HOMEWORK 2 SOLUTIONS, MATH 175 - FALL 2010

This homework assignment covers Sections 13.5 - 13.6 in the book.

1. Find an equation for a plane which contains the line given by $(x, y, z) = t(1, 2, 0)$ and which forms an angle of $\pi/3$ with the xy-plane.

We need to first find a normal vector $\mathbf{n} = (x_0, y_0, z_0)$ for this plane. Note that since any vector parallel to a normal vector is again a normal vector, we may assume that $\mathbf{n}$ has length 1. We know that $\mathbf{n}$ must form an angle of $\pi/3$ with $\mathbf{k}$, the normal vector of the xy-plane, this gives rise to the equation $z_0 = \mathbf{n} \cdot \mathbf{k} = |\mathbf{n}||\mathbf{k}| \cos(\pi/3) = 1/2$.

Since the vector $(1, 2, 0)$ is parallel to the above given line, which is contained in the plane, we also know that $\mathbf{n}$ must be perpendicular to $(1, 2, 0)$. This give rise to the equation $x_0 + 2y_0 = (1, 2, 0) \cdot \mathbf{n} = \mathbf{0}$.

Hence $\mathbf{n}$ is of the form $(-2y_0, y_0, 1/2)$, and since this must have length 1, we have $4y_0^2 + y_0^2 + (1/4) = 1$, hence $y_0 = \pm\sqrt{3/20}$. Note that there are two possibilities for y_0 since there are two planes which satisfy the above conditions.

Since the plane contains the origin $\mathbf{0}$, after simplifying, we may find an equation for the plane given by

$$\mathbf{r} \cdot (-2\sqrt{3}, \sqrt{3}, \sqrt{5}) = 0, \qquad \text{or} \qquad \mathbf{r} \cdot (2\sqrt{3}, -\sqrt{3}, \sqrt{5}) = 0.$$

We may also rewrite the above equations as

$$\boxed{-2\sqrt{3}x + \sqrt{3}y + \sqrt{5}z = 0, \qquad \text{or} \qquad 2\sqrt{3}x - \sqrt{3}y + \sqrt{5}z = 0.}$$

2. Find parametric equations for the line of intersection of the planes $x + y - z = 1$ and $3x + 2y - z = 0$. Also find the angle between these two planes.

To find a point on this line we can for instance set $z = 0$ and then use the above equations to solve for x and y. In this case we get $x = -2$ and $y = 3$ so $(-2, 3, 0)$ is a point on the line. Also the direction of the line lives in both planes and so in particular is perpendicular to both normal vectors, therefore a vector which is parallel to the line is given by $(1, 1, -1) \times (3, 2, -1) = (1, -2, -1)$.

Thus an equation of the line is given by the vector equation

$$(x, y, z) = (-2, 3, 0) + t(1, -2, -1),$$

or the parametric equations

$$\boxed{x = -2 + t, \quad y = 3 - 2t, \quad z = -t,}$$

or the symmetric equations

$$x + 2 = \frac{y - 3}{-2} = -z.$$

Then angle between these planes is given by $\cos\theta = |(1, 1, -1)|^{-1}|(3, 2, -1)|^{-1}(1, 1, -1) \cdot (3, 2, -1) = 6/\sqrt{42}$, and so $\theta = \cos^{-1}(6/\sqrt{42})$.

3. Find the distance between the parallel planes $2x - 2y - 6z = 1$ and $x - y - 3z = -2$.

To find the distance between the planes we may take a point on the first plane (how about $(0,0,\frac{-1}{6})$ and find the distance from this point to the second plane. We can do this by the formula we derived in class so that the distance is

$$\frac{|0+0+3/6+2|}{\sqrt{1+1+9}} = \boxed{\frac{5}{2\sqrt{11}}}.$$

4. Find the cross-sections of the surface $2x^2+2y^2+z^2=1$ in the planes $x=k$, $y=k$ and $z=k$. Sketch the surface.

When x is a constant k the cross-section is given by $2y^2+z^2=1-2k^2$ which is the equation of an ellipse. Similarly when y, or z is constant we get another ellipse. And so our equation describes an ellipsoid which I leave to you to sketch.

5. Find the cross-sections of the surface $-x^2-2y^2+3z^2=1$ in the planes $x=k$, $y=k$ and $z=k$. Sketch the surface.

When x is a constant k the cross-section is given by $-2y^2+3z^2=1+k^2$ which is a hyperbola, note that $k^2>0$ and so for no value of k will we obtain the asymptotes of this hyperbola. Similarly when y is a constant we also get a hyperbola. When z is a constant k we get $-x^2+2y^2=1-3k^2$ which describes an ellipse.

Thus our surface is a hyperboloid in the direction of the z-axis, which has two sheets. I leave it to you to sketch this.

6. Show that the curve of intersection of the surfaces $x^2+2y^2-z^2+3x=1$ and $2x^2+4y^2-2z^2-5y=0$ lies in a plane.

If we take the second equation, and subtract from it twice the first equation we obtain

$$-5y-6x=-2.$$

This we recognize as the equation of a plane, and since any point on the intersection of the above surfaces must satisfy this equation also, we have shown that the intersection of the surfaces lies on this plane.

7. A surface consists of all the points P such that the distance from P to the plane $z=1$ is the same as the distance from P to the point $(0,1,-2)$. Find an equation for, and sketch this surface.

Given a point $P=(x,y,z)$, the distance from P to the plane $z=1$ is $|1-z|$. Also, the distance from P to the point $(0,1,-2)$ is $\sqrt{x^2+(y-1)^2+(z+2)^2}$. If these distances are equal then we have the equation

$$(1-z)^2 = x^2+(y-1)^2+(z+2)^2.$$

Simplifying this equation gives

$$x^2+(y-1)^2+(6z+3)=0,$$

which we recognize as the equation of an elliptic paraboloid. I leave this to you to sketch.