Homework 2, Section 1.5, Problem 38.

Problem 1.5.38. Consider the cascade of two tanks where $V_1 = 100$ (gal) and $V_2 = 200$ (gal) the volumes of brine in the two tanks. Each tank initially contains 50 lb of salt. The three flow rates are each 5 gal/min, with pure water flowing into tank 1.

(**a**) Find the amount $x(t)$ of salt in tank 1 at time t.

(**b**) Suppose that $y(t)$ is the amount of salt in tank 2 at time t. Show first that

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200},$$

and then solve for $y(t)$, using the function $x(t)$ found in part (a).

(**c**) Finally, find the maximum amount of salt ever in tank 2.

Solution. Given a tank with an initial amount v_0 of solution and such that we are pumping in brine with a concentration c_i of salt at a rate r_i, and such that we are pumping out the solution at a rate of r_o we know that the amount of salt $x(t)$ in the solution at time t is governed by the differential equation:

$$\frac{dX}{dt} = r_i c_i - \frac{r_o X}{v + (r_i - r_o)t}.$$

For the first tank in part (a) we have the values $r_i = r_o = 5$ gal/min, $v = 100$ gal, and $c_i = 0$ since pure water is flowing into tank 1. Thus the differential equation we obtain is

$$\frac{dx}{dt} = -\frac{5x(t)}{100}.$$

By separation of variables we can solve this differential equation.

$$\ln|x| = \int \frac{1}{x}\frac{dx}{dt}dt = \int \frac{-1}{20}dt = \frac{-t}{20} + C,$$

hence $x(t) = |x(t)| = \exp(\frac{-t}{20})C_1$. Initially we have $C_1 = x(0) = 50$ and so the final solution for part (a) is:

$$x(t) = 50\exp(\frac{-t}{20}).$$

For part (b) if $y(t)$ is the amount of salt in tank 2 then $y(t)$ satisfies the same differential equation above however we now have that the concentration c_i of salt in the incoming water changes as a function of time, in fact it is just the total amount of salt $x(t)$ in tank 1 divided by the volume 100 of the solution in tank 1. For tank 2 we check that the other quantities are given by $r_i = r_0 = 5$, $v = 200$, and hence the differential equation we obtain is:

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200} = \frac{5}{2}\exp(\frac{-t}{20}) - \frac{1}{40}y.$$

This is a linear equation $\frac{dy}{dt} + \frac{1}{40}y = \frac{5}{2}\exp(\frac{-t}{20})$ hence if we multiply both sides of the equation by $\exp(\frac{t}{40})$ and use Leibniz's rule we have

$$\frac{d}{dx}(\exp(\frac{t}{40})y) = \exp(\frac{t}{40})\frac{dy}{dx} + \frac{1}{40}\exp(\frac{t}{40})y = \frac{5}{2}\exp(\frac{-t}{40}).$$

Integrating we have

$$\exp(\frac{t}{40})y = \int \frac{5}{2}\exp(\frac{-t}{40})dt = -100\exp(\frac{-t}{40}) + C.$$

Initially we have $y(0) = 50$ and so we see that $50 = -100 + C$ hence $C = 150$, solving for y gives the solution:

$$y = -100\exp(\frac{-t}{20}) + 150\exp(\frac{-t}{40}).$$

For part (c) we want to find the maximum of y, the slope of y at the maximum will be equal to 0 thus we must set the derivative equal to 0 and solve for t. Setting the derivative equal to 0 gives the formula

$$0 = \frac{dy}{dx} = \frac{d}{dx}(-100\exp(\frac{-t}{20}) + 150\exp(\frac{-t}{40})) = 5\exp(\frac{-t}{20}) - \frac{15}{4}\exp(\frac{-t}{40}).$$

Hence simplifying we see $\exp(\frac{-t}{40}) = \frac{3}{4}$, so that the maximum for y is attained at $t = -40\ln(\frac{3}{4}) \approx 11.51$, and hence the maximum is given by

$$y(-40\ln(\frac{3}{4})) = -100(\frac{3}{4})^2 + 150\frac{3}{4} = \frac{225}{4} = 56.25.$$