HOMEWORK 1, MATH 362A - FALL 2010

1. Let $\mathcal{H}$ be a Hilbert space. Show that if $X \subset \mathcal{H}$ is a bounded set then X has a unique Chebyshev center, i.e., there exists a unique element $h_0 \in \mathcal{H}$ such the function $h \mapsto \sup_{x \in X} \|h - x\|$ obtains its minimum at h_0.

We can prove this using the parallelogram identity similar to our use of the identity in proving that closed convex sets have a unique element of minimal norm.

Let $d = \inf_{h \in \mathcal{H}} \sup_{x \in X} \|h - x\|$. Note that, by hypothesis, X is bounded and hence $d < \infty$. Suppose $h_n \in \mathcal{H}$ is a sequence such that $\lim_{n\to\infty} \sup_{x \in X} \|h_n - x\| = d$.

Using the parallelogram identity we can estimate the size of $\|h_n - h_m\|$ by

$$\|h_n - h_m\|^2 = 2(\|h_n - x\|^2 + \|h_m - x\|^2) - 4\|(h_n + h_m)/2 - x\|^2$$

where x is any fixed element in $\mathcal{H}$.

In particular, if $\varepsilon > 0$ is fixed, then by choosing N such that $\sup_{x \in X} \|h_n - x\|^2 < d^2 + (\varepsilon^2/4)$, for all $n \geq N$, and choosing x such that $4\|(h_n + h_m)/2 - x\|^2 \geq 4d^2$ (which is possible by the definition of d) we have that

$$\|h_n - h_m\|^2 \leq 2(d^2 + (\varepsilon^2/4) + d^2 + (\varepsilon^2/4)) - 4d^2 = \varepsilon,$$

for all $n, m \geq N$.

This shows that $\{h_n\}_n$ is Cauchy and hence converges to an element $h_0 \in \mathcal{H}$ which (because of continuity of the norm) minimizes the function $h \mapsto \sup_{x \in X} \|h - x\|$. Uniqueness follows from the fact that we started with an arbitrary sequence h_n such that $\lim_{n\to\infty} \sup_{x \in X} \|h_n - x\| = d$. Or alternately, if we have another such element then we may again use the parallelogram identity to show that it coincides with h_0.

One application of the above property is that if G is a group of affine transformations on a Hilbert space $\mathcal{H}$ ($\|g \cdot h_1 - g \cdot h_2\| = \|h_1 - h_2\|$ for all $g \in G$, $h_1, h_2 \in \mathcal{H}$), then G fixes a point if and only if G has a bounded orbit. Indeed, if $X = G \cdot h$ is a bounded orbit, then we know that X has a unique Chebyshev center x_0, moreover if $g \in G$, then since g is affine we have that $g \cdot x_0$ is a Chybyshev center of $g \cdot X = X$, hence $g \cdot x_0 = x_0$ for all $g \in G$.

2. Let $\mathcal{H}$ be an infinite dimensional Hilbert space, show that any Hamel (algebraic) basis for $\mathcal{H}$ must be uncountable.

Consider the vector space $X = \oplus_{\mathbb{N}} \mathbb{F}$ of functions from $\mathbb{N}$ to $\mathbb{F}$ which have finite support. Suppose $\| \cdot \|$ is a norm on X. For each $n \in \mathbb{N}$ consider the finite dimensional subspace X_n consisting of functions which vanish on $k > n$. This is a finite dimensional subspace and hence is closed. Moreover, $\cup_{n \in \mathbb{N}} X_n = X$, and it is easy to see that each X_n is nowhere dense (if $x \in X_n$ and δ_{n+1} is the function which is 0 everywhere except for $n + 1$ where it is 1, then $x + t\delta_{n+1} \to x$ as $t \to 0$). Hence, by the Baire Category Theorem X cannot be complete.

3. Let B be a Banach space with norm $\| \cdot \|$. Show that B is isometrically isomorphic to a Hilbert space if and only if the parallelogram identity holds: $\|x + y\|^2 + \|x - y\|^2 = 2(\|x\|^2 + \|y\|^2)$ for all $x, y \in B$.

We will prove this for complex Hilbert spaces, the proof for real Hilbert spaces is similar.

That Hilbert spaces satisfy the parallelogram identity was shown in class and so we will only prove the converse. Suppose B is a Banach space which satisfies the parallelogram identity. We define an inner product on B by the formula

$$4\langle x, y\rangle = \Sigma_{k=0}^{3} i^k \|x + i^k y\|^2.$$

Note that $4\langle x, x\rangle = \|2x\|^2 + i\|(1+i)x\|^2 + 0 - i\|(1-i)x\|^2 = 4\|x\|^2$. So all that remains is to show that $\langle \cdot, \cdot \rangle$ is sesquilinear.

Suppose $x, y, z \in B$, first note that

$$4\langle x/2, z\rangle = \Sigma_{k=0}^{4} i^k \|(x/2) + i^k z\|^2,$$

which by the parallelogram identity is equal to

$$\Sigma_{k=0}^{3} i^k (2(\|(x/2) + i^k(z/2)\|^2 + \|i^k(z/2)\|^2) - \|x/2\|^2)$$

$$= \Sigma_{k=0}^{3} i^k (2(\|(x/2) + i^k(z/2)\|^2 + \|i^k(z/2)\|^2) = 2\langle x, z\rangle.$$

Similarly we can show that $\langle x, z/2\rangle = \frac{1}{2}\langle x, z\rangle$.

Next, we have

$$4\langle x + y, z\rangle = \Sigma_{k=0}^{3} i^k \|(x + y) + i^k z\|^2.$$

Using the parallelogram identity this is equal to

$$\Sigma_{k=0}^{3} i^k (2(\|x + i^k(z/2)\|^2 + \|y + i^k(z/2)\|^2) - \|x - y\|^2)$$
$$\Sigma_{k=0}^{3} i^k 2(\|x + i^k(z/2)\|^2 + \|y + i^k(z/2)\|^2),$$

and from above we showed that this is equal to

$$= 4\langle x, z\rangle + 4\langle y, z\rangle.$$

Similarly, it follows that $\langle x, y + z\rangle = \langle x, y\rangle + \langle x, z\rangle$.

We leave it as a straightforward exercise to show that $\langle ix, z\rangle = \langle x, -iz\rangle = i\langle x, z\rangle$.

Combining the above facts, if we consider the function $\alpha \mapsto \langle \alpha x, z\rangle - \alpha\langle x, z\rangle$ then we have shown that this function is 0 whenever $\alpha = a + ib$ with a, and b dyadic rationals. Since this function is continuous and such numbers are dense it follows that $\langle \alpha x, z\rangle = \alpha\langle x, z\rangle$ for all $\alpha \in \mathbb{C}$. A similar argument shows that $\langle x, \alpha z\rangle = \overline{\alpha}\langle x, z\rangle$ for all $\alpha \in \mathbb{C}$.

4. Show that an inner-product space H is complete if and only if any maximally orthogonal set is an orthogonal basis.

If H is a Hilbert space, β is a maximally orthogonal set, and $x \in H$, then consider the subspace $\mathcal{K}$ spanned by β. Then $x - \mathrm{Proj}_{\mathcal{K}}(x)$ is orthogonal to every element in β and hence since β is maximal it follows that $x - \mathrm{Proj}_{\mathcal{K}}(x) = 0$, i.e., $x \in \mathcal{K}$. Since x was arbitrary this shows that $\mathcal{K} = H$ and hence β is a orthogonal basis for H.

On the other hand if H is not complete, then denote by $\overline{H}$ the Hilbert space completion of H and take $x \in \overline{H} \setminus H$. Let β be a maximally orthogonal set in $H \cap \{x\}^\perp = \{y \in H \mid \langle y, x\rangle = 0\}$.

If $y_1, y_2 \in H$ such that y_1, and y_2 are orthogonal to every element in β, then $\langle y_2, x\rangle y_1 - \langle y_1, x\rangle y_2$ is also orthogonal to every element in β and in addition is orthogonal to x. By definition of β we must then have $\langle y_2, x\rangle y_1 = \langle y_1, x\rangle y_2$. In particular, this shows that if β is not maximally orthogonal in H, then there exists a non-zero vector $y_0 \in H$ such that $\beta \cup \{y_0\}$ is maximally orthogonal in H.

Note that β is not an orthogonal basis since x is not in the closure of the span of β (hence, since H is dense in $\overline{H}$, there also exists an element in H which is not in the closure of the span of β). If we are in the case that β is not maximally orthogonal then it is also easy to see that $\beta \cup \{y_0\}$ is also not an orthogonal basis for the same reason, if x were in the closure of the span of this then x would be a scalar multiple of y_0 and it would then follow that $x \in H$.

(**Bonus Problem**). From the Gram-Schmidt process we know that if $\mathcal{H}$ is a separable Hilbert space and $V \subset \mathcal{H}$ is a dense subspace then V contains an orthonormal basis for $\mathcal{H}$. Show that this is not true in general.

Recall that it follows from Bessel's inequality that if $\mathcal{H}$ is a Hilbert space, $\mathcal{K} \subset \mathcal{H}$ a separable subspace, and $\beta \subset \mathcal{H}$ an orthogonal set, then $\{h \in \beta \mid \mathrm{Proj}_{\mathcal{K}}(h) \neq 0\}$ is countable.

Let $\mathcal{K} = \ell^2\mathbb{N}$ and let $\beta_0 \subset \mathcal{K}$ be a Hamel basis for $\mathcal{K}$, (which is uncountable by problem 2 above). Consider the Hilbert space $\ell^2\beta_0 \oplus \mathcal{K}$, and the non-closed subspace V which is algebraically spanned by $\beta = \{\delta_x \oplus x \mid x \in \beta_0\}$.

Since β_0 is a linearly independent set it follows that $\{h \in V \mid \mathrm{Proj}_{\mathcal{K}}(h) = 0\} = \{0\}$. Hence, from the remark above we see that any orthogonal subset of V has to be countable.

On the other hand, $\overline{V}$ is not separable, (for instance because $\mathrm{Proj}_{\ell^2\beta_0}(V)$ is dense in the non-separable Hilbert space $\ell^2\beta_0$), thus V does not contain an orthonormal basis for $\mathcal{H} = \overline{V}$.