Math 175 - Exam 3, December 2, 2009 Name:________________Key___________________________

Problem 1 (20 points). Evaluate the integral $\int_C xe^y\ ds$ where C is portion of the circle $x^2+y^2=4$ going counterclockwise from $(2,0)$ to $(\sqrt{2},\sqrt{2})$.

We may parameterize C by $r(t)=(2\cos t, 2\sin t)$ for $0\le t\le \pi/4$. We then have

$$\begin{aligned}\int_C xe^y\ ds &= \int_0^{\pi/4} 2\cos t e^{2\sin t}\sqrt{4\sin^2 t+4\cos^2 t}\ dt\\ &= 2[e^{2\sin t}]_{t=0}^{\pi/4} = 2(e^{\sqrt{2}}-1).\end{aligned}$$

Problem 2 (20 points). Evaluate $\iiint_E (x^2+y^2+z^2)^2\ dV$ where E is the region above the xy-plane and in between the spheres $x^2+y^2+z^2=1$ and $x^2+y^2+z^2=4$.

If we convert to spherical coordinates then the region E is given by $E=\{(\rho,\theta,\phi)|1\le\rho\le 2, 0\le \theta<2\pi, 0\le\phi\le\pi/2\}$. Hence the integral becomes

$$\begin{aligned}\int_0^{\pi/2}\int_0^{2\pi}\int_1^2 \rho^4\rho^2\sin\phi\ d\rho\ d\theta\ d\phi &= \int_0^{\pi/2}\sin\phi\ d\phi\int_0^{2\pi}\ d\theta\int_1^2\rho^6\ d\rho\\ &= [-\cos\phi]_0^{\pi/2}2\pi[\rho^7/7]_1^2 = 2\pi(2^7-1)/7 = 254\pi/7.\end{aligned}$$

Problem 3 (20 points). Evaluate $\int_C F\cdot\ d\mathbf{r}$, where $F(x,y,z)=yz\mathbf{i}+(x+1)z\mathbf{j}+((x+1)y+1)\mathbf{k}$ and C is the curve given by $r(t)=(\cos t,\sin t,t)$, for $0\le t\le 2\pi$.

If F is a conservative vector field then for some function f we have $f_x=yz$ hence $f=xyz+g(y,z)$. We also have $(x+1)z=f_y=xz+\partial g/\partial y$ and so $g(y,z)=yz+h(z)$. Therefore $f=xyz+yz+h(z)$. We also have $((x+1)y+1)=f_z=xy+y+h'(z)$ and so $h(z)=z+K$. Hence we have shown that $F=\nabla(xyz+yz+z)$ and so by the Fundamental Theorem of Line Integrals we have

$$\int_C F\cdot\ d\mathbf{r} = f(1,0,2\pi)-f(1,0,0)=2\pi.$$

Problem 4 (20 points). Evaluate the integral $\int_C -y^3\ dx+x^3\ dy$ where C is the curve orineted positively which is made of the line segment from $(0,0)$ to $(\sqrt{2},0)$, the counterclockwise arc on $x^2+y^2=2$ from $(\sqrt{2},0)$ to $(1,1)$, and also the line segment from $(1,1)$ to $(0,0)$.

This is a closed curve and so if we let D be the region enclose by C then we may apply Green's Theorem to conclude

$$\int_C -y^3\ dx+x^3\ dy = \iint_D (3x^2+3y^2)\ dA.$$

By converting to polar coordinates we compute

$$\begin{aligned}\iint_D (3x^2+3y^2)\ dA &= 3\int_0^{\pi/4}\int_0^{\sqrt{2}} r^2 r\ dr\ d\theta\\ &= \frac{3\pi}{4}\frac{1}{4}(\sqrt{2})^4 = \frac{3\pi}{4}.\end{aligned}$$

Problem 5 (20 points). Find the curl and divergence of the vector field $F(x,y,z)=e^x\mathbf{i}+e^{xy}\mathbf{j}+e^{xyz}\mathbf{k}$.

We may plug in the formulas to conclude

$$\text{curl } F = \nabla \times F = xze^{xyz}\mathbf{i} - yze^{xyz}\mathbf{j} + ye^{xy}\mathbf{k}.$$

We also have

$$\text{div } F = \nabla \cdot F = e^x + xe^{xy} + xye^{xyz}.$$

Problem 6 (Extra Credit - 10 points, no partial credit). Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function $f(x, y, z) = ze^{x^2+y^2+z^2}$. Find a vector field G such that div $G = f$ and $G(0, 0, 0) = \mathbf{i} + \mathbf{j}$.

Given a function f, we can always find a vector field H such that div $H = f$ by setting $H = (\int f(x, y, z)\ dz)\mathbf{k}$. In this case this gives $H = (e^{x^2+y^2+z^2}/2)\mathbf{k}$. Note that $H(0, 0, 0) = \frac{1}{2}\mathbf{k}$.

If we add to H any conservative vector field than we still have that the divergence gives us f. In particular we can add the contant (and also conservative) vector field $\mathbf{i} + \mathbf{j} - \frac{1}{2}\mathbf{k}$ to obtain the desired vector field $G = \mathbf{i} + \mathbf{j} + ((e^{x^2+y^2+z^2} 11)/2)\mathbf{k}$.

Of course there are many other solutions as well, but this one is the easiest to write down.