HOMEWORK 5, MATH 175 - FALL 2009

This homework assignment covers Sections 15.4 - 15.6 in the book.

1. Find an equation of the tangent plane to the surface given by $z = x^2 e^{x^2-y^2}$ at the point $(-1, 1, 1)$.

We know the general formula for the tangent plane to the surface given by $z = f(x, y)$ at the point $(x_0, y_0, f(x_0, y_0))$ is

$$z - f(x_0, y_0) = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Here we have $x_0 = -1$, $y_0 = 1$, and $f(x, y) = x^2 e^{x^2-y^2}$. Hence $f_x = 2xe^{x^2-y^2} + 2x^3 e^{x^2-y^2}$ and $f_y = -2x^2 y e^{x^2-y^2}$. Thus the equation of the tangent plane at the point $(-1, 1, 1)$ is

$$\boxed{z - 1 = -4(x + 1) - 2(y - 1).}$$

2. Find the linear approximation of the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ at the point $(2, 3, -1)$.

The linear approximation is the linear function we get from the equation of the tangent space.

Here we have $f_x = 2x/\sqrt{x^2 + y^2 + z^2}$, $f_y = 2y/\sqrt{x^2 + y^2 + z^2}$, and $f_z = 2z/\sqrt{x^2 + y^2 + z^2}$, and so the linear approximation at the point $(2, 3, -1)$ is

$$\sqrt{2^2 + 3^2 + (-1)^2} + \frac{2 \cdot 2}{\sqrt{2^2 + 3^2 + (-1)^2}}(x - 2) + \frac{2 \cdot 3}{\sqrt{2^2 + 3^2 + (-1)^2}}(y - 3) + \frac{2 \cdot (-1)}{\sqrt{2^2 + 3^2 + (-1)^2}}(z - (-1))$$

$$= \boxed{\frac{\sqrt{14}}{7}(-7 + 2x + 3y - z).}$$

3. Suppose $z = x^2 e^{yx}$, where $x = \sin t$, and $y = \ln t$. Find dz/dt.

By the chain rule we have $\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$. Here we have $\frac{\partial z}{\partial x} = 2xe^{yx} + x^2 y e^{yx} = xe^{yx}(2 + x)$, $\frac{\partial z}{\partial y} = x^3 e^{yx}$. Also $\frac{dx}{dt} = \cos t$, and $\frac{dy}{dt} = 1/t$. Thus

$$\frac{dz}{dt} = xe^{yx}(2 + x)\cos t + x^3 e^{yx}/t = \boxed{t^{\sin t}(\sin t \cos t(2 + \sin t) + \frac{\sin^3 t}{t}).}$$

4. Suppose $z = e^{x-y}$, where $x = st$, and $y = s/t$. Find $\partial z/\partial s$ and $\partial z/\partial t$.

Just as above we may use the chain rule to find

$$\frac{\partial z}{\partial s} = e^{x-y}t - \frac{e^{x-y}}{t} = \boxed{e^{s(t-\frac{1}{t})}(t - \frac{1}{t}).}$$

Also we have

$$\frac{\partial z}{\partial t} = e^{x-y}s + \frac{e^{x-y}}{t^2} = \boxed{e^{s(t-\frac{1}{t})}(s + \frac{1}{t^2}).}$$

5. Suppose $z = \sqrt{r^2 + s^2}$, where $r = y + x\cos t$, and $s = x + y\sin t$. Find $\partial z/\partial x$, $\partial z/\partial y$, and $\partial z/\partial t$.

Again we use the chain rule:

$$\frac{\partial z}{\partial x} = \frac{2r}{\sqrt{r^2 + s^2}}\cos t + \frac{2s}{\sqrt{r^2 + s^2}}$$

$$= \boxed{2(y\cos t + x\cos^2 t + x + y\sin t)/\sqrt{(y + x\cos t)^2 + (x + y\sin t)^2}.}$$

$$\frac{\partial z}{\partial y} = \frac{2r}{\sqrt{r^2+s^2}} + \frac{2s}{\sqrt{r^2+s^2}}\sin t$$

$$= \boxed{2(y + x\cos t + x\sin t + y\sin^2 t)/\sqrt{(y+x\cos t)^2 + (x+y\sin t)^2}.}$$

$$\frac{\partial z}{\partial t} = -\frac{2r}{\sqrt{r^2+s^2}}x\sin t + \frac{2s}{\sqrt{r^2+s^2}}y\cos t$$

$$= \boxed{2(-xy\sin t - x^2\cos t\sin t + xy\cos t + y^2\cos t\sin t)/\sqrt{(y+x\cos t)^2 + (x+y\sin t)^2}.}$$

6. Suppose $y^5 - x^2y^2 = 1 - e^{xy}$. Find dy/dx.

If we define $F(x,y) = y^5 - x^2y^2 + e^{xy} - 1$ then $F(x,y) = 0$ and hence by the chain rule $F_x\frac{dx}{dx} + F_y\frac{dy}{dx} = 0$ and so

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \boxed{\frac{2xy^2 - ye^{xy}}{5y^4 - 2x^2y + xe^{xy}}.}$$

7. Suppose $xyz = \cos(x+y+z)$. Find $\partial z/\partial x$, and $\partial z/\partial y$.

If we define $F(x,y,z) = xyz - \cos(x+y+z)$ then by the chain rule $F_x\frac{\partial x}{\partial x} + F_y\frac{\partial y}{\partial x} + F_z\frac{\partial z}{\partial x} = 0$. Hence

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \boxed{\frac{-yz - \sin(x+y+z)}{xy + \sin(x+y+z)}.}$$

Similarly

$$\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \boxed{\frac{-xz - \sin(x+y+z)}{xy + \sin(x+y+z)}.}$$

8. Consider the function $f(x,y) = y^3/x^2$.

(a). Find the gradient of f.

(b). Find the directional derivative of f in the direction of $(4,-1)$ at the point $(1,1)$.

(a). The gradient of f is $\nabla f = (f_x, f_y)$, hence in this case

$$\boxed{\nabla f = (-2y^3/x^3, 3y^2/x^2).}$$

(b). The unit vector in the direction of $(4,-1)$ is $(4,-1)/\sqrt{17}$, hence the directional derivative at the point $(1,1)$ is

$$(\nabla f(1,1)) \cdot (4,-1)/\sqrt{17} = \frac{-2\cdot 8 - 3\cdot 1}{\sqrt{17}} = \boxed{\frac{-19}{\sqrt{17}}.}$$

9. Find the directions in which the directional derivative of the function $f(x,y) = x\cos y$ at the point $(-1,\pi/4)$ has the value -1.

The gradient of f is $\nabla f = (\cos y, -x\sin y)$ hence the directional derivative at $(-1,\pi/4)$ in the direction of a unit vector $u = (\cos t, \sin t)$ is

$$D_u f(-1,\pi/4) = (1/\sqrt{2}, 1/\sqrt{2}) \cdot (\cos t, \sin t) = \frac{1}{\sqrt{2}}(\cos t + \sin t).$$

This will be -1 whenever $h(t) = \cos t + \sin t = -\sqrt{2}$, $0 \le t < 2\pi$. By finding the critical points of $h(t)$ we see that $-\sqrt{2}$ is the minimum of h and it is obtained only when $t = 5\pi/4$. Hence the only time that the directional derivative is -1 at the point $(-1,\pi/4)$ is when we are taking the derivative in the direction of $\boxed{(\cos 5\pi/4, \sin 5\pi/4) = (-1/\sqrt{2}, -1/\sqrt{2})}$.

10. Find the maximal value of the directional derivative $D_u f$ at the point $(1,2,-1)$ to the function $f(x,y,z) = \frac{1}{1+2x^2+3y^2+4z^2}$. In which direction is this maximum attained?

The gradient of this function is

$$\nabla f = (\frac{-4x}{(1+2x^2+3y^2+4z^2)^2}, \frac{-6y}{(1+2x^2+3y^2+4z^2)^2}, \frac{-8z}{(1+2x^2+3y^2+4z^2)^2}).$$

At the point $(1, 2, -1)$ this gives

$$\nabla f(1, 2, -1) = (-4, -12, 8)/361.$$

The maximum value of a directional derivative at this point is then

$$\|\nabla f(1, 2, -1)\| = \boxed{\frac{4\sqrt{14}}{361}.}$$

The direction that this maximum is achieved is in the direction of the gradient $(-4, -12, 8)/361$, or in terms of unit vectors

$$\boxed{\frac{1}{\sqrt{14}}(-1, -3, 2).}$$