This homework assignment covers Sections 14.1 - 14.3 in the book.

1. Find \( \lim_{t \to 0} r(t) \) where \( r : \mathbb{R} \to \mathbb{R}^3 \) is the vector function given by \( r(t) = (e^{-t^2}, t^2, \cos(t - \pi)) \).

   We know that the limit of a vector function is the vector which is made up of the limits of the coordinate functions if they exist. We have \( \lim_{t \to 0} e^{-t^2} = 1 \), \( \lim_{t \to 0} \frac{t^2}{\sin^2 t} = 1 \) (using L’hôpital’s rule), and \( \lim_{t \to 0} \cos(t - \pi) = -1 \). Therefore

\[
\lim_{t \to 0} r(t) = (1, 1, -1).
\]

2. Find a vector function that represents the curve of intersection of the two surfaces given by \( x^2 + y^2 = 4 \) and \( z = xy \).

   If \( r : \mathbb{R} \to \mathbb{R}^3 \) is a vector function which gives us the curve of intersection then we see that \( r_1(t)^2 + r_2(t)^2 = 4 \) in order for \( r \) to live on the cylinder \( x^2 + y^2 = 4 \). Thus we might want to look at \( r_1(t) = 2 \sin t \), and \( r_2(t) = 2 \cos t \), in this case \( r_3(t) = r_1(t)r_2(t) = 4 \sin t \cos t \). And indeed the resulting vector function does the job

\[
r(t) = (2 \sin t, 2 \cos t, 4 \sin t \cos t).
\]

3. Consider the vector functions \( r(t) = (|t|, \sin t, -\sqrt{|t|}) \) and \( s(t) = (t^2, \sin t, 1 - \cos t) \). Determine where the functions \( r \), \( s \), and \( f(t) = r(t) \cdot s(t) \) are differentiable, and compute the derivatives.

   We know that a vector function is differentiable if each of the coordinate functions are differentiable. The function \( r_2(t) \) is differentiable everywhere and has derivative \( r_2'(t) = \cos t \), the functions \( r_1(t) \), and \( r_2(t) \) are differentiable whenever \( t \neq 0 \) and has derivatives \( r_1'(t) = -1 \), for \( t < 0 \), \( r_1'(t) = 1 \), for \( t > 0 \), \( r_3'(t) = \frac{1}{2 \sqrt{t}} \), for \( t < 0 \), and \( r_3'(t) = -\frac{1}{2 \sqrt{t}} \), for \( t > 0 \).

   Hence \( r \) is differentiable when ever \( t \neq 0 \) and has derivative

\[
r'(t) = \begin{cases} 
(\frac{1}{2 \sqrt{|t|}}, \frac{1}{2 \sqrt{|t|}}, \frac{1}{2 \sqrt{|t|}}) & \text{for } t < 0 \\
(1, \cos t, -\frac{1}{2 \sqrt{|t|}}) & \text{for } t > 0.
\end{cases}
\]

   Similarly we can see that \( s(t) \) is differentiable everywhere and has derivative

\[
s'(t) = (2t, \cos t, \sin t).
\]

   For \( f(t) = r(t) \cdot s(t) \) we may use the formula for the derivative of a dot product \( (r \cdot s)'(t) = r'(t) \cdot s(t) + r(t) \cdot s'(t) \) to calculate the derivative when \( t \neq 0 \) as

\[
f'(t) = \begin{cases} 
-2t^2 + 2 \cos t \sin t + \frac{1}{2 \sqrt{|t|}}(1 - \cos t) - \sqrt{t} \sin t & \text{for } t < 0 \\
2 \cos t \sin t - \frac{1}{2 \sqrt{|t|}}(1 - \cos t) - \sqrt{t} \sin t & \text{for } t > 0.
\end{cases}
\]

   For \( t = 0 \) we have to look at \( f(t) = |t|^3 + \sin^2 t - \sqrt{|t|}(1 - \cos t) \) directly. The first two terms of the sum are differentiable at \( t = 0 \) and both have derivatives 0, for the third term we may try to use the definition of the derivative which in this case is

\[
\lim_{t \to 0} \sqrt{|t|}(1 - \cos t)/t = \lim_{t \to 0} (1 - \cos t)/|t| = 0,
\]

   which we find by using L’hôpital’s rule. Thus \( f \) is differentiable at 0 and has derivative

\[
f'(0) = 0.
\]

4. Find parametric equations for the tangent line to the curve given by \( x = 1 + t, y = \sqrt{t}, z = e^t - t \) at the point \( (2, 1, e - 1) \).
If we let $r(t) = (1 + t, \sqrt{7}, e^t - t)$ then $r$ traces the curve and $r'(t) = (1, \frac{1}{2\sqrt{7}}, e^t - 1)$. A vector which is parallel to the tangent line is then given by the derivative $r'(1) = (1, \frac{1}{2}, e - 1)$. Since we already know a point on the line we may then write the equation of the line as

$$(x, y, z) = (2, 1, e - 1) + t(1, \frac{1}{2}, e - 1).$$

Writing this in parametric equations gives

$$x = 2 + t, \quad y = 1 + \frac{t}{2}, \quad z = e - 1 + t(e - 1).$$

5. Let $r(t) = (2t + 3, e^t - t, e^t + 3t - 7)$ find $f(t) = r(t) \cdot (r'(t) \times r''(t))$. (Hint: First find $f'(t)$ and $f(0)$ and then solve a differential equation.)

First note that $r'(t) = (3, 3e^t - 1, e^t + 3)$ and $r''(t) = (0, 3e^t, e^t) = r'''(t)$. Also $r(0) = (0, 3, -6)$, $r'(0) = (3, 2, 4)$, and $r''(0) = (0, 3, 1)$ Using the rules for differentiating dot and cross products we have

$$f'(t) = r'(t) \cdot (r'(t) \times r''(t)) + r(t) \cdot (r'(t) \times r''(t))' = r'(t) \cdot (r'(t) \times r''(t)) + r(t) \cdot (r''(t) \times r''(t)) + r(t) \cdot (r'(t) \times r'''(t)).$$

Note that the first two terms are 0 and the last term is the same as $f$ since $r''' = r''$, thus $f' = f$ and so we know that $f$ must be of the form $f(t) = Ce^t$.

To find $C$ we need to just plug in $t = 0$ so that

$$C = f(0) = (0, 3, -6) \cdot (0, 3, 1) = -63.$$

Hence

$$f(t) = -63e^t.$$

6. Find the length of the curve $r(t) = (\cos t, \sin t, \ln \cos t)$, for $0 \leq t \leq \pi/4$.

To find the length we use the formula

$$L(0, \pi/4) = \int_0^{\pi/4} ||r'(t)|| dt = \int_0^{\pi/4} \sqrt{1 + \tan^2 t} dt = \int_0^{\pi/4} \sec t dt.$$

The anti-derivative of $\sec t$ is $\ln(\sec t + \tan t)$, Hence by the Fundamental Theorem of Calculus

$$L(0, \pi/4) = \ln(\sec(\pi/4) + \tan(\pi/4)) - \ln(\sec 0 + \tan 0) = \ln(\sqrt{2} + 1).$$

7. Find the unit tangent vector, the unit normal vector, and the curvature at time $t$ of the curve given by $r(t) = (\sin t, e^t, t^2)$.

The unit tangent vector is given by $T(t) = r'(t)/||r'(t)||$ where $r'(t) = (\cos t, e^t, 2t)$ and $||r'(t)|| = \sqrt{\cos^2 t + e^{2t} + 4t^2}$, so that

$$T(t) = \frac{1}{\sqrt{\cos^2 t + e^{2t} + 4t^2}} (\cos t, e^t, 2t).$$

The unit normal vector is given by $N(t) = T'(t)/||T'(t)||$, unfortunately $T$ does not have a very nice form and so differentiating it while possible is going to be a mess, therefore I will not continue the calculation here.

To find the curvature we may use the formula $\kappa(t) = ||r'(t) \times r''(t)||/||r'(t)||^3$. We have that $r''(t) = (-\sin t, e^t, 2)$ hence

$$r'(t) \times r''(t) = (2e^t(1 - t), 2(t \sin t - \cos t), e^t(\cos t + \sin t)),$$

and

$$||r'(t) \times r''(t)|| = \sqrt{4e^{2t}(1 - t)^2 + 4(t \sin t - \cos t)^2 + e^{2t}(\cos t + \sin t)^2}.$$