Homework 3, Section 1.6, Problem 59.

*Problem 1.6.59.*

Solve the differential equation

$$\frac{dy}{dx} = \frac{x - y - 1}{x + y + 3}$$

by finding $h$ and $k$ so that the substitutions $x = u + h$, $y = v + k$ transform it into the homogeneous equation

$$\frac{dv}{du} = \frac{u - v}{u + v}.$$

*Solution.* If $h$, and $k$ are constants and we make the substitution $x = u + h$ and $y = v + k$ then we have that $\frac{du}{dx} = \frac{dv}{dy} = 1$, hence

$$\frac{dv}{du} = \frac{dv}{dy}\frac{dy}{dx}\frac{dx}{du} = \frac{u + h - v - k - 1}{u + h + v + k + 3}.$$

If we wanted this to be equal to $\frac{u-v}{u+v}$ the most natural conditions to look at would be

$$u - v = u + h - v - k - 1,$$

and

$$u + v = u + h + v + k + 3.$$

Hence we need $h - k = 1$ and $h + k = -3$. We see that indeed this is the case if we set $h = -1$ and $k = -2$.

Once we have the equation $\frac{dv}{du} = \frac{u-v}{u+v} = \frac{1-v/u}{1+v/u}$ we may make the substitution $w = v/u$, so that $v = uw$ and

$$w + u\frac{dw}{du} = \frac{dv}{du} = \frac{1 - w}{1 + w},$$

hence by simplifying we have

$$\frac{1 + w}{1 - 2w - w^2}\frac{dw}{du} = \frac{1}{u}.$$

Integrating both sides gives us

$$\frac{-1}{2}\ln(1 - 2w - w^2) = \ln u + C_0.$$

Hence $1 - 2w - w^2 = \frac{C}{u^2}$, for some constant $C$.

Substituting back in $w = v/u$ and multiplying both sides by $u^2$ gives us

$$u^2 - 2uv - v^2 = C.$$

Substituting back for $x$ and $y$ gives us the implicit solution:

$$(y + 2)^2 + 2(y + 2)(x + 1) - (x + 1)^2 = C_1.$$

Note that we can also solve this differential equation using the fact that it is exact.