HOMEWORK 4, MATH 175 - FALL 2009

This homework assignment covers Sections 15.1 - 15.3 in the book.

1. Find and sketch the domain of the function $f(x,y) = \sqrt{x-y} + \ln(y^2)$. What is the range of this function?

In order for the function to be defined we need that $\ln(y^2)$ is defined and so $y \neq 0$, and $\sqrt{x-y}$ needs to be defined and so $x \geq y$. Hence the domain is

$$\boxed{D(f) = \{(x,y) \in \mathbb{R}^2 \mid x \geq y, y \neq 0\}.}$$

If $a \in \mathbb{R}$ then $\ln((e^{a/2})^2) = r$ and so $f(e^{a/2}, e^{a/2}) = a$ hence the range of f is all of $\mathbb{R}$.

2. Sketch the contour map and the graph of the function $f(x,y) = 4x^2 + y^2 + 1$.

The contour curves are the curves on where f is constant, i.e. they are given by the equation $k = 4x^2+y^2+1$ where k is a constant. These equations describe a family of ellipses centered at the origin,

We can then see that the graph of f will be an elliptic paraboloid opening up in the positive z-direction.

3. Sketch the domain and contour map of the function $f(x,y) = \ln(y/x)$.

In order for the function to be defined we need that $y/x > 0$, hence either $x,y > 0$, or $x,y < 0$ so the domain is the first and third quadrant of $\mathbb{R}^2$ not including the axes.

$$\boxed{D(f) = \{(x,y) \in \mathbb{R}^2 \mid x,y > 0, \quad \text{or} \quad x,y < 0\}.}$$

The contour curves are described by the equations $k = \ln(y/x)$ where k is a constant. Solving for y we see that this is equivalent to $y = e^k x$, $x \neq 0$. This describes a family of lines going through the origin having positive slope e^k. Of course we leave out the point $(0,0)$ in each of these lines since it is not in the domain of f.

4. Find the limit and prove it exists or prove that it does not exist for:

(a). $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2+2y^2}$.

(b). $\lim_{(x,y)\to(0,0)} \frac{x \sin y}{x^2+2y^2}$.

(c). $\lim_{(x,y)\to(1,1)} \frac{e^x \ln y}{x^2+2y^2}$.

(d). $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2 y - z^2}{x^2+y^2+z^2}$.

(a). First consider the function $g(t) = t - \sin t$ for $t \geq 0$. Note that since $g(0) = 0$ and $|\sin t| \leq 1$ if we want to find the minimum of this function we can restrict ourselves to the interval $[0, \pi/2)$. Since this is a smooth function the minimum lives at a critical point and so we differentiate to get $g'(t) = 1 - \cos t$. Thus the critical points are where $\cos t = 1$ in the interval $[0, \pi/2)$. This only happens when $t = 0$ and so the minimum of g is $g(0) = 0$, i.e. we have shown that $\sin t \leq t$, for $t \geq 0$.

Since $\sin -t = -\sin t$ it follows that $|\sin t| \leq |t|$, for $t \in \mathbb{R}$, and so $\sin^2 t \leq t^2$, for $t \in \mathbb{R}$.

As we saw in class $x^2 \leq x^2 + 2y^2$ and so $\frac{x^2}{x^2+2y^2} \leq 1$, for $(x,y) \neq (0,0)$. Thus if we are given $\varepsilon > 0$ and we set $\delta = \sqrt{\varepsilon}$ then if $\sqrt{x^2+y^2} = \|(x,y) - (0,0)\| < \delta$ we have

$$\begin{aligned} \left|\frac{x^2 \sin^2 y}{x^2+2y^2} - 0\right| &= \frac{x^2}{x^2+2y^2} \sin^2 y \\ &\leq \sin^2 y \leq y^2 \leq x^2 + y^2 \\ &= (\sqrt{x^2+y^2})^2 < \delta^2 = \varepsilon. \end{aligned}$$

Hence $\lim_{(x,y)\to(0,0)} \frac{x^2\sin^2 y}{x^2+2y^2}$ does exist, and in fact it equals $\boxed{0}$.

(b). If we consider the limit along the curve given by $y = x$, then we have

$$\lim_{x\to 0} \frac{x\sin x}{x^2+2x^2} = 1/3,$$

which can be found using L'Hôpital's rule. Also in this same way if we look at the limit along the curve given by $y = -x$ then we have

$$\lim_{x\to 0} \frac{-x\sin x}{x^2+2x^2} = -1/3.$$

Therefore we get two different limits along two different curves and so $\boxed{\text{the limit does not exist}}$.

(c). We know from class that the functions $f(x,y) = e^x$, $g(x,y) = \ln y$, and $h(x,y) = \frac{1}{x^2+2y^2}$ are all continuous at the point $(1,1)$. Since the product of continuous functions is continuous we have that the limit exists and equals the function at the point $(1,1)$, i.e.

$$\lim_{(x,y)\to(1,1)} \frac{e^x \ln y}{x^2+2y^2} = \frac{e^1\ln 1}{1^2+2\cdot 1^2} = \boxed{0}.$$

(d). If we consider the limit along the curve $x = y = 0$ then we have

$$\lim_{z\to 0} \frac{-z^2}{z^2} = -1.$$

On the other hand if we consider the limit along the curve $x = z = 0$ then we have

$$\lim_{y\to 0} \frac{0}{y^2} = 0.$$

Since we get two different limits along two different curves $\boxed{\text{the limit does not exist}}$.

5. Let $f(x,y) = \begin{cases} 0 & if \quad y \le 0 \quad or \quad y \ge x^4 \\ 1 & if \quad 0 < y < x^4. \end{cases}$

(a). Show that $f(x,y)\to 0$ as $(x,y)\to(0,0)$ along any path through $(0,0)$ of the form $y = mx^a$ with $a < 4$.

(b). Show that f is discontinuous at $(0,0)$.

(a). If we look at the curve $y = mx^a$, then we need that $a > 0$ in order for the curve to pass through the origin. Also if $m < 0$ then the curve is approaching the origin from below where f is identically 0 and hence the limit is 0. Thus we may assume that $m > 0$, and $0 < a < 4$.

In this case if $|x| < m^{1/(4-a)}$ then since $4 - a > 0$ we may exponentiate both sides and preserve the inequality, hence $|x|^{4-a} < m$. By multiplying both sides by $|x|^a > 0$ we have $x^4 < m|x|^a$.

What we have shown is that if $|x| < m^{1/(4-a)}$ then $x^4 < m|x|^a$ and so by definition of f we have $f(x, mx^a) = 0$. Therefore if we take the limit along the curve given by $y = mx^a$ we have

$$\lim_{x\to 0} f(x, mx^a) = 0.$$

(b). If we consider the curve $y = x^4/2$ then $0 < y(x) < x^4$ for all $x \ne 0$. Hence $f(x, x^4/2) = 1$ and so

$$\lim_{x\to 0} f(x, x^4/2) = 1.$$

Since we get two different limits along two different curves we have that the limit does not exist.

6. Find the partial derivatives of the following functions:

(a). $f(x,y) = y^4 - 3xy - e^x$.

(b). $f(x,y) = \tan xy$.

(c). $f(x,y,z) = z^2 e^{xy^2 z}$.

(d). $f(x_1, x_2, \ldots, x_n) = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$.

(a). $\partial f/\partial x(x,y) = -3y - e^x$,

$\partial f/\partial y(x,y) = 4y^3 - 3x$.

(b). $\partial f/\partial x(x,y) = y(1+\tan^2 xy) = y\sec^2 xy$,
$\partial f/\partial y(x,y) = x(1+\tan^2 xy) = x\sec^2 xy$.

(c). $\partial f/\partial x(x,y,z) = z^3y^2e^{xy^2z}$,
$\partial f/\partial y(x,y,z) = 2z^3xye^{xy^2z}$,
$\partial f/\partial z(x,y,z) = 2ze^{xy^2z} + z^2xy^2e^{xy^2z} = ze^{xy^2z}(2+zxy^2)$.

(d). $\partial f/\partial x_j(x_1,x_2,\ldots,x_n) = 2x_j/\sqrt{x_1^2+x_2^2+\cdots+x_n^2}$, for all $1 \leq j \leq n$.

7. Find all second partial derivatives of the function $f(x,y) = \sin^2(x-y)$.

Note all 2nd order partial derivatives of f are continuous hence $f_{xy} = f_{yx}$.
$f_x(x,y) = 2\sin(x-y)\cos(x-y)$, and $f_y(x,y) = -2\sin(x-y)\cos(x-y)$ hence
$f_{xx}(x,y) = 2\cos^2(x-y) - 2\sin^2(x-y)$,
$f_{yx}(x,y) = f_{xy}(x,y) = 2\sin^2(x-y) - 2\cos^2(x-y)$,
$f_{yy}(x,y) = 2\cos^2(x-y) - 2\sin^2(x-y)$.