This homework assignment covers Sections 17.1-17.4 in the book.

1. Sketch the vector field \( F(x, y) = \frac{1}{2}i + yj \).

2. Find the gradient vector field for \( f(x, y) = x^2 - y \) and sketch it.
   The gradient vector field is just \( \nabla f(x, y) = 2xi - j \).

3. Evaluate the line integral \( \int_C x \sin y \, ds \) where \( C \) is the line segment from \((0, 3)\) to \((4, 6)\).
   The curve \( C \) can be parametrized by \( r(t) = (0, 3) + t(4, 3) \) where \( 0 \leq t \leq 1 \), and then we have \( ||r'(t)|| = \sqrt{4^2 + 3^2} = 5 \). Hence
   \[
   \int_C x \sin y \, ds = \int_0^1 20t \sin(3 + 3t) \, dt,
   \]
   integration by parts \((u = t \text{ and } dv = \sin(3 + 3t)dt)\) then gives
   \[
   \int_0^1 20t \sin(3 + 3t) \, dt = 20\left[ -\frac{1}{3}t \cos(3 + 3t) + \frac{1}{9} \sin(3 + 3t) \right]_0^1 = \frac{20}{9}(\sin 6 - 3 \cos 6 - \sin 3).
   \]

4. Evaluate the line integral \( \int_C \sin x \, dx + \cos y \, dy \), where \( C \) consists of the top half of the circle \( x^2 + y^2 = 1 \) from \((1, 0)\) to \((-1, 0)\) and the line segment from \((-1, 0)\) to \((-2, 3)\).
   If we split the curve into two parts we can find a parameterization for each part and then continue as in 3. Let’s instead use the Fundamental Theorem of Line Integrals.
   Note that \( F(x, y) = \sin xi + \cos yj \) is a conservative vector field. Indeed if \( f_x = \sin x \) then \( f = -\cos x + g(y) \) where \( g \) is a function of \( y \). Then we have \( \cos y = f_y = g'(y) \) so that \( g(y) = \sin y + K \) where \( K \) is some constant.
   In particular we have \( F = \nabla(-\cos x + \sin y) \) and hence we have
   \[
   \int_C \sin x \, dx + \cos y \, dy = \int_C \nabla f \cdot dr
   = f(-2, 3) - f(1, 0) = -\cos 2 + \sin 3 + \cos 1.
   \]

5. Evaluate the line integral \( \int_C F \cdot dr \) where \( F(x, y, z) = (x + y)i + (y - z)j + z^2k \) and \( C \) is given by the vector function \( r(t) = t^2i + t^3j + t^2k \), \( 0 \leq t \leq 1 \).
   \( F \) is not a conservative vector field and so we cannot use the Fundamental Theorem of Line Integrals. We will have to compute this directly. Since \( r(t) = t^2i + t^3j + t^2k \) we have \( r'(t) = 2ti + 3t^2j + 2tk \). Therefore
   \[
   \int_C F \cdot dr = \int_0^1 F(r(t)) \cdot r'(t) \, dt
   = \int_0^1 (2t^3 + 2t^4 + 3t^5 - 3t^4 + 2t^5) \, dt
   = \int_0^1 (5t^5 - t^4 + 2t^3) \, dt
   = \frac{5}{6} - \frac{1}{5} + \frac{1}{2} = \frac{17}{15}.
   \]
6. Evaluate the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F}(x,y,z) = (2xz + y^2)i + 2xyj + (x^2 + 3z^2)k \) and \( C \) is given by \( x = t^2, y = t + 1, z = 2t - 1, 0 \leq t \leq 1 \).

If this is a conservative vector field then we have
\[
f_x = 2xz + y^2,
\]
hence \( f = x^2z + xy^2 + g(y,z) \) where \( g \) is some function. Therefore we have
\[
2xy = f_y = 2xy + \partial g/\partial y,
\]
and so \( g(y,z) = h(z) \) for some function \( h \). We then have \( f = x^2z + xy^2 + h(z) \) and so
\[
x^2 + 3z^2 = f_z = x^2 + h'(z),
\]
hence \( h(z) = z^3 + K \) for some constant \( K \).

In particular we have shown that \( \nabla(x^2z + xy^2 + z^3) \) and so by the Fundamental Theorem of Line Integrals we have
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,2,1) - f(0,1,-1) = (1 + 4 + 1) - (0 + 0 - 1) = 7.
\]

7. Evaluate the line integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) where \( \mathbf{F}(x,y,z) = e^yi + xe^yj + (z + 1)e^z k \), and \( C \) is given by \( \mathbf{r}(t) = ti + t^2j + t^3k, 0 \leq t \leq 1 \).

Just as above, if \( \mathbf{F} \) is a conservative vector field then we have
\[
f_x = e^y,
\]
hence \( f = xe^y + g(y,z) \). Therefore
\[
xe^y = f_y = xe^y + \partial g/\partial y,
\]
and so \( g(y,z) = h(z) \). We then have \( f = xe^y + h(z) \) and so
\[
(z + 1)e^z = f_z = h'(z),
\]
therefore \( h(z) = ze^z + K \) and in particular we have \( \nabla(xe^y + ze^z) \) and so by the Fundamental Theorem of Line Integrals we have
\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(1,1,1) - f(0,0,0) = 2e.
\]

8. Evaluate the line integral \( \int_C \cos y \; dx + x^2 \sin y \; dy \), where \( C \) is the rectangle with vertices \( (0,0) \), \( (5,0) \), \( (5,2) \), and \( (0,2) \) oriented positively.

Let \( D \) be the region enclosed by the curve \( C \). Using Green’s Theorem we have that
\[
\int_C \cos y \; dx + x^2 \sin y \; dy = \iint_{D} (2x \sin y + \sin y) \; dA
\]
\[
= \int_{0}^{5} \int_{0}^{2} (2x + 1) \sin y \; dy \; dx = [x^2 + x]_{0}^{5}[- \cos y]_{0}^{2} = 30(1 - \cos 2).
\]

9. Evaluate the line integral \( \int_C \sin y \; dx + x \cos y \; dy \), where \( C \) is given by the ellipse \( x^2 + xy + y^2 = 1 \), oriented positively.

Let \( D \) be the region enclosed by the curve \( C \). Using Green’s Theorem we have that
\[
\int_C \sin y \; dx + x \cos y \; dy = \iint_{D} (\cos y - \cos y) \; dA = 0.
\]