This homework assignment covers Sections 13.1 - 13.4 in the book.

1. Find an equation of the sphere which intersects the origin and whose center is \((1, -1, 3)\).

   Since the center of the sphere is \((1, -1, 3)\) and the sphere intersects the origin \((0, 0, 0)\) we know that the radius of the sphere must be \(||(1, -1, 3) - (0, 0, 0)|| = \sqrt{1^2 + (-1)^2 + 3^2} = \sqrt{11}||. Hence from the general formula for a sphere we know that the equation must be:

   \[ (x - 1)^2 + (y + 1)^2 + (z - 3)^2 = 11. \]

2. Find an equation describing all the points which are equidistant from the points \((1, 1, 1)\) and \((-1, -1, -1)\), describe this set.

   Recall from class that the square distance from a point \(P = (a_1, a_2, a_3)\) to a point \(Q = (b_1, b_2, b_3)\) is \(|PQ|^2 = (b_1 - a_1)^2 + (b_2 - a_2)^2 + (b_3 - a_3)^2\). Thus we are looking for all points \((x, y, z)\) such that

   \[ (x - 1)^2 + (y - 1)^2 + (z - 1)^2 = (x + 1)^2 + (y + 1)^2 + (z + 1)^2. \]

   Expanding both sides of the above equation then gives

   \[ x^2 - 2x + 1 + y^2 - 2y + 1 + z^2 + 2z + 1 = x^2 - 2x + 1 + y^2 - 2y + 1 + z^2 + 2z + 1, \]

   and simplifying gives

   \[ -2x - 2y - 2z = 2x + 2y + 2z \]

   or

   \[ x + y + z = 0, \]

   which describes a plane going through the origin.

3. Find the unit vectors that are parallel to the tangent line to the parabola \(y = x^2\) at the point \((2, 4)\).

   At the point \((2, 4)\) we know from single variable calculus that the slope of the tangent line is \(y'(2) = 4\), hence any vector with slope 4 in \(\mathbb{R}^2\) will be parallel to the tangent line at this point, e.g \((1, 4)\). Of course we are looking for unit vectors and so we divide by the length and get \(\frac{1}{\sqrt{17}}\langle1, 4\rangle (1, 4)\). Given any non-zero vector there are only two unit vectors which are parallel and so we obtain the other parallel vector by multiplying by \(-1\). So we have only

   \[ \langle \frac{1}{\sqrt{17}}, \frac{4}{\sqrt{17}} \rangle \text{ and } \langle -\frac{1}{\sqrt{17}}, -\frac{4}{\sqrt{17}} \rangle. \]

4. Find the orthogonal projection of the vector \(v = (2, -1, 3)\) in the direction of the vector \(w = (2, 1, 1)\).

   We have a nice formula for the projection of one vector \(v\) in the direction of another vector \(w\), it’s given by \((v \cdot \frac{w}{||w||}) \frac{w}{||w||}\).
In this case we have $|w| = \sqrt{2^2 + 1^2 + 1^2} = \sqrt{6}$ and so $\frac{w}{|w|} = \frac{1}{\sqrt{6}}(2, 1, 1)$. Hence the projection of $v$ in the direction of $w$ is given by

$$((2, -1, 3) \cdot \frac{1}{6}(2, 1, 1)) \frac{1}{\sqrt{6}}(2, 1, 1) = \frac{(4 - 1 + 3)}{\sqrt{6}} = \frac{(2, 1, 1)}{\sqrt{6}}.$$ 

5. If $r = (x, y, z)$, $a = (2, 1, -1)$, and $b = (1, 1, 0)$ then show that the equation $(r - a) \cdot (r - b) = 0$ describes a sphere and find it’s center and radius.

Expanding out the equation gives us:

$$0 = (r - a) \cdot (r - b) = (x - 2, y - 1, z + 1) \cdot (x - 1, y - 1, z)$$

$$= (x - 2)(x - 1) + (y - 1)(y - 1) + (z + 1)z = x^2 - 3x + 2 + y^2 - 2y + 1 + z^2 + z.$$

Completing each square then gives us

$$0 = (x - \frac{3}{2})^2 - \frac{9}{4} + 2 + (y - 1)^2 - 1 + 1 + (z + \frac{1}{2})^2 - \frac{1}{4}$$

$$= (x - \frac{3}{2})^2 + (y - 1)^2 + (z - \frac{-1}{2})^2 - \frac{1}{2}.$$

This then is the usual form of an equation describing a sphere with center $\left(\frac{3}{2}, 1, \frac{-1}{2}\right)$ and radius $\sqrt{\frac{1}{2}} = \frac{\sqrt{2}}{2}$.

6. Find all vectors $v = (v_1, v_2, v_3)$ such that $i \times (j \times v) = (i \times j) \times v$.

Computing the left hand side of the equation gives

$$i \times (j \times v) = i \times (v_3i - v_1k) = v_1j.$$

The right hand side of the equation gives

$$(i \times j) \times v = k \times v = -v_2i + v_1j.$$

If these are equal then we have $(0, v_1, 0) = -v_2i + v_1j = (-v_3, v_1, 0)$ and so the coordinate must also be equal, i.e. $0 = -v_2$, $v_1 = v_1$, and $0 = 0$. Thus $v_2 = 0$ while $v_1$ and $v_3$ can be any real numbers, hence our solution is:

$$v \in \{(t, 0, s) \mid t, s \in \mathbb{R}\}.$$

7. Find all vectors $v = (v_1, v_2, v_3)$ such that $i \cdot (j \times v) = (i \times j) \cdot v$.

We know from class that there is a general formula $a \cdot (b \times c) = (a \times b) \cdot c$ and hence the above equation will hold for all vectors in $\mathbb{R}^3$. 
