HOMEWORK 3, MATH 175 - FALL 2009

This homework assignment covers Sections 14.1 - 14.3 in the book.

1. Find $\lim_{t\to 0} r(t)$ where $r : \mathbb{R} \to \mathbb{R}^3$ is the vector function given by $r(t) = (e^{-t^2}, \frac{t^2}{\sin^2 t}, \cos(t-\pi))$.

We know that the limit of a vector function is the vector which is made up of the limits of the coordinate functions if they exists. We have $\lim_{t\to 0} e^{-t^2} = 1$, $\lim_{t\to 0} \frac{t^2}{\sin^2 t} = 1$ (using L'hôpital's rule), and $\lim_{t\to 0} \cos(t-\pi) = -1$. Therefore

$$\boxed{\lim_{t\to 0} r(t) = (1, 1, -1).}$$

2. Find a vector function that represents the curve of intersection of the two surfaces given by $x^2 + y^2 = 4$ and $z = xy$.

If $r : \mathbb{R} \to \mathbb{R}^3$ is a vector function which gives us the curve of intersection then we see that $r_1(t)^2+r_2(t)^2 = 4$ in order for r to live on the cylinder $x^2 + y^2 = 4$. Thus we might want to look at $r_1(t) = 2\sin t$, and $r_2(t) = 2\cos t$, in this case $r_3(t) = r_1(t)r_2(t) = 4\sin t \cos t$. And indeed the resulting vector function does the job

$$\boxed{r(t) = (2\sin t, 2\cos t, 4\sin t\cos t).}$$

3. Consider the vector functions $r(t) = (|t|, \sin t, -\sqrt{|t|})$ and $s(t) = (t^2, \sin t, 1-\cos t)$. Determine where the functions r, s, and $f(t) = r(t)\cdot s(t)$ are differentiable, and compute the derivatives.

We know that a vector function is differentiable if each of the coordinate functions are differentiable. The function $r_2(t)$ is differentiable everywhere and has derivative $r_2'(t) = \cos t$, the functions $r_1(t)$, and $r_2(t)$ are differentiable whenever $t \neq 0$ and has derivatives $r_1'(t) = -1$, for $t < 0$, $r_1'(t) = 1$, for $t > 0$, $r_3'(t) = \frac{1}{2\sqrt{-t}}$, for $t < 0$, and $r_3'(t) = -\frac{1}{2\sqrt{t}}$, for $t > 0$.

Hence r is differentiable when ever $t \neq 0$ and has derivative

$$\boxed{r'(t) = \begin{cases} (-1, \cos t, \frac{1}{2\sqrt{-t}}) & \text{for } \ t < 0 \\ (1, \cos t, -\frac{1}{2\sqrt{t}}) & \text{for } \ t > 0. \end{cases}}$$

Similarly we can see that $s(t)$ is differentiable everywhere and has derivative

$$\boxed{s'(t) = (2t, \cos t, \sin t).}$$

For $f(t) = r(t)\cdot s(t)$ we may use the formula for the derivative of a dot product $(r\cdot s)'(t) = r'(t)\cdot s(t) + r(t)\cdot s'(t)$ to calculate the derivative when $t \neq 0$ as

$$\boxed{f'(t) = \begin{cases} -2t^2 + 2\cos t\sin t + \frac{1}{2\sqrt{-t}}(1-\cos t) - \sqrt{-t}\sin t & \text{for } \ t < 0 \\ 2\cos t \sin t - \frac{1}{2\sqrt{t}}(1-\cos t) - \sqrt{t}\sin t & \text{for } \ t > 0. \end{cases}}$$

For $t = 0$ we have to look at $f(t) = |t^3| + \sin^2 t - \sqrt{|t|}(1-\cos t)$ directly. The first two terms of the sum are differentiable at $t = 0$ and both have derivatives 0, for the third term we may try to use the definition of the derivative which in this case is

$$\lim_{t\to 0} \sqrt{|t|}(1-\cos t)/t = \lim_{t\to 0}(1-\cos t)/\sqrt{|t|} = 0,$$

which we find by using L'hôpital's rule. Thus f is differentiable at 0 and has derivative

$$\boxed{f'(0) = 0.}$$

4. Find parametric equations for the tangent line to the curve given by $x = 1+t$, $y = \sqrt{t}$, $z = e^t - t$ at the point $(2, 1, e-1)$.

If we let $r(t) = (1+t, \sqrt{t}, e^t - t)$ then r traces the curve and $r'(t) = (1, \frac{1}{2\sqrt{t}}, e^t - 1)$. A vector which is parallel to the tangent line is then given by the derivative $r'(1) = (1, \frac{1}{2}, e-1)$. Since we already know a point on the line we may then write the equation of the line as

$$(x, y, z) = (2, 1, e-1) + t(1, \frac{1}{2}, e-1).$$

Writing this in parametric equations gives

$$\boxed{x = 2+t, \quad y = 1 + \frac{t}{2} \quad z = e - 1 + t(e-1).}$$

5. Let $r(t) = (2t + t, 3e^t - t, e^t + 3t - 7)$ find $f(t) = r(t) \cdot (r'(t) \times r''(t))$. (Hint: First find $f'(t)$ and $f(0)$ and then solve a differential equation.)

First note that $r'(t) = (3, 3e^t - 1, e^t + 3)$ and $r''(t) = (0, 3e^t, e^t) = r'''(t)$. Also $r(0) = (0, 3, -6)$, $r'(0) = (3, 2, 4)$, and $r''(0) = (0, 3, 1)$ Using the rules for differentiating dot and cross products we have

$$\begin{aligned} f'(t) &= r'(t) \cdot (r'(t) \times r''(t)) + r(t) \cdot (r'(t) \times r''(t))' \\ &= r'(t) \cdot (r'(t) \times r''(t)) + r(t) \cdot (r''(t) \times r''(t)) + r(t) \cdot (r'(t) \times r'''(t)). \end{aligned}$$

Note that the first two terms are 0 and the last term is the same as f since $r''' = r''$, thus $f' = f$ and so we know that f must be of the form $f(t) = Ce^t$.

To find C we need to just plug in $t = 0$ so that

$$C = f(0) = (0, 3, -6) \cdot ((3, 2, 4) \times (0, 3, 1)) = -63.$$

Hence

$$\boxed{f(t) = -63e^t.}$$

6. Find the length of the curve $r(t) = (\cos t, \sin t, \ln \cos t)$, for $0 \leq t \leq \pi/4$.

To find the length we use the formula

$$\begin{aligned} L(0, \pi/4) &= \int_0^{\pi/4} \|r'(t)\| dt \\ &= \int_0^{\pi/4} \sqrt{1 + \tan^2 t} dt = \int_0^{\pi/4} \sec t dt. \end{aligned}$$

The anti-derivative of $\sec t$ is $\ln(\sec t + \tan t)$, Hence by the Fundamental Theorem of Calculus

$$\boxed{L(0, \pi/4) = \ln(\sec(\pi/4) + \tan(\pi/4)) - \ln(\sec 0 + \tan 0) = \ln(\sqrt{2} + 1).}$$

7. Find the unit tangent vector, the unit normal vector, and the curvature at time t of the curve given by $r(t) = (\sin t, e^t, t^2)$.

The unit tangent vector is given by $T(t) = r'(t)/\|r'(t)\|$ where $r'(t) = (\cos t, e^t, 2t)$ and $\|r'(t)\| = \sqrt{\cos^2 t + e^{2t} + 4t^2}$, so that

$$\boxed{T(t) = \frac{1}{\sqrt{\cos^2 t + e^{2t} + 4t^2}} (\cos t, e^t, 2t).}$$

The unit normal vector is given by $N(t) = T'(t)/\|T'(t)\|$, unfortunately T does not have a very nice form and so differentiating it while possible is going to be a mess, therefore I will not continue the calculation here.

To find the curvature we may use the formula $\kappa(t) = \|r'(t) \times r''(t)\|/\|r'(t)\|^3$. We have that $r''(t) = (-\sin t, e^t, 2)$ hence

$$r'(t) \times r''(t) = (2e^t(1-t), 2(t \sin t - \cos t), e^t(\cos t + \sin t)),$$

and

$$\|r'(t) \times r''(t)\| = \sqrt{4e^{2t}(1-t)^2 + 4(t \sin t - \cos t)^2 + e^{2t}(\cos t + \sin t)^2}.$$

Putting this all together we have

$$\boxed{\kappa(t) = \frac{\sqrt{4e^{2t}(1-t)^2 + 4(t \sin t - \cos t)^2 + e^{2t}(\cos t + \sin t)^2}}{(\cos^2 t + e^{2t} + 4t^2)^{3/2}}.} \quad \text{Blegh.}$$