Math 196 - Quiz 1, September 4, 2008 Name:__________________Key______________________

Problem 1. A Car accelerating at a constant rate goes from 0 to 60 miles per hour in 6 seconds, how far does the car travel in this time?

Solution 1. If we let $X(t)$, $V(t)$, and $A(t)$ be respectively the position, velocity, and acceleration of the car at time t, then we know that the differential equations that they satisfy are $\frac{dX}{dt} = V$, and $\frac{d^2X}{dt^2} = \frac{dV}{dt} = A$. We know that the acceleration is some constant a_0, so solving these differential equations by integration give us the formulas:

$$X(t) = \frac{1}{2}a_0t^2 + C_1t + C_2,$$

and

$$V(t) = a_0t + C_1,$$

for some constants C_1 and C_2.

The car starts at rest and so we have the initial conditions $X(0) = 0$ and $V(0) = 0$, from which we conclude that $C_1 = C_2 = 0$, therefore $X(t) = \frac{1}{2}a_0t^2 = \frac{1}{2}tV(t)$. We also have that after 6 seconds or $\frac{6}{(60)^2}$ hours the car has reached 60 miles per hour, i.e. $V(\frac{6}{(60)^2}) = 60$. Thus after 6 seconds the car has traveled $X(\frac{6}{(60)^2}) = \frac{1}{2}(\frac{6}{(60)^2})60 = \frac{1}{20}$ miles.

Problem 2. Find a solution (or give a reason why it does not exist) to the differential equation $x^2\frac{dy}{dx} = y + 1$ under each of the following initial conditions:

a) $x = 1, y = 0$
b) $x = 1, y = -2$
c) $x = 2, y = -1$

Solution 2. Note that the function $g(x, y) = \frac{y+1}{x^2}$ and its partial $\frac{\partial g}{\partial y}(x, y) = \frac{1}{x^2}$ are continuous off the line $x = 0$, hence since none of the initial value problems above have $x = 0$ we know that for each of the cases above there locally exists a unique solution to the differential equation.

To solve the differential equation let us separate the variables to get the new equation:

$$\frac{1}{y+1}\frac{dy}{dx} = \frac{1}{x^2},$$

when $y + 1 \neq 0$, and $x \neq 0$, (note when $y + 1 = 0$ we get the singular solution which is the contant function $y = -1$.

Integrating both sides with respect to x gives us

$$\ln|y + 1| = -\frac{1}{x} + C.$$

For a) y + 1 is positive and the initial condition gives us $0 = \ln|0 + 1| = -1 + C$ so $C = 1$, hence exponentiating both sides gives us $y = \exp(-\frac{1}{x} + 1) - 1$.

For b) y + 1 is negative (hence $|y + 1| = -(y + 1)$)and the initial condition gives us $0 = \ln|-2+1| = -1+C$ so again $C = 1$, exponentiating both sides gives us $y = -\exp(-\frac{1}{x} + 1) - 1$.

For c) we have the singular solution $y = -1$.