

Problem 1. For the following matrix A , determine if it is diagonalizable and if so diagonalize it, i.e. find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

$$A = \begin{pmatrix} 1 & -1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{pmatrix}.$$

Solution 1. Note that A is an upper triangular matrix and the eigenvalues of A (which are the roots of the characteristic polynomial) are just the diagonal entries of A , i.e. the eigenvalues of A are 1, 0, and -1 . Note that since they are distinct we know that A has 3 linearly independent eigenvectors and thus must be diagonalizable. Furthermore if we find 3 linearly independent eigenvectors then we can choose P to be the matrix whose columns are these vectors and P will satisfy the equation above.

Thus we need to find eigenvectors for A . A is simple enough that we might (correctly) guess the eigenvectors $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, but let's continue without trying to do guess work.

An eigenvector v associated to the eigenvalue 1 is a solution to the equation $(A - I)v = 0$, and thus we just need to solve this equation. We do this by putting the associated matrix into row reduced echelon form:

$$\begin{pmatrix} 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Thus we see that the solution system is given by $\left\{ \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \mid t \in \mathbb{R} \right\}$ and so in particular $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ is an eigenvector.

Similarly, if we solve the equation $(A - 0)v = 0$ we have:

$$\begin{pmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and so $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector.

Also, solving the equation $(A - (-1)I)v = 0$ we have:

$$\begin{pmatrix} 2 & -1 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

and so $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ is an eigenvector.

Setting $P = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$ it is then simple to find $P^{-1} = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$, and we can check that $P^{-1}AP = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, which is diagonal.