Math 196 - Exam 3, November 18, 2008 Name:____

Problem 1 (20 points). In each of the following cases, matrices A, and B are given. Determine if there exists a matrix S such that $SA = B$. If so then find such an S, and if not then give a reason why not.

1. $A = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$.

2. $A = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 0 & 3 \\ 1 & 1 & 1 \end{pmatrix}$.

3. $A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 2 & -1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & 1 \\ 2 & -1 & 0 \\ 2 & -1 & 0 \end{pmatrix}$.

Solution 1. 1. If A is invertible then we have $(BA^{-1})A = BI = B$ and so $S = BA^{-1}$ (note that in this case S is unique). For part 1, A is invertible since $\det A \neq 0$ (check this) and so we first find A^{-1}.

$$(A|I) = \left(\begin{array}{ccc|ccc} 2 & 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{array}\right) \overset{\text{R1} \leftrightarrow \text{R3}}{\longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 2 & 0 & -1 & 1 & 0 & 0 \end{array}\right) \overset{\text{R2} - \text{R1}}{\longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 2 & 0 & 1 & -1 \\ 2 & 0 & -1 & 1 & 0 & 0 \end{array}\right)$$

$$\overset{\text{R3} - 2\text{R1}}{\longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -1 & 2 & 0 & 1 & -1 \\ 0 & -2 & -3 & 1 & 0 & -2 \end{array}\right) \overset{-\text{R2}, \text{R3} - 2\text{R2}}{\longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & -7 & 1 & -2 & 0 \end{array}\right)$$

$$\overset{-(1/7)\text{R3}}{\longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1/7 & 2/7 & 0 \end{array}\right) \overset{\text{R1} - \text{R2}}{\longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1/7 & 2/7 & 0 \end{array}\right)$$

$$\overset{\text{R1} - 3\text{R3}}{\longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/7 & 1/7 & 0 \\ 0 & 1 & -2 & 0 & -1 & 1 \\ 0 & 0 & 1 & -1/7 & 2/7 & 0 \end{array}\right) \overset{\text{R2} + 2\text{R3}}{\longrightarrow} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/7 & 1/7 & 0 \\ 0 & 1 & 0 & -2/7 & -3/7 & 1 \\ 0 & 0 & 1 & -1/7 & 2/7 & 0 \end{array}\right) = (I|A^{-1})$$

Hence

$$S = BA^{-1} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & 0 & -1 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 3/7 & 1/7 & 0 \\ -2/7 & -3/7 & 1 \\ -1/7 & 2/7 & 0 \end{pmatrix} = \begin{pmatrix} 1/7 & 5/7 & 0 \\ 1 & 0 & 0 \\ 3/7 & 1/7 & 0 \end{pmatrix}$$

2. Here we have $\det A = 0$, and so if $SA = B$ then $\det B = \det(SA) = \det S \det A = 0$, however $\det B \neq 0$, and so no such S exists.

3. In this case we have $\det A = \det B = 0$ and so the argument in part 2 does not work. In fact we notice that B is obtained from A by adding the third row to the second row, and we know that this elementary row operation corresponds to multiplication on the left by the matrix $S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$.

Problem 2 (10 points). Find a basis for each of the eigenspaces of the matrix $A = \begin{pmatrix} 2 & 0 & -1 & 1 \\ 0 & 2 & 1 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

Solution 2. As A is upper-triangular we see immediately that the characteristic polynomial is $\chi_A(x) = \det(A - xI) = (2-x)^2(1-x)^2$ and hence the eigenvalues of A are 2 and 1. The eigenspace corresponding to 2 is the space of all solutions to the equation $(A - 2I)v = 0$. By performing Gaussian elimination on the matrix $(A - 2I)$ we find the the solution set:

$$V_2 = \left\{ \begin{pmatrix} t \\ s \\ 0 \\ 0 \end{pmatrix} \middle| t, s \in \mathbb{R} \right\}.$$

Hence a basis for the eigenspace V_2 is given by:

$$\beta_2 = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

The eigenspace corresponding to 1 is the space of all solutions to the equation $(A - I)v = 0$. By performing Gaussian elimination on the matrix $(A - I)$ we find the solution set:

$$V_1 = \left\{ \begin{pmatrix} t \\ -t \\ t \\ 0 \end{pmatrix} \middle| t \in \mathbb{R} \right\}.$$

Hence a basis for the eigenspace V_1 is given by:

$$\beta_1 = \left\{ \begin{pmatrix} 1 \\ -1 \\ 1 \\ 0 \end{pmatrix} \right\}.$$

Problem 3 (20 points). For each of the cases below determine if the matrix A is diagonalizable with real coefficients. (You do not need to actually find a P such that $P^{-1}AP$ is diagonal, but you do need to give a reason why such a P does or does not exist.)

1. $A = \begin{pmatrix} \sqrt{2} & \pi \\ \pi & \sqrt{2} \end{pmatrix}$

2. $A = \begin{pmatrix} 3 & 2 & -1 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$

3. $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

4. $A = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 0 & 2 & 3 & 4 & 5 & 6 \\ 0 & 0 & 3 & 4 & 5 & 6 \\ 0 & 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 0 & 0 & 5 & 6 \\ 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix}$

Solution 3. 1. The characteristic polynomial for A is $\chi_A(x) = \det(A - xI) = (\sqrt{2} - x)^2 - \pi^2 = (\sqrt{2} - x + \pi)(\sqrt{2} - x - \pi)$, hence A is a 2×2 matrix which has two distinct eigenvalues and hence must have 2 linearly independent eigenvectors and so is diagonalizable.

2. Here we have $\chi_A(x) = \det(A - xI) = (3-x)(-x(1-x)-0) - 2(0-0) + (-1)(0-(-x)) = -x^3 + 4x^2 - 4x = x(x-2)^2$, hence A has a repeated eigenvalue for 2. The eigenspace associated to the eigenvalue 2 is the space of solutions to the equation $(A - 2I)v = 0$. Solving this by Gaussian elimination we see that the eigenspace is:

$$V_2 = \left\{ \begin{pmatrix} t \\ 0 \\ t \end{pmatrix} \middle| t \in \mathbb{R} \right\}.$$

Since this space is 1-dimensional we see that A has only 2 linearly independent eigenvectors and hence is not diagonalizable.

3. Here A is upper triangular and hence the eigenvalues are the diagonal entries, i.e. A has only one eigenvalue and it is 1. Solving the equation $(A - I)v = 0$ we see that the corresponding eigenspace is:

$$V_1 = \left\{ \begin{pmatrix} t \\ 0 \\ 0 \end{pmatrix} \middle| t \in \mathbb{R} \right\}.$$

Hence just as in part 2 we see that A is not diagonalizable.

4. Here A is again upper triangular and so the eigenvalues are the diagonal entries. Since they are distinct we know that A must have six eigenvectors corresponding to different eigenvalues (and hence linearly independent). Thus A is diagonalizable.

Problem 4 (15 points). Find the solution to the homogeneous linear differential equation $y'' - 2y' + 2y = 0$ which satisfies the initial conditions $y(0) = 1, y'(0) = 0$.

Solution 4. The associated polynomial to this differential equation is $p(x) = x^2 - 2x + 2$. By the binomial formula we see that the roots of p are $1 + i$ and $1 - i$. Therefore we know that the general solution to this equation is given by:

$$y = c_1 e^x \sin x + c_2 e^x \cos x.$$

Substituting in the initial conditions we have $1 = y(0) = c_2$, and $0 = y'(0) = c_1 + c_2$, so that $c_1 = -c_2 = -1$. Thus the solution is given by:

$$y = e^x (\cos x - \sin x).$$

Problem 5 (20 points). Consider the homogeneous linear differential equaiton $y^{(3)} - xy'' + 9y' - 9xy = 0$ together with solutions $f_1(x) = \sin 3x$, $f_2(x) = \cos 3x$, $f_3(x) = 3\sin x - 4\sin^3 x$. Determine whether or not f_1, f_2, and f_3 are linearly independent. Show your work.

Solution 5. As f_1, f_2, and f_3 are solutions to a homogeneous linear differential equation we know that they will be linearly dependent if and only if the Wronskian is identically zero, which happens if and only if the Wronskian is zero at some point.

The Wronskian is given by:

$$W(x) = \det \begin{pmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{pmatrix} = \det \begin{pmatrix} \sin 3x & \cos 3x & 3\sin x - 4\sin^3 x \\ 3\cos 3x & -3\sin 3x & 3\cos x - 12\sin^2 x \cos x \\ -9\sin 3x & -9\cos 3x & -3\sin x - 24\sin x\cos^2 x + 12\sin^3 x \end{pmatrix}.$$

Hence

$$W(0) = \det \begin{pmatrix} 0 & 1 & 0 \\ 3 & 0 & 3 \\ 0 & 9 & 0 \end{pmatrix} = 0,$$

and so the functions are linearly dependent.

In fact $f_1 = f_2$ by the triple angle formula:

$$\sin 3x = 3\sin x - 4\sin^3 x.$$

To see this formula just use Euler's identity $e^{ix} = \cos x + i\sin x$ and then see what happens when you cube both sides and equate the imaginary parts.

Problem 6 (15 points). Find the general solution to linear differential equation

$$y^{(3)} + 3y'' + 3y' + y = 2\cos x - 2\sin x$$

knowing that a particular solution is given by $y_0(x) = \sin x$.

Solution 6. The general solution is just the sum of a particular solution and the general solution of the associated homogeneous equation. Thus we need to find the general solution to:

$$y^{(3)} + 3y'' + 3y' + y = 0.$$

The associated polynomial is $x^3 + 3x^2 + 3x + 1 = (x + 1)^3$ and so we know the general solution to the homogeneous equation is given by

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x}.$$

Hence since we are given a particular solution $y_0(x) = \sin x$ we know that the general solution to the above differential equation is given by

$$y(x) = c_1 e^{-x} + c_2 x e^{-x} + c_3 x^2 e^{-x} + \sin x,$$

where c_1, c_2, and c_3 are constants.