

Character rigidity for lattices in higher-rank groups

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www.math.vanderbilt.edu/~peters10/ncgoa2016slides.pdf

www.math.vanderbilt.edu/~peters10/ViennaLecture.pdf

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Characters

Definition

Let Γ be a discrete group.

A **character** on Γ is a function $\tau : \Gamma \rightarrow \mathbb{C}$ such that

- $\tau(e) = 1$.
- $\tau(ghg^{-1}) = \tau(h)$.
- $[\tau(g_j^{-1}g_i)]$ is non-negative definite for $g_1, \dots, g_n \in \Gamma$.

τ is **extremal** if it is an extreme point in the convex space of characters.

Examples

- $\pi : \Gamma \rightarrow U(n)$ irreducible, then $\tau(g) = \frac{1}{n} \text{Tr}(\pi(g))$ is an extremal character. (These are almost periodic, i.e., $\{L_g(\tau) \mid g \in G\}$ is uniformly pre-compact in $\ell^\infty \Gamma$).
- Γ virtually abelian iff every extremal character is almost periodic. (Thoma '64).
- δ_e is an extreme character iff Γ has infinite non-trivial conjugacy classes.

Theorem (P; Conjectured by Connes, early 1980's)

Suppose G is a higher-rank simple Lie group with trivial center (e.g., $G = \mathrm{PSL}_n(\mathbb{R})$, $n \geq 3$), and $\Gamma < G$ is a lattice, then every extremal character is either almost periodic or else δ_e .

GNS-construction

- $\tau : \Gamma \rightarrow \mathbb{C}$ a character.
- $\langle \xi, \eta \rangle = \sum_{g \in \Gamma} \tau(g) (\overline{\eta} * \xi)(g)$, $\xi, \eta \in \mathbb{C}\Gamma$.
- $\pi : \Gamma \rightarrow \mathcal{U}(\overline{\mathbb{C}\Gamma})$.
- $\tau(g) = \langle \pi(g)\delta_e, \delta_e \rangle$, extends to a trace on $L(\Gamma, \tau) = \overline{\mathrm{sp}}(\pi(\Gamma))$.

- $L(\Gamma, \tau)$ is a factor iff τ is extremal.
- $L(\Gamma, \tau)$ is type I iff τ is almost periodic.
- $L\Gamma = L(\Gamma, \delta_e)$.

Theorem (P; Conjectured by Connes, early 1980's)

Suppose G is a higher-rank simple Lie group with trivial center, and $\Gamma < G$ is a lattice, then every extremal character is either almost periodic or else δ_e .

Theorem (Equivalent formulation)

Suppose G is a higher-rank simple Lie group with trivial center, and $\Gamma < G$ is a lattice, then $\Gamma < \mathcal{U}(L\Gamma)$ is **Operator Algebraic Superrigid**:

- If M is a finite factor;
- $\pi : \Gamma \rightarrow \mathcal{U}(M)$ a homomorphism such that $\pi(\Gamma)'' = M$,

then either

- $\overline{\pi(\Gamma)}$ is compact;
- π extends to an isomorphism $\tilde{\pi} : L\Gamma \rightarrow M$.

Theorem (Margulis '77)

Suppose G is a higher-rank simple Lie group with trivial center, and $\Gamma < G$ is a lattice, then $\Gamma < G$ is *superrigid*:

- If H is a simple Lie group;
- $\pi : \Gamma \rightarrow H$ is a homomorphism such that $\pi(\Gamma)$ is Zariski dense,

then either

- $\overline{\pi(\Gamma)} = H$ is compact;
- π extends to a homomorphism $\tilde{\pi} : G \rightarrow H$.

Operator Algebraic superrigidity

Theorem (Equivalent formulation)

Suppose G is a higher-rank simple Lie group with trivial center, and $\Gamma < G$ is a lattice, then $\Gamma < \mathcal{U}(L\Gamma)$ is *Operator Algebraic Superrigid*:

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Theorem (Segal-von Neumann '50)

If G is a simple real Lie group then there is no non-trivial continuous homomorphism into a finite von Neumann algebra.

Equivalently, there is no non-constant continuous character.

Just infinite groups

OA-superrigidity



just infinite (non-trivial normal subgroups are finite index).

Proof.

- If $\Sigma \triangleleft \Gamma$, consider $\tau(g) = 1_\Sigma(g) = \begin{cases} 1 & \text{if } g \in \Sigma; \\ 0 & \text{otherwise.} \end{cases}$
- Or consider $\lambda_\Sigma : \Gamma \rightarrow \mathcal{U}(L(\Gamma/\Sigma))$. □

Theorem (Margulis normal subgroup theorem '79, '80; Kazhdan '67)

Irreducible lattices in higher rank groups are just infinite.

Theorem (Bader-Shalom '06, Shalom '00)

Most irreducible lattices in products of simple groups are just infinite.

OA-superrigidity



Every ergodic p.m.p. action on a diffuse space is free.

Proof.

- Suppose $\Gamma \curvearrowright (X, \mu)$ ergodic p.m.p.
- $\text{Stab}: X \rightarrow \text{Sub}(\Gamma)$, $\nu = \text{Stab}_* \mu$ gives an **invariant random subgroup**.
- Consider $\tau(g) = \mathbb{P}(g \in \nu) = \int 1_\Sigma(g) d\nu(\Sigma)$.
- Or consider $\Gamma \rightarrow [\mathcal{R}_{\Gamma \curvearrowright X}] \subset \mathcal{U}(L(\mathcal{R}_{\Gamma \curvearrowright X}))$. □

Theorem (Stuck-Zimmer '94, Creutz-P '12)

For irreducible lattices in G where every factor of G is higher-rank, then every ergodic p.m.p. action on a diffuse space is free.

Previous results for lattices:

Theorem (Bekka '07)

$\Gamma = \mathrm{PSL}_n(\mathbb{Z})$, $n \geq 3$, is OA-superrigid.

Theorem (P-Thom '13)

$\Gamma = \mathrm{PSL}_2(\mathbb{Z}[\sqrt{2}])$ is OA-superrigid.

Theorem (Creutz-P '13)

If $\Gamma < G = \mathrm{PSL}_n(\mathbb{R}) \times \mathrm{PSL}_n(\mathbb{Q}_p)$, $n \geq 3$, p a prime, is an **arbitrary irreducible lattice** then Γ is OA-superrigid.

Kazhdan's property (T)

Definition

- Γ has *property (T)* if almost invariant vectors \implies invariant vectors.
- If $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$, and $\xi_n \in \mathcal{H}$, $\|\xi_n\| = 1$, $\|\pi(g)\xi_n - \xi_n\| \rightarrow 0$, for $g \in \Gamma$.
- Then there exists $\eta \in \mathcal{H}$, $\eta \neq 0$, such that $\pi(g)\eta = \eta$ for $g \in \Gamma$.

Kazhdan '67

- Lattices in higher-rank simple groups have property (T).
- Property (T) passes to quotients.

Amenability (Von Neumann '29)

Definition

- Γ is **amenable** if there is an invariant state on $\ell^\infty\Gamma$.
- Equivalently (Følner '55) there exists $F_n \subset \Gamma$ finite such that $\frac{|F_n \Delta gF_n|}{|F_n|} \rightarrow 0$ for all $g \in \Gamma$.

Note

- Γ is finite iff Γ is both amenable and has property (T).

Proof.

- Amenable implies $\ell^2\Gamma$ has almost invariant vectors.
- Property (T) then implies $\ell^2\Gamma$ has a non-zero invariant vector. \square

Margulis' strategy

Theorem (Margulis normal subgroup theorem '79; Kazhdan '67)

Lattices in higher rank simple groups with trivial center are just infinite.

Outline.

- Suppose $\Sigma \triangleleft \Gamma$, is a non-trivial normal subgroup.
- Γ/Σ has property (T). (Kazhdan '67)
- Take P the minimal parabolic subgroup. Then P is amenable and so $\Gamma \curvearrowright G/P$ is **amenable**. (Zimmer '77)

I.e., there exists an invariant conditional expectation

$$E : L^\infty((G/P) \times (\Gamma/\Sigma)) \rightarrow L^\infty(G/P).$$

- Σ acts trivially on the range $E(\ell^\infty(\Gamma/\Sigma))$.
But $\Sigma \curvearrowright G/P$ is ergodic (**Margulis factor theorem**), hence $E|_{\ell^\infty(\Gamma/\Sigma)}$ is an invariant mean, and so Γ/Σ is amenable. \square

Amenable von Neumann algebras

Definition

A von Neumann algebra $B \subset \mathcal{B}(\mathcal{H})$ is **amenable** (or *injective*) if there exists a conditional expectation $E : \mathcal{B}(\mathcal{H}) \rightarrow B$.

Theorem (Schwartz '63)

If H is an amenable group and $\sigma : H \rightarrow \text{Aut}(B)$ with B amenable, then $B^H = \{x \in B \mid \sigma_h(x) = x, h \in H\}$ is amenable.

Corollary (Zimmer '77)

If $P < G$ is an amenable subgroup, $\Gamma < G$ a lattice, and $\pi : \Gamma \rightarrow \mathcal{U}(\mathcal{H})$, then $B = L^\infty(G/P; \mathcal{B}(\mathcal{H}))^\Gamma$ is amenable

Proof.

$B \cong L^\infty(G/\Gamma; \mathcal{B}(\mathcal{H}))^P$ for an induced action of P . □

Von Neumann algebras with property (T)

Definition

A finite factor M has **property (T)** if every Hilbert bimodule having almost central vectors has a non-zero central vector.

Theorem (Connes-Jones '85)

If Γ has property (T), M is a finite factor and $\pi : \Gamma \rightarrow \mathcal{U}(M)$ such that $\pi(\Gamma)'' = M$, then M has property (T).

(If π is the left-regular representation then also the converse holds.)

Note

- A finite factor M is finite dimensional iff M is both amenable and has property (T).

Theorem

Lattices in higher rank simple groups with trivial center are operator algebraic superrigid.

Outline.

- Suppose $\pi : \Gamma \rightarrow \mathcal{U}(M)$ is a finite factor representation which does not extend to $L\Gamma$.
- M has property (T). (Kazhdan '67, Connes-Jones '85)
- Take P the minimal parabolic subgroup. Then $B = L^\infty(G/P; \mathcal{B}(L^2M))^\Gamma$ is amenable. (Note, if π were the left-regular representation then $B \cong L^\infty(G/P) \rtimes \Gamma$.)
- We show that $M = B$ (**Ergodicity type result**), and so M is amenable. □

Theorem (P)

Suppose

- G_1, G_2 compactly generated with trivial amenable radical, $G = G_1 \times G_2$,
- $\Gamma < G$ a lattice,
- If $H \triangleleft G$ is any proper normal subgroup then $H \cap \Gamma = \{e\}$,

then for any finite factor representation $\pi : \Gamma \rightarrow \mathcal{U}(M)$

- either M is amenable, or π extends to an isomorphism $\tilde{\pi} : L\Gamma \rightarrow M$.

Sketch for $G = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$.

- Suppose $\pi : \Gamma \rightarrow \mathcal{U}(M)$ is a factor representation.
- $SL_2(\mathbb{R}) \curvearrowright \mathbb{R}(= SL_2(\mathbb{R})/P)$ by linear fractional transformations.
- Set $B = L^\infty(\mathbb{R}^2; \mathcal{B}(L^2 M))^\Gamma$, and $B_2 = L^\infty(\mathbb{R}; \mathcal{B}(L^2 M))^\Gamma$,
(where, for B_2 , Γ factors through the second copy of $SL_2(\mathbb{R})$).

Theorem (P)

Suppose

- G_1, G_2 non-compact, $G = G_1 \times G_2$,
- $\Gamma < G$ a lattice,
- If $G \curvearrowright (M, \tau)$ is ergodic, then $\Gamma \curvearrowright (M, \tau)$ is properly outer, then for any finite factor representation $\pi : \Gamma \rightarrow \mathcal{U}(M)$
- either M is amenable, or π extends to an isomorphism $\tilde{\pi} : L\Gamma \rightarrow M$.

Sketch for $G = SL_2(\mathbb{R}) \times SL_2(\mathbb{R})$.

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- Set $B = L^\infty(\mathbb{R}^2; \mathcal{B}(L^2 M))^\Gamma$, and $B_2 = L^\infty(\mathbb{R}; \mathcal{B}(L^2 M))^\Gamma$, (where, for B_2 , Γ factors through the second copy of $SL_2(\mathbb{R})$).

Lemma

If $f \in B = L^\infty(\mathbb{R}; L^\infty(\mathbb{R}; \mathcal{B}(L^2 M)))^\Gamma$, and f_0 is in the (SOT)-essential range of f , then there exists $\tilde{f} \in B_2$, such that $P_{\hat{1}} f_0 P_{\hat{1}} = P_{\hat{1}} \tilde{f} P_{\hat{1}}$.

Proof.

- **Lebesgue density:** If $E \subset R$ has positive measure, then there exists $\{\gamma_n\} \subset \Gamma$, such that $\gamma_n E \rightarrow R$ in measure, and $p_2(\gamma_n) \rightarrow e$.
- Let $E_n \subset \mathbb{R}$ be positive measure such that $f|_{E_n} \sim f_0$.
- Take $\gamma_n \in \Gamma$ so that $\gamma_n E_n \rightarrow \mathbb{R}$ in measure and $p_2(\gamma_n) \rightarrow e$.
- Take \tilde{f} to be any wot-cluster point of $\{\pi(\gamma_n) f \pi(\gamma_n^{-1})\}$.
- Using that $\pi(\gamma_n) P_{\hat{1}} = J \pi(\gamma_n^{-1}) J P_{\hat{1}}$ check that \tilde{f} works. □

Ergodicity type results

Sketch continued.

- Suppose $\gamma_0 \in \Gamma \setminus \{e\}$, such that $\tau(\pi(\gamma_0)) \neq 0$. To complete the proof we must show that $B_2 = M$.
- For each open set $O \subset G_1$ set $\mathcal{K}_O = \overline{\text{co}}\{\pi(\lambda\gamma_0\lambda^{-1}) \mid \lambda \in p_1(\Gamma) \cap O\}$. Set $\mathcal{K} = \bigcap_{\{O \text{ nbhd of } e\}} \mathcal{K}_O$.
- Then $\alpha = \tau(\pi(\gamma_0))$ is the unique element of minimal $\|\cdot\|_2$ in \mathcal{K} . (**Convexity argument à la Popa's intertwining, etc.**)
- For $\sigma_{\gamma_0}^0 \in \mathcal{U}(L^2(\mathbb{R}))$ we have

$$\begin{aligned}\sigma_{\gamma_0}^0 \otimes \alpha &\sim \sum \sigma_{\gamma_0}^0 \otimes \alpha_i J\pi(\lambda_i\gamma_0\lambda_i^{-1})J \\ &\sim \sum \alpha_i \sigma_{\lambda_i\gamma_0\lambda_i^{-1}}^0 \otimes J\pi(\lambda_i\gamma_0\lambda_i^{-1})J \in B'_2.\end{aligned}$$

- Hence, for any g in the closure of the normal subgroup of Γ generated by γ_0 , we have $\sigma_g^0 \otimes 1 \in B'_2$. (By hypothesis this is all of $SL_2(\mathbb{R})$).
- By ergodicity of $SL_2(\mathbb{R}) \curvearrowright \mathbb{R}$ we conclude that $B_2 = M$. □