HOMEWORK 1 SOLUTIONS, MATH 175 - FALL 2010

This homework assignment covers Sections 13.1 - 13.4 in the book.

1. Find an equation of the sphere which intersects the origin and whose center is $(1, -1, 3)$.

Since the center of the sphere is $(1, -1, 3)$ and the sphere intersects the origin $(0, 0, 0)$ we know that the radius of the sphere must be $|(1, -1, 3) - (0, 0, 0)| = \sqrt{1^1 + (-1)^2 + 3^2} = \sqrt{11}$. Hence from the general formula for a sphere we know that the equation must be:

$$\boxed{(x-1)^2 + (y+1)^2 + (z-3)^2 = 11.}$$

2. Find an equation describing all the points which are equidistant from the points $(1, 1, 1)$ and $(-1, -1, -1)$, describe this set.

Recall from class that the square distance from a point $P = (a_1, a_2, a_3)$ to a point $Q = (b_1, b_2, b_3)$ is $|PQ|^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2$. Thus we are looking for all points (x, y, z) such that

$$(x-1)^2 + (y-1)^2 + (z-1)^2 = (x+1)^2 + (y+1)^2 + (z+1)^2.$$

Expanding both sides of the above equation then gives

$$x^2 - 2x + 1 + y^2 - 2y + 1 + z^2 + 2z + 1 = x^2 - 2x + 1 + y^2 - 2y + 1 + z^2 + 2z + 1,$$

and simplifying gives

$$-2x - 2y - 2z = 2x + 2y + 2z$$

or

$$\boxed{x + y + z = 0,}$$

which describes a plane going through the origin.

3. Find the unit vectors that are parallel to the tangent line to the parabola $y = x^2$ at the point $(2, 4)$.

At the point $(2, 4)$ we know from single variable calculus that the slope of the tangent line is $y'(2) = 4$, hence any vector with slope 4 in $\mathbb{R}^2$ will be parallel to the tangent line at this point., e.g $(1, 4)$. Of course we are looking for unit vectors and so we divide by the length and get $\frac{1}{\sqrt{1+4^2}}(1, 4)$. Given any non-zero vector there are only two unit vectors which are parallel and so we obtain the other parallel vector by multiplying by -1. So we have only

$$\boxed{(\frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}}) \text{ and } (\frac{-1}{\sqrt{17}}, \frac{-4}{\sqrt{17}}).}$$

4. Find the orthogonal projection of the vector $\mathbf{v} = (2, -1, 3)$ in the direction of the vector $\mathbf{w} = (2, 1, 1)$.

We have a nice formula for the projection of one vector $\mathbf{v}$ in the direction of another vector $\mathbf{w}$, it's given by $(\mathbf{v} \cdot \frac{\mathbf{w}}{|\mathbf{w}|})\frac{\mathbf{w}}{|\mathbf{w}|}$.

In this case we have $|\mathbf{w}| = \sqrt{2^2+1^2+1^2} = \sqrt{6}$ and so $\frac{\mathbf{w}}{|\mathbf{w}|} = \frac{1}{\sqrt{6}}(2,1,1)$. Hence the projection of $\mathbf{v}$ in the direction of $\mathbf{w}$ is given by

$$((2,-1,3)\cdot\frac{1}{6}(2,1,1))\frac{1}{\sqrt{6}}(2,1,1) = \frac{(4-1+3)}{\sqrt{6}}\frac{1}{\sqrt{6}}(2,1,1) = \boxed{(2,1,1).}$$

5. If $\mathbf{r} = (x,y,z)$, $\mathbf{a} = (2,1,-1)$, and $\mathbf{b} = (1,1,0)$ then show that the equation $(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b}) = 0$ describes a sphere and find it's center and radius.

Expanding out the equation gives us:

$$0 = (\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b}) = (x-2,y-1,z+1)\cdot(x-1,y-1,z)$$
$$= (x-2)(x-1)+(y-1)(y-1)+(z+1)z = x^2-3x+2+y^2-2y+1+z^2+z.$$

Completing each square then gives us

$$0 = (x-\frac{3}{2})^2 - \frac{9}{4} + 2 + (y-1)^2 - 1 + 1 + (z+\frac{1}{2})^2 - \frac{1}{4}$$
$$= \boxed{(x-\frac{3}{2})^2 + (y-1)^2 + (z-\frac{-1}{2})^2 - \frac{1}{2}.}$$

This then is the usual form of an equation describing a sphere with center $(\frac{3}{2}, 1, \frac{-1}{2})$ and radius $\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$.

6. Find all vectors $\mathbf{v} = (v_1,v_2,v_3)$ such that $\mathbf{i}\times(\mathbf{j}\times\mathbf{v}) = (\mathbf{i}\times\mathbf{j})\times\mathbf{v}$.

Computing the left hand side of the equation gives

$$\mathbf{i}\times(\mathbf{j}\times\mathbf{v}) = \mathbf{i}\times(v_3 i - v_1 k) = v_1 j.$$

The right hand side of the equation gives

$$(\mathbf{i}\times\mathbf{j})\times\mathbf{v} = k\times\mathbf{v} = -v_2 i + v_1 j.$$

If these are equal then we have $(0,v_1,0) = v_1 j = -v_2 i + v_1 j = (-v_2, v_1, 0)$ and so the coordinate must also be equal, i.e. $0 = -v_2$, $v_1 = v_1$, and $0 = 0$. Thus $v_2 = 0$ while v_1 and v_3 can be any real numbers, hence our solution is:

$$\boxed{\mathbf{v}\in\{(t,0,s) \mid t,s\in\mathbb{R}\}.}$$

7. Find all vectors $\mathbf{v} = (v_1,v_2,v_3)$ such that $\mathbf{i}\cdot(\mathbf{j}\times\mathbf{v}) = (\mathbf{i}\times\mathbf{j})\cdot\mathbf{v}$.

We know from class that there is a general formula $a\cdot(b\times c) = (a\times b)\cdot c$ and hence the above equation will hold for all vectors in $\boxed{\mathbb{R}^3}$.