Math 208 - Exam 2, Take home exam due 11:00am March 17, 2009

Regulations:

1. You may either turn the exam in at the beginning of class on March 17th, or if you are unable to attend class that day then you may email me your exam in a pdf file or give your exam to the math department secretary. In any case the exam is due by **11:00am, March 17**, no exceptions.

2. Each problem should begin on a separate page and should have your name written at the top of the page. The problem should be clearly stated before you give your solution. Your exam should be stapled together, no loose pages, no paper clips.

3. Please use standard 8.5x11 inch blank paper. **Do not use lined paper**. Write in black ink or pencil only and leave at least a 1 inch margin on all sides. You may type your exam if you prefer, in this case I recommend using the program LaTeX, I will put the LaTeX version of this exam on the course website. You can find information about LaTeX on the internet.

4. To receive full credit your solutions should be expressed clearly and make sense grammatically. If you use a particular theorem or method then this should be stated. You should say "using the method of undetermined coefficients we have ..." or "by the Fundamental Theorem of Calculus ...", etc. See the solution to Exam 1 for an example of how this should be done. Your exam should look professional and I will take off points for sloppiness.

5. If the final solution to a problem is a formula or quantity then your final answer should be clearly visible. Please express your answer in a simple closed form. You will not need to use a calculator, for example an answer of the form $\sqrt{2\pi}$ is better than an answer of the form 2.506628...

6. You are welcome to use whichever resource you like so long as it does not involve interaction with another person. For example you are welcome to use the book or to look something up on the internet, but you are not allowed to ask a friend for help. **No group work**. Note that even a fairly innocent comment such as "number 3 was easy" or "I was stuck for a long time on number 5" contains a wealth of information. Please refrain from making such comments. If you do use a source other than your notes then you should indicate so by adding a footnote or a list of references.

7. Justify all of your work. Each step in your solution should follow logically from the previous step. If the methods used in your solution are incorrect then the solution is wrong, even if the final answer is correct.

8. If you have a question about a homework problem or something we went over in class then you are welcome to ask me about it. I will not however help you with an exam problem. Note that I will be not be around after Friday and after that I will check my email intermittently at best. If you have questions they should be brought to my attention this week in office hours and no later than Friday.

Problems.

Problem 1. Find the general solution on $\mathbb{R}$ to each of the following differential equations:

(a). $y^{(4)} + 2y'' + y = 0$.

(b). $y^{(4)} - 8y'' + 16y = \sin t + e^{2t}$.

(c). $y^{(4)} + 2y = t$

(d). $y^{(3)} + y'' - y = te^{-2t}$

Problem 2. Consider the differential equation $(2-t)y''' + (2t-3)y'' - ty' + y = 0$ for $t < 2$. Notice that $y_1(t) = e^t$ is a solution to this differential equation. Find a solution y which satisfies the initial value conditions $y(0) = -1$, $y'(0) = 1$, and $y''(0) = 1$.

Problem 3. (a). Prove that the set of functions $\{\sin nt, \cos nt\}_{n\in\mathbb{N}}$ is linearlly independent, i.e. there is no linear combination with non-zero coefficients which equals 0.

(b). Determine whether or not the set $\{\sin t \sin \frac{3t}{2}, \sin t \cos \frac{3t}{2}, \sin \frac{t}{2}, \cos \frac{t}{2}\}$ is linearly dependent or independent. Note that each function is a solution to the differential equation $16y^{(4)} - 104y'' + 25y = 0$.

(c). Determine whether or not the set $\{\sin t \sin \frac{3t}{2}, \cos t \cos \frac{3t}{2}, \sin \frac{t}{2}, \cos \frac{t}{2}\}$ is linearly dependent. Note that each function is a solution to the differential equation $16y^{(4)} - 104y'' + 25y = 0$.

Problem 4. Let $a, b \in \mathbb{R}$ with $b > 0$ and consider the matrix $A = \begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix}$. Given $n \in \mathbb{N}$, determine closed expressions (only depending on a, b, and n) for the coefficients of the matrix A^n. For the specific case of $a = -1$ and $b = 6$ use this to give an explicit expression for $e^A = \Sigma_{n=0}^{\infty} \frac{1}{n!} A^n = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + A + \frac{1}{2}A^2 + \frac{1}{6}A^3 + \cdots$.

Hint: You may want to first consider the case when $a = b = 1$ to get a feeling for the situation.

Problem 5. Fix $n \in \mathbb{N}$ and consider the differential equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$.

(a). Determine two linearly independent solutions in powers of x for $|x| < 1$.

(b). Show that there is a polynomial solution P_n such that $P_n(1) = 1$, calculate P_0, P_1, P_2, and P_3.

(c). Show that $P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$, for all $n \in \mathbb{N}$.

(d). Calculate $\int_{-1}^{1} P_n(x) P_m(x) dx$, for all $n, m \in \mathbb{N}$.

(e). Show that $\{P_n\}_{n\in\mathbb{N}}$ forms a basis for the space of polynomials, i.e. show that they are linearly independent and that every polynomial is a linear combination of P_n's.