Math 175 - Exam 2, October 28, 2009 Name:_______________Key__________________________

Problem 1 (20 points). Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x,y) = \begin{cases} \frac{x^2}{2x^2+y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$ Draw the contour map (level curves) for f and find where the function is continuous.

If we set $x^2/(2x^2+y^2) = k$ (note that the range of f is $[0,1/2]$ hence we should take $0 \le k \le 1/2$) then by solving for y we have $y = \pm x\sqrt{\frac{1}{k} - 2}$ if $k \neq 0$ and $x = 0$ if $k = 0$. Hence the contours of this function will just be lines going through the origin with slope $\pm\sqrt{\frac{1}{k} - 2}$ where $0 < k \le 1/2$ or the line $x = 0$ if $k = 0$.

Since $x^2/(2x^2+y^2)$ is a rational function it is continuous on its domain $\{(x,y)|(x,y) \neq (0,0)\}$. At the point $(0,0)$ the function is not continuous since if we take a limit along the line $y = 0$ we get $\frac{1}{2} \neq 0 = f(0,0)$. Therefore f is continuous on $\{(x,y)|(x,y) \neq (0,0)\}$.

Problem 2 (20 points). Find an equation for the tangent plane to the surface given by the equation $z = xe^{x(y-1)}$ at the point $(1,1,1)$.

If we set $f(x,y) = xe^{x(y-1)}$ then $f_x = e^{x(y-1)} + x(y-1)e^{x(y-1)}$ and $f_y = x^2e^{x(y-1)}$. Thus the equation of the tangent line at $(1,1,f(1,1))$ is given by $z - 1 = z - f(1,1) = f_x(1,1)(x-1) + f_y(1,1)(y-1) = (x-1) + (y-1)$. Simplifying, we have $z = x + y - 1$.

Problem 3 (20 points). Consider the function $g(x,y,z) = x^2 + y^2 + z^2 - xyz$, where x, y, and z are all functions of s, and t which satisfy the equations $x = s$, $y = t$, and $s^3 + t^3 + z^3 + 6stz = 0$. Find $\partial g/\partial t$ when $s = 1$ and $t = 0$.

First note that when $s = 1$ and $t = 0$ we have $x = 1$, $t = 0$, and $(1)^3 + (0)^3 + z^3 + 6(1)(0)z = 0$ so that $z = -1$. To find $\partial g/\partial t$ we use the chain rule:

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial g}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial t}.$$

We have $\frac{\partial g}{\partial x} = 2x - yz$, $\frac{\partial g}{\partial y} = 2y - xz$, and $\frac{\partial g}{\partial z} = 2z - xy$ so that when $s = 1$ and $t = 0$ we have $\frac{\partial g}{\partial x} = 2$, $\frac{\partial g}{\partial y} = 1$, and $\frac{\partial g}{\partial z} = -2$.

We also have $\frac{\partial x}{\partial t} = 0$ and $\frac{\partial y}{\partial t} = 1$. To find $\frac{\partial z}{\partial t}$ we use implicit differentiation to the function $0 = F(x,y,z) = s^3 + t^3 + z^3 + 6stz = 0$ to conclude

$$\frac{\partial z}{\partial t} = -\frac{F_t}{F_z} = -\frac{3t^2 + 6sz}{3z^2 + 6st},$$

so that when $s = 1$ and $t = 0$ we have $\frac{\partial z}{\partial t} = 2$.

Now we can solve for $\frac{\partial g}{\partial t}$:

$$\frac{\partial g}{\partial t} = (2)(0) + (1)(1) + (-2)(2) = -3.$$

Problem 4 (20 points). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by $f(x,y) = x^3 + y^3 - 3xy$.

(a). Find all local minimums, maximums, and saddle points of f.

(b). Find the absolute minimum and maximum of f when restricted to the domain $D = \{(x, y) \mid 0 \leq y \leq x \leq 4\}$ i.e., D is the region above the x-axis, to the left of the line $x = 4$ and under the line $x = y$.

For part (a) lets find the critical points. If $0 = f_x = 3x^2 - 3y$ and $0 = f_y = 3y^2 - 3x$ (note that f_x and f_y both exists and are continuous everywhere) then we have $y = x^2 = y^4$ so that $y = 0$ in which case $x = 0$, or $y = 1$ in which case $x = 1$. Therefore there are two critical points $(0, 0)$ and $(1, 1)$.

If $D = f_{xx}f_{yy} - (f_{xy})^2 = 36xy - 9$ then at $(0, 0)$ we have $D = -9 < 0$ and so $(0, 0)$ is a saddle point at which point we have $f(0, 0) = 0$. At $(1, 1)$ we have $D = 27 > 0$ and $f_{xx} = 6 > 0$, and so $(1, 1)$ is a local minimum at which point we have $f(1, 1) = -1$.

We therefore have that there are no critical points on the interior of D and so for part (b) we need only to check the boundary. On the line $y = 0$, $0 \leq x \leq 4$ we have $f(x, 0) = x^3$ which is increasing and hence has a minimum of 0 at $(0, 0)$ and a maximum of 64 at $(4, 0)$.

On the line $x = 4$, $0 \leq y \leq 4$ we have $h(y) = f(4, y) = 64 + y^3 - 12y$ so that $h'(y) = 3y^2 - 12$ hence we have a critical point when $y = 2$, where we have $f(4, 2) = 48$, and we also have on the boundary $f(4, 0) = 64$ and $f(4, 4) = 80$.

On the line $x = y$ we have $g(x) = f(x, x) = 2x^3 - 3x^2$ so that $g'(x) = 6x^2 - 6x$ we therefore have a critical point when $x = 0$, or $x = 1$. When $x = 1$ we have $f(1, 1) = -1$, and on the boundary we have $f(0, 0) = 0$ and $f(4, 4) = 80$.

Summarizing, we have

$$f(0, 0) = 0, \quad f(1, 1) = -1, \quad f(4, 4) = 80, \quad f(4, 2) = 48, \quad f(4, 0) = 64.$$

We know that the absolute minimum and maximum on D is among this set so that we have an absolute minimum of -1 at $(1, 1)$ and an absolute maximum of 80 at $(4, 4)$.

Problem 5 (20 points). Find the volume of the solid which is bounded by the surface $z = xye^{x^2y}$, and the planes $x = 0$, $x = 1$, $y = 1$, $y = 2$, and $z = 0$.

We know that the volume of this solid is nothing but the double integral of xye^{x^2y} over the region D which is bounded by the lines in the xy-plane $x = 0$, $x = 1$, $y = 1$, and $y = 2$. Hence the volume is

$$\int_0^1 \int_1^2 xye^{x^2y} \, dy \, dx.$$

We may use Fubini's Theorem to rewrite this as

$$\int_1^2 \int_0^1 xye^{x^2y} \, dx \, dy = \int_1^2 [e^{x^2y}/2]_{x=0}^1 \, dy$$

$$= \frac{1}{2} \int_1^2 (e^y - 1) \, dy = \frac{1}{2}[e^y - y]_{y=1}^2 = \frac{1}{2}(e^2 - e - 1).$$

Problem 6 (Extra Credit - 10 points, no partial credit). Compute $\int_0^1 \int_0^1 f(x, y) dx dy$ where $f(x, y) = \frac{1}{x+y}$. (Hint: You may want to use the fact that $f(x, y) = f(y, x)$.)

The square $[0, 1] \times [0, 1]$ is symmetric about the line $x = y$ as is the function $\frac{1}{x+y}$, hence we half the integral if we just integrate over the triangle given by the lines $y = 0$, $x = 1$, and $y = x$. Thus we may rewrite the above integral as

$$2 \int_0^1 \int_0^x \frac{1}{x + y} \, dy \, dx = 2 \int_0^1 [\ln(x + y)]_{y=0}^x \, dx = 2 \int_0^1 (\ln(2x) - \ln x) \, dx = 2 \int_0^1 \ln 2 \, dx = 2 \ln 2.$$