Math 175 - Exam 1, September 30, 2009 Name:_______________Key___________________________

Problem 1 (20 points). Let three points be given by $P = (2, 1, 5)$, $Q = (-1, 3, 4)$, and $R = (3, 0, 6)$.

(a). Find an equation of the plane which contains P, Q, and R.

(b). Find the area of the triangle PQR.

(a). Both the vectors $PQ = (-3, 2, -1)$ and $PR = (1, -1, 1)$ lie in the plane and hence $n = PQ \times PR = (1, 2, 1)$ is normal to the plane. Since P lies on the plane we may describe the plane as $r \cdot n = P \cdot n = 9$ or $x + 2y + z = 9$.

(b). The area of the triangle PQR is just 1/2 the area of the parallelogram generated by the vectors PQ and PR, hence the area is $\frac{1}{2}\|PQ \times PR\| = \frac{1}{2}\|(1, 2, 1)\| = \frac{\sqrt{6}}{2}$.

Problem 2 (20 points). Consider the plane P given by the equation $x - y + 3z = 1$, and the line L given by $(x, y, z) = (1, 1, 0) + t(3, -1, -1)$, $t \in \mathbb{R}$.

(a). Find the angle between the plane P and the line L.

(b). Find an equation of a parallel plane P' such that the line segment given by the part of L which lies between the two planes P and P' has length $3\sqrt{11}$.

(a). The angle θ between the plane and the line is just $\pi/2$ minus the angle between a vector parallel to the line (for example $v = (3, -1, -1)$) and a normal vector to the plane (for example $n = (1, -1, 3)$). Hence $\cos(\frac{\pi}{2} - \theta) = |v \cdot n|/\|v\|\|n\|$. We have $|v \cdot n| = 1$, $\|v\| = \sqrt{11}$, and $\|n\| = \sqrt{11}$, hence

$$\theta = \frac{\pi}{2} - \cos^{-1}(1/11).$$

(b). The line L intersects P at the point $Q = (4, 0, -1)$, if $x - y + 3z = d$ is a parallel plane P' then L intersects P' at the point $R = (1 + 3d, 1 - d, -d)$, hence the length of the middle line segment is just the distance between these two points, which is

$$d(R, Q) = \sqrt{(3d - 3)^2 + (1 - d)^2 + (-d + 1)^2} = \sqrt{11(1 - d)^2} = \sqrt{11}|1 - d|.$$

This will be $3\sqrt{11}$ when d is either -2, or 4. Hence the plane $x - y + 3z = -2$, or the plane $x - y + 3z = 4$ does the job.

Problem 3 (20 points). Classify and sketch the surface given by the equation $4y^2 + z^2 - x - 16y - 4z + 20 = 0$.

By completing the squares we can rewrite this as

$$4(y - 2)^2 + (z - 2)^2 - x = 0.$$

Looking at the cross-sections we then have an ellipse, a parabola, and a parabola. Hence the surface is an elliptic parabola.

Since the x term is the one which is not squared we have that the elliptic parabola opens up in the direction of the x-axis, also since we have $-x$ instead of x this means that we open up in the positive x direction. Also since we have $(y-2)$, and $(z-2)$ the vertex of our elliptic parabola should be at the point $(0, 2, 2)$.

Problem 4 (20 points).

(a). Write the equations $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = z$ in spherical coordinates.

(b). Describe in words and sketch the two surfaces given by the equations.

(a). Recall the formulas

$$\rho^2 = x^2 + y^2 + z^2, \quad z = \rho \cos \phi,$$

hence $x^2 + y^2 + z^2 = z$ transforms to $\rho^2 = \rho \cos \phi$, or $\rho = \cos \phi$.

Also note

$$x^2 + y^2 = \rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) = \rho^2 \sin^2 \phi,$$

hence $z = \sqrt{x^2 + y^2}$ transforms to $\rho^2 \cos^2 \phi = \rho^2 \sin^2 \phi = \rho^2 (1 - \cos^2 \phi)$, thus $\cos^2 \phi = 1/2$ and so $\phi = \pi/4$ (remember that $0 \leq \phi \leq \pi$).

(b). $\rho = \cos \phi$ gives a sphere of radius $1/2$ centered at $(0, 0, 1/2)$. $\phi = \pi/4$ gives a cone opening up in the direction of the positive z-axis.

Problem 5 (20 points). Let $r : [0, \infty) \to \mathbb{R}^3$ be given as $r(t) = (\frac{4}{3} t^{3/2}, 2t, \frac{1}{2} t^2)$.

(a). Find the length of the resulting curve from the point $(0, 0, 0)$ to $(\frac{32}{3}, 8, 8)$.

(b). Find the curvature of the resulting curve at the point $(\frac{4}{3}, 2, \frac{1}{2})$.

(a). We have $r'(t) = (2t^{1/2}, 2, t)$, and $\|r'(t)\| = \sqrt{4t + 4 + t^2} = |t + 2|$ Thus the arc length is given by

$$\int_0^4 \|r'(u)\| du = \int_0^4 (u + 2) du = \frac{1}{2} 4^2 + 8 = 16.$$

(b). We also have $r''(t) = (t^{-1/2}, 0, 1)$ and so $r'(t) \times r''(t) = (2, -t^{1/2}, -2t^{-1/2})$. Thus

$$\|r'(t) \times r''(t)\| = \sqrt{4 + t + \frac{4}{t}} = (t + 2)/\sqrt{t}.$$

The curvature function is then given by

$$\kappa(t) = \|r'(t) \times r''(t)\| / \|r'(t)\|^3 = 1/\sqrt{t}(t + 2)^2.$$

When $t = 1$ we get

$$\kappa(1) = 1/9.$$

Problem 6 (Extra Credit - 10 points, no partial credit). Suppose that $r : \mathbb{R} \to \mathbb{R}^3$ defines a smooth curve which is parameterized with respect to arc length. Show that $r'(t)$ and $r''(t)$ are always perpendicular. Find an example where this is false if r is not parameterized with respect to arc length.

If r is parameterized with respect to arc length then $\|r'(t)\| = \frac{ds}{dt} = 1$ and hence $r'(t) = T(t)$ lives on a sphere, thus we know from class that it is always perpendicular to its derivative $r''(t)$.

If r is not parameterized with respect to arc length then there are many examples where this can fail. For instance $r(t) = (t^2, 0, 0)$ then $r'(t) = (2t, 0, 0)$ and $r''(t) = (2, 0, 0)$. Then $r'(1) \cdot r''(1) = 4 \neq 0$.