$y = \frac{2}{5}e^{2t} + \frac{3}{5}e^{-3t}.$$

Problem 7 (Extra Credit - 10 points). Find an explicit solution to the second order differential equation: $y'' = e^y y'$ for $t < 1$, subject to the initial conditions $y(0) = 0$, $y'(0) = 1$.

Hint: Use the substitution $v = \frac{dy}{dt}$ and solve for v in terms of y.

Solution 7. Using the substitution $v = \frac{dy}{dt}$ gives $\frac{d^2y}{dt^2} = \frac{dv}{dt} = \frac{dv}{dy}\frac{dy}{dt} = v\frac{dv}{dy}$. Hence the above differential equation becomes

$$v\frac{dv}{dy} = e^y v.$$

This is a separable equation and so we can solve for v as

$$\frac{dy}{dt} = v = e^y + C_1.$$

When $t = 0$ we have that $y = 0$ and $\frac{dy}{dt} = 1$, hence we have $1 = 1 + C_1$ and so $C_1 = 0$.

Thus $\frac{dy}{dt} = e^y$ and solving this differential equation gives $-e^{-y} = \int e^{-y} dy = \int dt = t + C_2$, i.e. $y = -\ln(-C_2 - t)$.

The initial condition $y(0) = 0$ gives $0 = -\ln(-C_2)$ hence $C_2 = -1$ and so we get our solution:

$$y = -\ln(1 - t).$$