Name:____

Problem 1 (15 points). Find all solutions to the following system of linear equations, check your work:

$$x_1 + x_2 - x_3 = 0$$

$$2x_2 - 2x_3 = -2$$

$$x_1 - x_2 + x_3 = 2.$$

Solution 1. Let's perform Gaussian elimination on the associated matrix.

$$\begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & -2 \\ 1 & -1 & 1 & 2 \end{pmatrix} \overset{\text{R3} - \text{R1}}{\longrightarrow} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & -2 \\ 0 & -2 & 2 & 2 \end{pmatrix} \overset{\text{R3} - \text{R2}}{\longrightarrow} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 2 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\overset{\frac{1}{2}\text{R2}}{\longrightarrow} \begin{pmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \overset{\text{R1} - \text{R2}}{\longrightarrow} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore x_3 is a free variable and we can then read off the solutions:

$$\left\{ \begin{pmatrix} 1 \\ t-1 \\ t \end{pmatrix} |t \in \mathbb{R} \right\}$$

Problem 2 (10 points).

1. Show that the set of functions $y(x)$ which satisfy the differential equation $y'' - 2y' + y = 0$ form a subspace of the vector space of all functions.

2. Using part 1 and the fact that the functions $y_1(x) = e^x$ and $y_2(x) = xe^x$ are solutions to the above differential equation, find a solution $y(x)$ such that $y(0) = 1$ and $y(1) = 0$.

Solution 2.

1. Let V be the set of solutions to the above differential equation. If $y = 0$ then $y = y' = y'' = 0$ and so

$$y'' - 2y' + y = 0 - 2(0) + 0 = 0,$$

hence $0 \in V$.

If $y_1, y_2 \in V$ then

$$\begin{aligned}(y_1 + y_2)'' - 2(y_1 + y_2)' + (y_1 + y_2) &= y_1'' + y_2'' - 2y_1' - 2y_2' + y_1 + y_2 \\ &= y_1'' - 2y_1' + y_1 + y_2'' - 2y_2' + y_2 = 0 + 0 = 0,\end{aligned}$$

hence $y_1 + y_2 \in V$.

If $y \in V$ and $a \in \mathbb{R}$ then

$$\begin{aligned}(ay)'' - 2(ay)' + (ay) &= ay'' - 2ay' + ay \\ &= a(y'' - 2y' + y) = a(0) = 0,\end{aligned}$$

hence $ay \in V$.

Thus since V contains 0 and is closed under addition and scalar multiplication it is indeed a subspace.

2. Since the set of solutions form a subspace, any linear combination of solutions is again a solution. In particular since both e^x and xe^x are solutions we have that $y(x) = ae^x + bxe^x$ is a solution $\forall a, b \in \mathbb{R}$.

Using the conditions above we see that $1 = y(0) = a$ and $0 = y(1) = (a + b)e$, hence $a = 1$ and $b = -a = -1$. Therefore the solution we are looking for is

$$y(x) = e^x - xe^x.$$

Problem 3 (20 points). Find a basis and calculate the dimension for each of the following vector spaces V:

1. $V = \mathbb{R}^5$.

2. $V = \{0\}$.

3. V is the set of all 2×2 matrices A such that $A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

4. V is the set of polynomials p of degree at most 3 such that $p(0) = p(1) = 0$.

5. V is the solution space to the following homogeneous system of linear equations:

$$x + z = 0$$
$$y - z = 0$$
$$x - y + 2z = 0.$$

Solution 3. Remember that the dimension of a vector space is just the number of elements in a basis.

1. A basis for $\mathbb{R}^5$ is given by $\beta = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$, hence $\dim V = 5$.

2. A basis for $\{0\}$ is given by $\beta = \emptyset = \{\}$, hence $\dim V = 0$.

3. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $\begin{pmatrix} 0 \\ 0 \end{pmatrix} = A\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} a+b \\ c+d \end{pmatrix}$. Thus $a = -b$ and $c = -d$.

We then have that $V = \left\{ \begin{pmatrix} t & -t \\ s & -s \end{pmatrix} | t, s \in \mathbb{R} \right\}$, a basis for V is then given by $\beta = \left\{ \begin{pmatrix} 1 & -1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix} \right\}$, hence $\dim V = 2$.

4. A polynomial $p(x)$ has a root at 0 and 1 if and only if $p(x) = x(x-1)q(x)$ for some polynomial $q(x)$ where the degree of $q(x)$ is 2 less than the degree of $p(x)$. Hence our vector space V is realized as:

$$V = \{x(x-1)(ax+b) | a, b \in \mathbb{R}\}.$$

A basis for V is then given by $\{x(x-1), x^2(x-1)\}$, hence $\dim V = 2$.

5. Let's solve this system of linear equations.

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & 2 & 0 \end{pmatrix} \overset{\text{R3} - \text{R1}}{\longrightarrow} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix} \overset{\text{R3} + \text{R2}}{\longrightarrow} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore the solution space is just

$$V = \left\{ \begin{pmatrix} -t \\ t \\ t \end{pmatrix} | t \in \mathbb{R} \right\}.$$

A basis for V is then given by $\beta = \left\{ \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} \right\}$, hence $\dim V = 1$.

Problem 4 (20 points). Determine whether or not the following set forms a basis for $\mathbb{R}^3$, justify your answer.

$$\left\{ \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix}, \begin{pmatrix} 4 \\ 8 \\ 5 \end{pmatrix} \right\}.$$

Solution 4. If we look at the equation $a \begin{pmatrix} 1 \\ 5 \\ -1 \end{pmatrix} + b \begin{pmatrix} 2 \\ 6 \\ 1 \end{pmatrix} + c \begin{pmatrix} 4 \\ 8 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ and try to solve this by Gaussian elimination then we find that there is a non-trivial solution.

Specifically we have that $a = 2$, $b = -3$, and $c = -1$ is a non-trivial solution. Hence the vectors are not linearly independent and hence do not form a basis.

Problem 5 (20 points). Determine wether or not the following are vector spaces, if they are then calculate the dimension, if they are not then give a reason why not:

1. V is the set of points $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ such that $x = y$ with the vector space structure coming from $\mathbb{R}^2$.

2. V is the set of points $\begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3$ such that $xy = yz$ with the vector space structure coming from $\mathbb{R}^3$.

3. V is the set of convergent sequences $\{x_n\}_n$ with the zero vector given by the zero sequence $\vec{0} = \{0\}_n$, addition of vectors given by entrywise addition $\{x_n\}_n \oplus \{y_n\}_n = \{x_n + y_n\}_n$, and scalar multiplicatoin given by entrywise multiplication $a \cdot \{x_n\}_n = \{ax_n\}_n$.

4. V is the set of positive real numbers $r > 0$ with the zero vector given by $\vec{0} = 1$, addition of vectors given by $r_1 \oplus r_2 = r_1 r_2$, and scalar multiplication given by $a \cdot r = r^a$.

5. V is the set of all linear operators $T : \mathbb{R}^2 \to \mathbb{R}$ with the zero vector given by the constant zero function $\vec{0} = 0$, addition of vectors given by pointwise addition $(T \oplus S)(v) = Tv + Sv$, and scalar multiplication given by pointwise multiplication $(a \cdot T)(v) = a(Tv)$.

Solution 5.

1. V is the solution set of a homogeneous system of linear equations hence it is a subspace of $\mathbb{R}^2$ and in particular is a vector space with basis given by $\left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}$, so that $\dim V = 1$.

2. $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ are both in V but the vector $\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ is not in V. Hence V is not closed under addition and is therefore not a vector space. (It does contain 0 and it is closed under scalar multiplication though).

3. Sequences are just functions from $\mathbb{N}$ to $\mathbb{R}$ and hence the set of all sequences form a vector space under the above operations. The set of convergent sequences contains the 0 sequence and is closed under addition and scalar multiplication hence forms a subspace, thus V is indeed a vector space.

If for each $k \in \mathbb{N}$ we look at the sequence which has $x_n = 0$, $\forall n \in \mathbb{N}$, $n \neq k$, and $x_k = 1$ then it is easy to see that these sequences form a linearly independent set. Thus V has no finite basis since it has an infinite linearly independent set and so $\dim V = \infty$.

4. Consider the map $\ln : V \to \mathbb{R}$, then by the properties of the natural logarithm we see that ln transforms the funky vector space structure on the positive reals to the usual vector space structure on $\mathbb{R}$. Hence V is indeed a vector space which is isomorphic to $\mathbb{R}$ and hence $\dim V = 1$.

5. As the addition and scalar multiplication on V is defined in terms of the addition and scalar multiplication on $\mathbb{R}^2$ it is not too difficult to see that the vector space properties are all satisfied. Thus V is indeed a vector space (in fact it is called the dual space of $\mathbb{R}^2$). Since linear operators preserve the vector space structure, any linear operator is completely determined by what it does on a basis. Therefore since $\mathbb{R}^2$ is 2 dimensional this causes V to be 2 dimensional also.

Problem 6 (15 points). Recall from calculus the power series expansions for e^x, $\sin x$, and $\cos x$:

$$e^x = \Sigma_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots,$$

$$\sin x = \Sigma_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$$

$$\cos x = \Sigma_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots.$$

Consider the matrix $A = \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}$.

1. Calculate A, A^2, A^3 and A^4.

2. Given a natural number $n \in \mathbb{N}$ calculate A^{2n} and A^{2n+1}.

3. Calculate $e^A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots$.

Solution 6.

1. $A = \begin{pmatrix} 0 & \pi \\ -\pi & 0 \end{pmatrix}$, $A^2 = \begin{pmatrix} -\pi^2 & 0 \\ 0 & -\pi^2 \end{pmatrix}$, $A^3 = \begin{pmatrix} 0 & -\pi^3 \\ \pi^3 & 0 \end{pmatrix}$, and $A^4 = \begin{pmatrix} \pi^4 & 0 \\ 0 & \pi^4 \end{pmatrix}$.

2. $A^2 n = (-1)^n \pi^{2n} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $A^{2n+1} = AA^{2n} = (-1)^n \pi^{2n+1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

3. From part 2 and the formulas above we have

$$e^A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 + \frac{1}{4!}A^4 + \cdots$$

$$= (1 - \frac{\pi^2}{2!} + \frac{\pi^4}{4!} - \frac{\pi^6}{6!} + \cdots) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + (\pi - \frac{\pi^3}{3!} + \frac{\pi^5}{5!} - \frac{\pi^7}{7!} + \cdots) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$= \cos \pi \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sin \pi \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Note that if we set $i = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ then what we have shown is the familiar Euler identity:

$$e^{i\pi} + 1 = 0.$$