Character rigidity for lattices and commensurators

Darren Creutz and Jesse Peterson

Abstract

We prove an operator algebraic superrigidity statement for homomorphisms of irreducible lattices, and also their commensurators, in certain higher-rank groups into unitary groups of finite factors. This extends the authors' previous work regarding non-free measure-preserving actions, and also answers a question of Connes for such groups.

1 Introduction

A seminal result in the theory of semisimple groups and their lattices is Margulis' superrigidity theorem [Mar75b]: Let Γ be an irreducible lattice in a center free higher-rank semisimple group G with no compact factors, let H be a simple algebraic group over a local field and let $\pi : \Gamma \to H$ be a homomorphism whose image is Zariski dense in H, then either $\pi(\Gamma)$ is precompact or else π extends to a continuous homomorphism $G \to H$.

Motivated by a conjecture of Selberg [Sel60], Margulis developed the superrigidity theorem as the central ingredient in the proof of the Arithmeticity Theorem [Mar75a] which states that every irreducible lattice in a higher-rank semisimple Lie or algebraic group is, in a suitable sense, the integer points of an algebraic group over a global field. Since then, the phenomenon of superrigidity has found a wide array of applications, notably Zimmer's orbit equivalence rigidity stating that if two such lattices admit probability-preserving actions that are orbit equivalent then the ambient groups are locally isomorphic (a consequence of the cocycle superrigidity theorem [Zim80] generalizing Margulis' work), and Furman's measure equivalence theorem [Fur99] stating that if a countable group is measure equivalent to a lattice in a higher-rank simple group then that group is in fact itself also a lattice.

There is a rich analogy between the interaction of a lattice in a group $\Gamma < G$, and the interaction between a countable group and its von Neumann algebra $\Gamma < \mathcal{U}(L\Gamma)$. In both situations the "analytic" properties of G or $L\Gamma$ are often reflected in corresponding properties of Γ. For example, Connes exhibited the first rigidity phenomenon in II_1 factors [Con80b] by exploiting Kazhdan's property (T) [Kaž67] for Γ. This analogy was made more precise when Connes introduced his theory of correspondences [Con80a, Pop86] which provided a proper setting for the "representation theory" of a finite von Neumann algebra.

Using this theory one is able to define analytic properties of a finite von Neumann algebra such as amenability, property (T), the Haagerup property, etc. More importantly, by inducing or restricting one is able to relate representations of the group Γ with correspondences for $L\Gamma$ and in this way show that such properties for the von Neumann algebra $L\Gamma$ are shared by Γ (e.g., [Con76a, Con76b, CJ85, Cho83]).

Based on the strong rigidity result of Mostow [Mos73], the superrigidity result of Margulis, and the cocycle version due to Zimmer, Connes suggested that the analogy could be pursued further and that there should be a similar superrigidity phenomenon for such groups embedded in their group von Neumann algebras. Connes further suggested that the first difficulty is to understand the role of the Poisson boundary in the setting of operator algebras. (See the discussion on page 86 in [Jon00]).

The first examples of lattices in higher-rank groups where this "operator algebraic superrigidity" was verified were obtained by Bekka [Bek07] who showed that this holds for the groups $SL_n(\mathbb{Z})$, for $n \geq 3$. Further examples were found in [PT13] where the same results were obtained for the groups $SL_2(A)$ where $A = \mathcal{O}$ is a ring of integers (or, more generally, $A = \mathcal{O}S^{-1}$ a localization) with infinitely many units. Despite this initial progress, the proofs in [Bek07] and [PT13] rely much too heavily on the structure of SL_n and as such do not generalize to arbitrary irreducible lattices. The purpose of this paper is to provide the first examples of higher-rank groups G such that operator algebraic superrigidity holds for arbitrary irreducible lattices.

Theorem A (Operator Algebraic Superrigidity for lattices)**.** *Suppose G is a semisimple connected Lie group with trivial center and no compact factors, such that at least one factor is higher-rank, and suppose H is a non-compact totally disconnected semisimple algebraic group over a local field with trivial center and no compact factors. Let $\Gamma < G \times H$ be a (strongly) irreducible lattice, and suppose $\pi : \Gamma \to \mathcal{U}(M)$ is a representation into the unitary group of a finite factor M such that $\pi(\Gamma)'' = M$. Then either M is finite dimensional, or else π extends to an isomorphism $L\Gamma \xrightarrow{\sim} M$.*

An example of G and H where the hypotheses in the previous theorem are satisfied is $G = PSL_n(\mathbb{R})$ and $H = PSL_n(\mathbb{Q}_p)$, for $n \geq 3$ and p a prime. The theorem above actually holds in a greater generality, e.g., in many cases G itself can also be totally disconnected (see Section 6 for the full generality). The above superrigidity result is a consequence of a corresponding superrigidity result for commensurators. To state this result we first recall that if $\Gamma < \Lambda$ is an inclusion of countable groups, then we say that Λ commensurates Γ if $[\Gamma : \Gamma \cap \lambda\Gamma\lambda^{-1}] < \infty$, for each $\lambda \in \Lambda$. Given such an inclusion we may consider the homomorphism of Λ into $\mathrm{Symm}(\Lambda/\Gamma)$ given by left multiplication. The relative profinite completion of Λ with respect to Γ is the closure of the image of Λ in $\mathrm{Symm}(\Lambda/\Gamma)$ where the latter is given the topology of pointwise convergence (see, e.g., [Sch80, SW13]). The relative profinite completion, denoted $\Lambda /\!\!/ \Gamma$, is a totally disconnected group which is locally compact since the image of Γ in $\mathrm{Symm}(\Lambda/\Gamma)$ generates a compact open subgroup.

Theorem B (Operator Algebraic Superrigidity for commensurators)**.** *Let G be as in Theorem A and suppose $\Lambda < G$ is a countable dense subgroup which contains and commensurates a lattice $\Gamma < G$ such that $\Lambda /\!\!/ \Gamma$ is a product of simple groups with the Howe-Moore property.*

If $\pi : \Lambda \to \mathcal{U}(M)$ is a finite factor representation such that $\pi(\Lambda)'' = M$, then either $M = \pi(\Gamma)''$ is finite dimensional, or else π extends to an isomorphism $L\Lambda \xrightarrow{\sim} M$.

These results, describing the types of homomorphisms from a lattice or commensurator into the unitary group of a finite factor, should be contrasted with Popa's rigidity results [Pop06b, Pop06c, Pop08, Pop07a] where he shows, for example, that under certain "malleability" conditions for a measure-preserving action, a cocycle for this action into the unitary group of a finite von Neumann algebra is always cohomologous to a homomorphism. Thus, combining our result with Popa's superrigidity results (see also [PS12]) one obtains a further rigidity statement about cocycles for such actions. For example, we obtain the following result.

Theorem C. *Let Λ and G be as in Theorem B, and consider the Bernoulli shift action $\Lambda \curvearrowright [0,1]^\Lambda$. If M is a separable finite von Neumann algebra, and $\alpha : \Lambda \times [0,1]^\Lambda \to \mathcal{U}(M)$ is a cocycle, then there exists a von Neumann subalgebra $N \subset M$, a central projection $p \in \mathcal{Z}(N)$, an isomorphism $\theta_1 : L\Lambda \to Np$ (if $p \neq 0$), and a homomorphism $\theta_2 : \Lambda \to \mathcal{U}(Np^\perp)$ such that $\theta_2(\Lambda)$ is precompact, and such that α is cohomologous to the homomorphism $\lambda \mapsto \theta_1(u_\lambda)p + \theta_2(\lambda)p^\perp$.*

As noted in [Bek07], operator algebraic superrigidity can also be described in the framework of characters. A character on a discrete group Λ is a conjugation invariant function τ which is of positive type, and is normalized so that $\tau(e) = 1$ (or equivalently, a trace on the full group C^*-algebra $C^*\Lambda$). If M is a finite von Neumann algebra with trace τ, and $\pi : \Lambda \to \mathcal{U}(M)$ is a representation then $\lambda \mapsto \tau(\pi(\lambda))$ gives a character. Moreover, the GNS-construction shows that every character arises in this way. It is also not hard to see that M may be chosen to be completely atomic if and only if τ is almost periodic, i.e., the set of translations $\{x \mapsto \tau(\lambda x) \mid \lambda \in \Lambda\}$ is precompact in $\ell^\infty\Lambda$. The space of characters forms a Choquet simplex, and the extreme points correspond to representations which generate a finite factor [Tho64b].

Theorem B is therefore equivalent to the following rigidity result for characters which reduces the classification to that of continuous characters on the Bohr compactification. There is a similar statement for Theorem A.

Theorem D (Character Rigidity for commensurators)**.** *Let Λ and G be as in Theorem B. If $\tau : \Lambda \to \mathbb{C}$ is an extreme point in the space of characters then either τ is almost periodic, or else $\tau = \delta_e$.*

For finite or compact groups the study of characters has a long and successful history, dating back to the work of Frobenius, Schur, Peter-Weyl, and others. Classification of characters for non-compact groups goes back to Segal and von Neumann's result in [SvN50] (see also [KS52]) where they show that connected simple Lie groups have no non-trivial continuous homomorphisms into a finite factor, and hence have no non-trivial continuous characters. For countably infinite groups the study of characters was initiated by Thoma [Tho64b, Tho64a, Tho66] who classified extreme characters for the group of finite permutations of $\mathbb{N}$. Since then, classification results for characters on non-compact groups have been extended to a wide range of settings. The emphasis first being on more "classical" type groups, e.g., [Kir65, Ovč71, Sku76, Voi76, VK81, VK82, Boy92, Boy93, Boy05], and then more recently to the less "classical" setting, e.g., [DN07, DN08b, DN08a, Dud11, DM13, DM12, EI13]. The only previous classification results focusing on lattices were obtained in [Bek07] and [PT13].

Another consequence of operator algebraic superrigidity is that it implies a rigidity phenomenon for the stabilizers of measure-preserving actions. Specifically, given a measure-preserving action of the lattice Γ, one naturally obtains a homomorphism of the group into the von Neumann algebra associated to the orbit equivalence relation [FM77] (see also [Ver10]). Applying operator algebraic superrigidity to this setting we obtain the main result from [CP12].

Corollary E ([CP12, Cre13])**.** *Let Λ and G be as in Theorem B. Then any probability measure-preserving ergodic action of Λ on a standard Lebesgue space is essentially free.*

The previous corollary (which generalizes the normal subgroup theorem in [CS12]), or rather its proof from [CP12], was the starting point for this current work.

2 Preliminaries and motivation

In this section we will recall some of the notions from ergodic theory and von Neumann algebras which we will use in the sequel, and we will also outline our argument. For a more detailed review of the ergodic theory of semisimple groups and von Neumann algebras we refer the reader to [Zim84] and [Dix81].

Let N be a finite von Neumann algebra with a normal faithful trace τ. The trace on N provides us with a positive definite inner product on N given by $\langle x, y\rangle = \tau(y^*x)$, and we denote by $L^2(N,\tau)$ the Hilbert space completion of N with respect to this inner product. We also have a norm on N given by $\|x\|_1 = \tau(|x|)$, and we denote by $L^1(N,\tau)$ the Banach space completion of N with this norm. Note that $\|x\|_1 \leq \|x\|_2 \leq \|x\|$ and so we consider inclusions $N \subset L^2(N,\tau) \subset L^1(N,\tau)$, moreover, each of these spaces is an N-bimodule, the trace extends continuously to $L^1(N,\tau)$, and we may identify $L^1(N,\tau)$ with the predual of N, by considering the linear functionals $N \ni x \mapsto \tau(x\xi)$, for $\xi \in L^1(N,\tau)$.

Left multiplication gives us a faithful representation of N in the space of bounded operators on $L^2(N,\tau)$, and so we will always consider $N \subset \mathcal{B}(L^2(N,\tau))$. Since we are also considering $N \subset L^2(N,\tau)$, to avoid confusion we shall sometimes use the notation $\hat{x} \in L^2(N,\tau)$ to identify an operator $x \in N$ as an element in $L^2(N,\tau)$.

Since τ is a trace it follows that conjugation $x \mapsto x^*$ on N extends to an anti-linear isometry $J : L^2(N, \tau) \to L^2(N, \tau)$. Note that if $x, y \in N$ then we have $Jx^*J\hat{y} = \widehat{yx}$ so that Jx^*J is multiplication on the right by x. Thus, $JNJ \subset N'$ and in fact we have $JNJ = N'$.

If Γ is a countable group and we consider the left-regular representation $\lambda : \Gamma \to \mathcal{U}(\ell^2\Gamma)$, then the group von Neumann algebra $L\Gamma$ is the von Neumann algebra generated by this representation, i.e., $L\Gamma = \{\lambda(\Gamma)\}''$ [MvN43]. Since Γ is discrete, the group von Neumann algebra will be a finite von Neumann algebra with a canonical normal faithful trace given by $\tau(x) = \langle x\delta_e, \delta_e \rangle$. As δ_e is a tracial and cyclic vector we may identify $\ell^2\Gamma$ with $L^2(L\Gamma, \tau)$ in such a way that the vector δ_e corresponds to the vector $\hat{1}$. Under this identification the von Neumann algebra $L\Gamma$ can then be described as the space of left-convolvers $\{\xi \in \ell^2\Gamma \mid \xi * \eta \in \ell^2\Gamma, \text{ for all } \eta \in \ell^2\Gamma\}$.

If we are given a quasi-invariant action on a probability space $\Gamma \curvearrowright (B, \eta)$ then we also have an action of Γ on the space of measurable functions given by $\sigma_\gamma(a) = a \circ \gamma^{-1}$. The Koopman representation of this action on $L^2(B, \eta)$ is the unitary representation given by the formula $\sigma^0_\gamma(\xi) = \sigma_\gamma(\xi)\sqrt{\frac{d\gamma\eta}{d\eta}}$. Note that by considering point-wise multiplication we may realize $L^\infty(B, \eta)$ as a von Neumann subalgebra of $\mathcal{B}(L^2(B, \eta))$, and under this identification the unitaries σ^0_γ normalize $L^\infty(B, \eta)$. Specifically, we have the formula $\sigma^0_{\gamma^{-1}} a \sigma^0_\gamma = \sigma_\gamma(a)$, for all $a \in L^\infty(B, \eta) \subset \mathcal{B}(L^2(B, \eta))$, and $\gamma \in \Gamma$. The group-measure space construction is the von Neumann algebra $L^\infty(B, \eta) \rtimes \Gamma \subset \mathcal{B}(L^2(X, \eta)\overline{\otimes}\ell^2\Gamma)$ generated by $L^\infty(B, \eta)$, together with the unitary operators $\{\sigma^0_\gamma \otimes \lambda(\gamma) \mid \gamma \in \Gamma\}$ [MvN37]. Note that by Fell's absorption principal we have a conjugacy of unitary representations $\sigma^0 \otimes \lambda \sim \mathrm{id} \otimes \lambda$, hence it follows that the von Neumann algebra $\{\sigma^0_\gamma \otimes \lambda(\gamma) \mid \gamma \in \Gamma\}''$ is canonically isomorphic to $L\Gamma$.

If the action $\Gamma \curvearrowright (B, \eta)$ is measure-preserving, then the crossed product $L^\infty(B, \eta) \rtimes \Gamma$ will be finite and have a canonical normal faithful trace given by $\tau(x) = \langle x(\hat{1} \otimes \delta_e), \hat{1} \otimes \delta_e \rangle$.

A useful property for finite von Neumann algebras is that for an arbitrary von Neumann subalgebra there always exists a normal conditional expectation [Ume54]. More specifically, if M is a finite von Neumann algebra with normal faithful trace τ, and if $N \subset M$ is a von Neumann subalgebra, then for each $x \in M$ the linear functional $y \mapsto \tau(xy)$ is normal, and hence there is a unique element $E(x) \in L^1(N, \tau)$ such that $\tau(xy) = \tau(E(x)y)$ for all $y \in N$. Since $E(x) \geq 0$ whenever $x \geq 0$ it follows that $E(x) \in N$ for all $x \in M$. This map $E : M \to N$ is a conditional expectation from M to N, i.e., it is a unital completely positive projection onto N, moreover, it is the unique such map satisfying the condition $\tau \circ E = \tau$ (and this condition also implies that it is normal).

2.1 On the method of proof

Before discussing the outline of our proof we recall Margulis' Normal Subgroup Theorem [Mar78, Mar79]: If Γ is an irreducible lattice in a center free higher-rank semisimple group G with no compact factors, then Γ is just infinite, i.e., every non-trivial normal subgroup has finite index.

Assuming $\Sigma \lhd \Gamma$ is non-trivial, Margulis' strategy for proving that $|\Gamma/\Sigma| < \infty$ consists of two "halves". The first, showing that Γ/Σ has Kazhdan's property (T) [Kaž67], and the second showing that Γ/Σ is amenable. Since for countable groups amenability and property (T) together imply finite, the result then follows. In the case when G has property (T), then property (T) for Γ/Σ is immediate since this passes to lattices and quotients. Significantly more difficult is the case when G has no factor with property (T), in which case property (T) can be obtained by relating the reduced cohomology spaces of Γ/Σ to those of G as in [Sha00].

The amenability half of the proof follows by exploiting the amenability properties of the Poisson boundary $G \curvearrowright (B, \eta)$ [Fur63], together with a "factor theorem" showing that any Γ-quotient of (B, η) must actually be a G-quotient.[1] More specifically, the action of Γ on (B, η) is amenable in the sense of Zimmer [Zim78], and so there exists a Γ-equivariant conditional expectation $E : L^\infty(B, \eta)\overline{\otimes}\ell^\infty(\Gamma/\Sigma) \to L^\infty(B, \eta)$. If we consider

$$L^\infty(B, \eta)^\Sigma = \{f \in L^\infty(B, \eta) \mid \sigma_{\gamma_0}(f) = f, \text{ for all } \gamma_0 \in \Sigma\},$$

then this is a Γ-invariant von Neumann subalgebra of $L^\infty(B, \eta)$ and so by Margulis' factor theorem it follows that $L^\infty(B, \eta)^\Sigma$ is also G-invariant. However, Σ acts trivially on the corresponding Koopman representation and since G is center free, irreducibility for $\Gamma < G$ easily implies that any non-trivial representation of G must be faithful for Γ. The conclusion is then that $L^\infty(B, \eta)^\Sigma = \mathbb{C}$, i.e., Σ acts ergodically on (B, η). However, we have $E(1 \otimes \ell^\infty(\Gamma/\Sigma)) \subset L^\infty(B, \eta)^\Sigma = \mathbb{C}$ and so we conclude that the restriction of E to $1 \otimes \ell^\infty(\Gamma/\Sigma)$ gives an invariant mean.

This general strategy of Margulis is remarkably flexible and has been employed to give rigidity results in a more abstract setting, e.g., [BS06, CS12], and also to give classification results beyond normal subgroups, e.g., [FSZ89] where normal subequivalence relations are considered, or [SZ94, CP12, Cre13, HT13] where non-free measure-preserving actions are considered.

Notions of property (T) and amenability are also of fundamental importance in the theory of II_1 factors (see, e.g., [Con76a, CJ85]), and hence it is natural to suspect that Margulis' strategy should have adaptations in this setting as well. This is especially the case given the emergence and success of Popa's deformation/rigidity theory where the major theme is to determine the structure of a II_1 factor by contrasting deformation properties (such as amenability) with rigidity properties (such as property (T)), (see [Pop06a, Pop07b, Vae10, Ioa12]).

Based on the ideas outlined above, if $\Gamma < \Lambda < G$ is as in Theorem B, then our strategy to prove operator algebraic superrigidity for Λ is to first show that if $\pi : \Lambda \to \mathcal{U}(M)$ is a

[1] To avoid confusion with terminology from von Neumann algebras we refer to a Γ-equivariant map between Γ-spaces as a Γ-quotient.

representation which generates a finite factor M, then either π extends to an isomorphism $L\Lambda \to M$ or else the von Neumann subalgebra $N = \pi(\Gamma)''$ is amenable (i.e., injective) and has property (T). Amenability and property (T) for N then imply that N is completely atomic. If N is completely atomic we then show there exists a continuous action of $\Lambda /\!/ \Gamma$ on a finite index von Neumann subalgebra of M which extends conjugation. However, $\Lambda /\!/ \Gamma$ is a product of non-discrete groups with the Howe-Moore property and we rule out the possibility of such actions which are non-trivial.

Just as property (T) passes to quotients of a countable group, it also passes to finite von Neumann algebras which the group generates [CJ85]. Thus, even in the finite factor setting if we assume that the ambient group G has property (T) then so does $\pi(\Gamma)''$ and hence the property (T) half of Margulis' strategy follows immediately. In the case when G has a non-compact factor with property (T), then using the notion of resolutions from [Cor06], in a similar fashion as in [Cre13], we show that any representation of Γ into an amenable finite von Neumann algebra must, upon passing to a finite index subalgebra, be given by a representation of G, which we again rule out using the Howe-Moore property. Thus, in this case also we are reduced to the amenability half of the argument.

The case when G has no non-compact factor with property (T) appears more subtle. (See also the related question at the end of Section 2 in [SZ94].) While a cohomological characterization of property (T) for finite factors was obtained in [Pet09], it is less clear if there is a characterization in terms of reduced cohomology. Consequently, it is unclear how to adapt reduced cohomology techniques, e.g., from [Sha00], in the setting of finite factors.

In order to adapt the amenability half of Margulis' strategy to the II_1 factor setting, we follow the suggestion of Connes by first understanding the role of the Poisson boundary in this setting. A significant first step in this direction was achieved by Izumi [Izu02, Izu04] who introduced the notion of a noncommutative Poisson boundary of a normal unital completely positive map $\phi : \mathcal{M} \to \mathcal{M}$ where $\mathcal{M}$ is a von Neumann algebra. Specifically, Izumi considers the space of "harmonic operators" $H^\infty(\mathcal{M}, \phi) = \{x \in \mathcal{M} \mid \phi(x) = x\}$ which is a weakly closed operator system. By fixing a free ultra-filter $\omega \in \beta\mathbb{N} \setminus \mathbb{N}$ one may construct a completely positive projection $E : \mathcal{M} \to H^\infty(\mathcal{M}, \phi)$ by the formula

$$E(x) = \mathrm{w-}\lim_{n\to\omega} \frac{1}{n} \sum_{k=0}^{n-1} \phi^k(x), \qquad x \in \mathcal{M}.$$

One may then use the Choi-Effros product $x \cdot y = E(xy)$ [CE77] to define a unique von Neumann algebraic structure on $H^\infty(\mathcal{M}, \phi)$. (See also the Appendix to [Izu12] where the multiplication structure on $H^\infty(\mathcal{M}, \phi)$ is constructed more directly by using the minimal dilation of ϕ).

As Izumi notes in [Izu04], if one starts with a countable group Γ, together with a probability measure $\mu \in \mathrm{Prob}(\Gamma)$, then one may consider the normal unital completely positive map $\phi_\mu : \mathcal{B}(\ell^2\Gamma) \to \mathcal{B}(\ell^2\Gamma)$ given by $\phi_\mu(T) = \int \rho(\gamma) T \rho(\gamma^{-1}) \, d\mu(\gamma)$, and in this case it is not hard to compute directly that $H^\infty(\mathcal{B}(\ell^2\Gamma), \phi_\mu)$ is canonically isomorphic to $L^\infty(B, \eta) \rtimes \Gamma$ where (B, η) is the commutative Poisson boundary from [Fur63].

To uncover the rigidity properties of a boundary one must not study only the crossed product $L^\infty(B, \eta) \rtimes \Gamma$ but rather the inclusion $L\Gamma \subset L^\infty(B, \eta) \rtimes \Gamma$. Thus, given an arbitrary finite von Neumann algebra M with a faithful normal trace τ, it is natural to want a notion of "boundary inclusions" $M \subset \mathcal{B}$ which play analogous roles to the inclusions $L\Gamma \subset L^\infty(B, \eta) \rtimes \Gamma$ in the case of group von Neumann algebras. Using Izumi's notion of a noncommutative Poisson boundary we will describe such inclusions, and the striking similarities to the case for discrete groups suggest that these inclusions should be the proper objects to consider as Poisson boundaries of the von Neumann algebra N.

The starting point for our construction is to consider the finite von Neumann algebra N together with a normal "hyperstate" φ on $\mathcal{B}(L^2(N, \tau))$, i.e., φ is a normal state on $\mathcal{B}(L^2(N, \tau))$ which restricts to the trace τ on N. Just as hypertraces are in one to one correspondence with conditional expectations (see p. 450 in [Con76b]), there is a one to one correspondence between (normal) hyperstates and (normal) unital N-bimodular completely positive maps $\phi : \mathcal{B}(L^2(N, \tau)) \to \mathcal{B}(L^2(N, \tau))$ given by the equation

$$\varphi(T) = \langle \phi(T)\hat{1}, \hat{1} \rangle, \quad T \in \mathcal{B}(L^2(N, \tau)).$$

Since $\phi_{|N} = \mathrm{id}$, the noncommutative Poisson boundary $\mathcal{B} = H^\infty(\mathcal{B}(L^2(N, \tau)), \phi)$ contains N as a von Neumann subalgebra. Note that by considering $\mathcal{B} \subset \mathcal{B}(L^2(N, \tau))$ as an operator system, then we obtain a normal "stationary" hyperstate φ_0 on $\mathcal{B}$ by restricting the hyperstate φ. Conversely, if we consider $N \subset \mathcal{B}$ as an abstract inclusion, together with the hyperstate φ_0, then considering the GNS-representation $\pi : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ we have a natural inclusion $L^2(N, \tau) \subset \mathcal{H}$, and if we denote $e : \mathcal{H} \to L^2(N, \tau)$ the orthogonal projection then we recover the embedding of $\mathcal{B}$ into $\mathcal{B}(L^2(N, \tau))$ through the "Poisson transform" $\mathcal{B} \ni T \mapsto e\pi(T)e \in \mathcal{B}(L^2(N, \tau))$.

Note that by construction of the boundary, there exists a completely positive projection $E : \mathcal{B}(L^2(N, \tau)) \to \mathcal{B}$, and so $\mathcal{B}$ is always injective [Arv69, CE77]. In the trivial case when $\phi = \mathrm{id}$ we have that $\mathcal{B} = \mathcal{B}(L^2(N, \tau))$. The more interesting case however is when $H^\infty(\mathcal{B}(L^2(N, \tau)), \phi) \cap N' = \mathcal{Z}(N)$ in which case we have $N' \cap \mathcal{B} = \mathcal{Z}(\mathcal{N})$ (this is related to the fact that commutative groups have only trivial Poisson boundaries). In particular, if in addition we have that N is a factor, then $\mathcal{B}$ will also be a factor and N will be an irreducible subfactor.

An example of particular interest in the sequel is the following generalization of the crossed product example given above. Suppose Γ is a countable group and N is a finite von Neumann algebra with normal faithful trace τ. Suppose further that $\pi : \Gamma \to \mathcal{U}(N)$ is a homomorphism such that $\pi(\Gamma)'' = N$. If we are given a probability measure $\mu \in \mathrm{Prob}(\Gamma)$ then as before we may consider the unital completely positive map $\phi_\mu : \mathcal{B}(L^2(N, \tau)) \to \mathcal{B}(L^2(N, \tau))$ given by $\phi_\mu(T) = \int (J\pi(\gamma)J)T(J\pi(\gamma^{-1})J)\, d\mu(\gamma)$. Since $JNJ = N'$ we have that ϕ_μ is N-bimodular, thus we have a corresponding boundary inclusion $N \subset \mathcal{B}$.

If $\Gamma \curvearrowright (B, \eta)$ is the commutative Poisson boundary corresponding to the measure μ then it is not hard to check that we have a completely isometric order isomorphism (and hence also a von Neumann algebra isomorphism) of $\mathcal{B}$ onto

$$\mathcal{B}_N = \{\sigma^0_\gamma \otimes (J\pi(\gamma)J) \mid \gamma \in \Gamma\}' \cap (L^\infty(B, \eta) \overline{\otimes} \mathcal{B}(L^2(N, \tau))), \tag{$\star$}$$

with inverse map $\int \otimes \operatorname{id}$.

The inclusion $N \subset \mathcal{B}_N$ (which we consider as a "noncommutative Γ-quotient" of (B, η)) is defined for any quasi-invariant action of Γ and it is useful to think of $\mathcal{B}_N$ as the space of essentially bounded Γ-equivariant functions from (B, η) to $\mathcal{B}(L^2(N, \tau))$ where the action on the latter is via conjugation by $J\pi(\gamma)J$. It follows from Theorem 5.1 in [Zim77] that $\mathcal{B}_N$ is injective whenever the action $\Gamma \curvearrowright (B, \eta)$ is amenable.

To finish the analogy with Margulis' normal subgroup theorem we have left to discuss the factor theorem for boundary actions. For this we strengthen the factor theorems for commensurators in [CS12] and [CP12] to the setting of II_1 factors. The key feature of boundaries we use here is that they are contractive (or SAT), i.e., for each measurable set $F \subset B$ we have $\inf_{\gamma \in \Gamma} \eta(\gamma F) \in \{0, 1\}$. This property was introduced by Jaworski [Jaw94, Jaw95], and we exploit this property to show in Theorem 3.2 that the inclusion of von Neumann algebras $N \subset \mathcal{B}_N$ is extremely rigid. For example, it follows that the only normal unital N-bimodular map from $\mathcal{B}_N$ to itself is the identity map. This rigidity theorem is a natural extension of Theorem 4.34 in [CP12] and allows us to show that the "noncommutative Γ-quotient" $\mathcal{B}_N$ is also invariant under the action of Λ (Proposition 5.1). If in addition we have that π does not extend to an isomorphism $L\Lambda \to M$, then we use an averaging argument together with the density of Λ in G to conclude that $\mathcal{B}_N$ must in fact be in the commutant of the action of G (Theorem 4.4). From this it follows easily that in fact $\mathcal{B}_N = N$ and hence N is injective, i.e., amenable.

3 Operator algebraic rigidity for contractive actions

Throughout this section Γ will be a countable group, $\Gamma_0 < \Gamma$ will be a finite index subgroup, $\Gamma \curvearrowright (B, \eta)$ will be a contractive action, N will be a finite von Neumann algebra with normal faithful trace τ, and $\pi : \Gamma \to \mathcal{U}(N)$ will be a homomorphism such that $N = \pi(\Gamma)''$. We let $\mathcal{B}_N$ be the corresponding boundary as defined by ($\star$).

Since $\Gamma \curvearrowright (B, \eta)$ is contractive, it is easy to show that for any $f \in L^\infty(B, \eta)$, with $f \geq 0$, there exists a sequence $\gamma_n \in \Gamma$ such that $\sigma_{\gamma_n}(f) \to \|f\|$ in the strong operator topology. Moreover, if $\tilde{f} \in L^\infty(B, \eta)$ is another function, then we can choose the sequence in such a way that $\sigma_{\gamma_n}(\tilde{f})$ converges strongly to a scalar. The following is an analogue of this fact for the noncommutative situation.

Lemma 3.1. *Using the notation above, for $x \in \mathcal{B}_N$, $\|x\| \leq 1$, $f \in L^\infty(B, \eta)$, $f \geq 0$, and $\varepsilon > 0$, there exists sequences $\{g_n\} \subset \Gamma_0$, $\{p_n\} \subset \mathcal{P}(N)$, with $\tau(p_n) > 1 - \varepsilon$, for all n, and*

$\{y_n\} \subset N$, such that $\{y_n\}$ is uniformly bounded, $\sigma_{g_n}(f)$ converges strongly to $\|f\|_\infty$, and $\pi(g_n)(p_n x - y_n)\pi(g_n^{-1})$ converges strongly to 0.

Proof. Let $F_n \subset \Gamma$ be an increasing sequence of finite sets such that $e \in F_1$, and $\cup_n F_n = \Gamma$. For each $n \in \mathbb{N}$ there exists a measurable subset $E \subset B$ with positive measure such that for $b \in E$ we have $\|f\|_\infty - f(b) < 1/n$, and such that for all $b_1, b_2 \in E$ we have $\|(x(b_1) - x(b_2))\hat{1}\|_2 \leq 1/n$.

Since the action of Γ_0 on (B, η) is contractive [CS12] there exists $g \in \Gamma_0$ such that $\eta(\cap_{\gamma \in F_n} \gamma^{-1} g E) > 1 - 1/n$. Fix a point $b_0 \in E$, and set $y_0 = x(b_0)\hat{1} \in L^2(N, \tau)$. Since $\|y_0^*\|_2 = \|y_0\|_2 \leq \|x\| \leq 1$, if we set $p = \mathbb{1}_{[0, \varepsilon^{-1/2}]}(|y_0^*|)$ then by Chebyshev's inequality we have $\tau(p) > 1 - \varepsilon$, and if $y = p y_0$, then $\|y\| \leq \varepsilon^{-1/2}$.

For $h \in F_n$, $b \in \cap_{\gamma \in F_n} \gamma^{-1} g E$, we have $\|f\|_\infty - \sigma_g(f)(b) < 1/n$, and

$$\begin{aligned} \|(\pi(g)(px - y)\pi(g^{-1}))(b)\pi(h)\hat{1}\|_2 &= \|(J\pi(g^{-1}h)J)(px(b) - y)(J\pi(h^{-1}g)J)\hat{1}\|_2 \\ &= \|(px(g^{-1}hb) - y)\hat{1}\|_2 \\ &= \|p(x(g^{-1}hb) - x(b_0))\hat{1}\|_2 < 1/n \end{aligned}$$

Since the bounds $\|x\| \leq 1$, and $\|y\| \leq \varepsilon^{-1/2}$ are independent of n, and as the span of $\pi(\Gamma)\hat{1}$ is dense in $L^2(N, \tau)$ it follows that by setting $g_n = g$, $p_n = p$, and $y_n = y$ we have that $\sigma_{g_n}(f)$ converges to $\|f\|_\infty$ strongly, and $\pi(g_n)(p_n x - y_n)\pi(g_n^{-1})$ converges strongly to 0. □

The following rigidity theorem for contractive actions strengthens Theorem 4.34 from [CP12].

Theorem 3.2. *Using the notation above, suppose $P \subset \mathcal{B}_N$ is a von Neumann subalgebra which contains N, set*

$$\widetilde{\mathcal{B}_N} = \{\sigma_\gamma^0 \otimes (J\pi(\gamma)J) \mid \gamma \in \Gamma_0\}' \cap (L^\infty(B, \eta) \overline{\otimes} \mathcal{B}(L^2(N, \tau))),$$

so that $N \subset P \subset \mathcal{B}_N \subset \widetilde{\mathcal{B}_N}$. If $\Phi : P \to \widetilde{\mathcal{B}_N}$ is a normal N-bimodular unital map, then $\Phi = \mathrm{id}$.

Proof. Fix $x \in P$, and $\varepsilon > 0$. By Lemma 3.1 there exists sequences $g_n \in \Gamma_0$, $p_n \in \mathcal{P}(N)$ with $\tau(p_n) > 1 - \varepsilon$, and $y_n \in N$ such that $\sigma_{g_n}(|\langle(\Phi(x) - x)\hat{1}, \hat{1}\rangle|) \in L^\infty(B, \eta)$ converges strongly to $\|\langle(\Phi(x) - x)\hat{1}, \hat{1}\rangle\|_\infty$, and $\pi(g_n)(p_n x - y_n)\pi(g_n^{-1})$ converges strongly to 0.

Since Φ is normal and N-bimodular we then have

$$\lim_{n\to\infty} \Phi(\pi(g_n)p_n x\pi(g_n^{-1})) - \pi(g_n)p_n x\pi(g_n^{-1}) = \lim_{n\to\infty} \Phi(\pi(g_n)y_n\pi(g_n^{-1})) - \pi(g_n)y_n\pi(g_n^{-1}) = 0,$$

where the limit is in the weak operator topology. Thus, using that $\Phi(x) - x \in \widetilde{\mathcal{B}_N}$ we have

$$\begin{aligned}
\|\langle(\Phi(x)-x)\hat{1},\hat{1}\rangle\|_\infty &= \lim_{n\to\infty}\left|\int \sigma_{g_n}(\langle(\Phi(x)-x)\hat{1},\hat{1}\rangle)\,d\eta\right| \\
&= \lim_{n\to\infty}\left|\int \langle (J\pi(g_n^{-1})J)(\Phi(x)-x)(J\pi(g_n)J)\hat{1},\hat{1}\rangle\,d\eta\right| \\
&= \lim_{n\to\infty}\left|\int \langle \pi(g_n)(\Phi(x)-x)\pi(g_n^{-1})\hat{1},\hat{1}\rangle\,d\eta\right| \\
&= \lim_{n\to\infty}\left|\int \langle (\Phi(\pi(g_n)x\pi(g_n^{-1})) - \pi(g_n)x\pi(g_n^{-1}))\hat{1},\hat{1}\rangle\,d\eta\right| \\
&= \lim_{n\to\infty}\left|\int \langle (\Phi(\pi(g_n)(1-p_n)x\pi(g_n^{-1})) - \pi(g_n)(1-p_n)x\pi(g_n^{-1}))\hat{1},\hat{1}\rangle d\eta\right| \\
&\leq \limsup_{n\to\infty}\|\Phi(x)-x\|_\infty\|1-p_n\|_2 < \|\Phi(x)-x\|_\infty\sqrt{\varepsilon}.
\end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\langle(\Phi(x)-x)\hat{1},\hat{1}\rangle$ is identically 0.

If $a, b \in N$, it then follows

$$\|\langle(\Phi(x)-x)a\hat{1},b\hat{1}\rangle\|_\infty = \|\langle(\Phi(b^*xa)-b^*xa)\hat{1},\hat{1}\rangle\|_\infty = 0,$$

and since $N \subset L^2(N,\tau)$ is dense we then have $\Phi(x) = x$, and so $\Phi = \mathrm{id}$ since x was arbitrary. □

Corollary 3.3. *Using the notation above, we have $J(\pi(\Gamma_0)' \cap N)J \subset \mathcal{B}_N'$. In particular, $\mathcal{B}_N \subset \mathcal{Z}(N)' \cap L^\infty(B,\eta)\overline{\otimes}\mathcal{B}(L^2(N,\tau))$.*

Proof. It is enough to show that each projection $p \in J(\pi(\Gamma_0)' \cap N)J \subset L^\infty(B,\eta)\overline{\otimes}\mathcal{B}(L^2(N,\tau))$, commutes with $\mathcal{B}_N$. Consider the map $\mathcal{B}_N \ni x \mapsto \Phi(x) = pxp + (1-p)x(1-p)$. Then Φ is a normal unital completely positive map which restricts to the identity on N and hence is N-bimodular. Moreover, $\Phi(x) \in \{\sigma_\gamma^0 \otimes (J\pi(\gamma)J) \mid \gamma \in \Gamma_0\}'$ since $p \in \mathcal{P}(J(\pi(\Gamma_0)' \cap N)J)$. Hence, by Theorem 3.2 we have $\Phi = \mathrm{id}$, i.e., $p \in \mathcal{B}_N'$. □

4 Actions of Howe-Moore groups on finite von Neumann algebras

Recall, that an automorphism θ of a von Neumann algebra M is properly outer if there is no non-zero element $v \in M$ such that $\theta(x)v = vx$ for all $x \in M$. An action $\alpha : \Lambda \to \mathrm{Aut}(M,\tau)$ of a countable group Λ is properly outer if α_λ is properly outer for each $\lambda \in \Lambda$.

Proposition 4.1. *Let G be a locally compact group, suppose that $\Lambda < G$ is a countable dense subgroup such that either* (*i*) *G is a product of non-compact connected simple groups with the Howe-Moore property, and the Λ intersection with any proper subproduct of G is trivial; or*

(*ii*) *G is a simple group with the Howe-Moore property, and Λ commensurates a lattice $\Gamma < G$, which has a non-torsion element.*

Suppose $\alpha : G \to \mathrm{Aut}(M_0, \tau)$ is a continuous ergodic, trace preserving action of G on a non-trivial finite von Neumann algebra M_0 with normal faithful trace τ, then the restriction of α to Λ is properly outer.

Proof. Suppose $g \in G$, and $v \in M_0$ such that $\alpha_g(x)v = vx$ for all $x \in M_0$, then it follows that $|v| \in \mathcal{Z}(M_0)$ and so replacing v with the partial isometry in its polar decomposition we may assume that v is a partial isometry, and that $v^*v \in \mathcal{Z}(M_0)$.

We then have $v = vv^*v = v^*vv$, and so $v^*v \geq vv^*$. As M_0 is finite we must have $v^*v = vv^*$, and so $\alpha_g(v^*v)vv^* = v(v^*v)v^* = vv^* = v^*v$. Thus, $\alpha_g(v^*v) \geq v^*v$ and as α is trace preserving we then have $\alpha_g(v^*v) = v^*v$.

Moreover, if $q \in \mathcal{Z}(M_0)$ such that $q \leq v^*v = vv^*$ then we have $\alpha_g(q) = \alpha_g(q)vv^* = vqv^* = q$. Therefore α_g acts trivially on $\mathcal{Z}(M_0)v^*v$. Hence, if Λ does not act properly outerly, and if $\mathcal{Z}(M_0) \neq \mathbb{C}$ then the restriction of the Λ-action to $\mathcal{Z}(M_0)$ is not free, which would contradict Theorem 7.2 in case (i), and Theorem 7.7 in case (ii) from [CP12]. Thus, we have left to consider the case when M_0 is a factor.

In this case, suppose $h \in G$ and $v \in M_0$ is non-zero partial isometry such that

$$\alpha_h(x)v = vx, \tag{1}$$

for all $x \in M_0$. Then since M_0 is a factor we have $v^*v = 1$, and so v is a unitary. If $g \in G$, then replacing x with $\alpha_g(x)$ and applying $\alpha_{g^{-1}}$ to (1) gives

$$\alpha_{g^{-1}hg}(x)\alpha_{g^{-1}}(v) = \alpha_{g^{-1}}(v)x, \tag{2}$$

for all $x \in M_0$.

If we assume by contradiction that the action restricted to Λ is not properly outer, then we have that $H = \{h \in G \mid \alpha_h \in \mathrm{Inn}(M_0)\}$ is a normal subgroup of G which has non-trivial intersection with Λ, and hence by hypothesis must be dense.

We now focus on cases (i) and (ii) separately. In case (i), by projecting down to the quotient $G/\ker(\alpha)$ we may assume that the action is faithful (note that by hypothesis we have that $\Lambda \cap \ker(\alpha) = \{e\}$). If $G_0 < G$ is a non-trivial simple factor of G, then we denote by H_0 the subgroup of H consisting of those elements $h \in H$ such that the unitary v in (1) is fixed by G_0. It follows easily from (2) that H_0 is a normal subgroup of G. Note that if we take the intersection over all such subgroups H_0 as we vary the factor, then by ergodicity we obtain the trivial group. Thus, we may choose a factor G_0, so that $H \neq H_0$.

We may decompose G as $G = \widehat{G_0} \times G_0$, and if $h \in H \setminus H_0$, and $v \in \mathcal{U}(M_0)$ such that $\alpha_h(x)v = vx$ for all $x \in M_0$, then writing h as $(g_1, g_2) \in \widehat{G_0} \times G_0$ we have

$$\alpha_h(x)\alpha_{g_2^{-1}}(v) = \alpha_{g_2^{-1}}(\alpha_h(\alpha_{g_2}(x))v) = \alpha_{g_2^{-1}}(v\alpha_{g_2}(x)) = \alpha_{g_2^{-1}}(v)x,$$

for all $x \in M_0$. Combining this with (1) then gives $v^*\alpha_{g_2^{-1}}(v) \in \mathcal{Z}(M_0) = \mathbb{C}$, and so we have $\alpha_g(v) \in \mathbb{T}v$ for all $g \in \overline{\langle g_2 \rangle}$. As G_0 has the Howe-Moore property and $h \notin H_0$ it then follows that $\overline{\langle g_2 \rangle}$ is compact. Since the projection of $H \setminus H_0$ onto G_0 is dense (H_0 is normal and so either the projection of H_0 to G_0 is trivial, or else the projection is dense and so a non-trivial H_0 coset will project densely) it then follows that G_0 has a dense set of elements which generate precompact subgroups. This then gives a contradiction since G_0 is connected and is a product of groups with the Howe-Moore property it must be a connected real Lie group [Rot80], and Theorem 3 in [Pla65] shows that there then cannot be a dense set of $g \in G_0$ such that $\overline{\langle g_0 \rangle}$ is compact.

For case (ii) we are assuming that Λ commensurates a lattice Γ which has a non-torsion element. By [CS12] we must have that $H \cap \Gamma$ has finite index in Γ (since $H \cap \Lambda$ is non-trivial and hence dense in G) and so must contain a non-torsion element γ_0. Thus there exists $v \in \mathcal{U}(M_0) \setminus \mathbb{T}$ such that $\sigma_\gamma(x)v = vx$ for all $\gamma \in \langle \gamma_0 \rangle$, and so $\sigma_\gamma(v) = v$ for all $\gamma \in \langle \gamma_0 \rangle$ showing that the action restricted to Γ is not mixing. But since G has the Howe-Moore property, the action must be mixing, and hence must be mixing when restricted to Γ, giving a contradiction. □

We remark that in the proof in part (i) of the previous theorem we did not actually need that Λ was dense. Thus, in this case if $\alpha : G \to \mathrm{Aut}(M_0, \tau)$ is a continuous, ergodic, trace preserving action on a non-trivial finite von Neumann algebra M_0, then α_g is properly outer for any $g \in G$ which is not contained in a proper subproduct of G.

Part (i) in the previous theorem also generalizes a result by Segal and von Neumann who showed that a simple real Lie group cannot embed continuously into a finite von Neumann algebra [SvN50]. While we will not use it in the sequel, we show here that Segal and von Neumann's result also holds in general for non-discrete groups with the Howe-Moore property.

Theorem 4.2. *Let G be a non-discrete totally disconnected simple group with the Howe-Moore property. Then there is no non-trivial continuous homomorphism of G into the unitary group of a finite von Neumann algebra.*

Proof. By [Tho64b] it is enough to consider the case when G generates a finite factor, and so suppose M is a finite factor with trace τ, and $\pi : G \to \mathcal{U}(M)$ is a continuous homomorphism such that $\pi(G)'' = M$. If we let $K < G$ be a compact open subgroup, then we have that $\pi \otimes \pi^{\mathrm{op}}$ has a non-trivial K-invariant vector $\xi \in L^2(M, \tau)\overline{\otimes}L^2(M, \tau)$. If we let $\mathcal{K}$ be the closed G-invariant subspace generated by ξ, then as $(\pi \otimes \pi^{\mathrm{op}})(G)'' \subset M\overline{\otimes}M^{\mathrm{op}}$ is again finite, it follows that there exists a unit tracial vector $\eta \in \mathcal{K}\overline{\otimes}\ell^2\mathbb{N}$ for $(\pi \otimes \pi^{\rm op} \otimes {\rm id})(G)$, i.e., for all $g, h \in G$ we have

$$\langle (\pi \otimes \pi^{\rm op} \otimes {\rm id})(gh)\eta, \eta\rangle = \langle (\pi \otimes \pi^{\rm op} \otimes {\rm id})(hg)\eta, \eta\rangle.$$

We may then consider a unit vector η_0 in the algebraic span ${\rm sp}\{\pi(G)\xi\} \otimes \ell^2\mathbb{N}$ such that $\|\eta - \eta_0\|_2 < 1/4$. Thus, the function of positive type $\varphi(g) = \langle (\pi \otimes \pi^{\rm op} \otimes {\rm id})(g)\eta_0, \eta_0\rangle$ satisfies $|\varphi(gh) - \varphi(hg)| < 1/2$ for all $g, h \in G$. Moreover, we have that φ is identically 1 on some compact open subgroup $K_0 < G$.

Since G is non-discrete and simple, and since K_0 is commensurated by G, Theorem 3 in [BL89] shows that the set of indices $\{[K_0 : K_0 \cap gK_0g^{-1}] \mid g \in G\}$ is unbounded, and from this it is not hard to see that $E = \cup_{g\in G} gK_0g^{-1}$ has infinite Haar measure. However, $|1 - \varphi(h)| < 1/2$ for all $h \in E$ and hence it follows that $\pi \otimes \pi \otimes {\rm id}$ is not mixing. By the Howe-Moore property there must then exist a non-zero invariant vector, showing that π has a finite dimensional invariant subspace, and again using the Howe-Moore property it follows that π has an invariant vector. If $p \in \mathcal{P}(M)$ is the projection onto the space of G-invariant vectors, then $p \in \mathcal{Z}(M) = \mathbb{C}$ and so $p = 1$, hence π is the trivial homomorphism. □

If G is a Polish group, Λ is a countable group, and we have a homomorphism $\iota : \Lambda \to G$ with dense image, then given a representation $\pi : \Lambda \to \mathcal{U}(M)$ into a finite von Neumann algebra we may consider

$$M_0 = \{x \in M \mid \|\pi(\lambda_n)x\pi(\lambda_n^{-1}) - x\|_2 \to 0 \text{ whenever } \iota(\lambda_n) \to e \text{ in } G\}.$$

Note that $M_0 \subset M$ is a von Neumann subalgebra such that $\pi(\lambda)M_0\pi(\lambda^{-1}) = M_0$ for all $\lambda \in \Lambda$. We therefore obtain a continuous action $\alpha : G \to {\rm Aut}(M_0, \tau)$ by defining $\alpha_g(x) = \lim_{\iota(\lambda)\to g, \lambda\in\Lambda} \pi(\lambda)x\pi(\lambda^{-1})$. We will call M_0 the G-algebra (with respect to the map $\iota : \Lambda \to G$) of the representation π.

Lemma 4.3. *Let G be a Polish group, suppose that $\Lambda < G$ is a countable dense subgroup such that any trace preserving ergodic action of G on a finite von Neumann algebra is properly outer when restricted to Λ. Then for any representation $\pi : \Lambda \to \mathcal{U}(M)$ into a finite factor M, such that $\pi(\Lambda)'' = M$, the G-algebra of π is $\mathbb{C}$.*

Proof. Let $M_0 \subset M$ be the G-algebra of π, and let $\alpha : G \to {\rm Aut}(M_0, \tau)$ be the associated continuous action as described above. Note that this is ergodic since M is a factor. We denote by E_0 the trace preserving conditional expectation from M to M_0. If $M_0 \neq \mathbb{C}$, then by assumption we have that the action α restricted to Λ is properly outer. If $x \in M_0$, and $\lambda \in \Lambda$ then we have $\alpha_\lambda(x)E_0(\pi(\lambda)) = E_0(\alpha_\lambda(x)\pi(\lambda)) = E_0(\pi(\lambda)x) = E_0(\pi(\lambda))x$, and hence since the action of Λ is properly outer we must have $E_0(\pi(\lambda)) = 0$ for each $\lambda \in \Lambda \setminus \{e\}$. Since $M = \pi(\Lambda)''$ we would then have $M_0 = E_0(M) = \mathbb{C}$, giving a contradiction. □

Theorem 4.4. *Let G be a Polish group, and $\Lambda < G$ a countable dense subgroup such that each proper closed normal subgroup of G intersects trivially with Λ. Suppose $G \curvearrowright (Y, \eta)$ is ergodic, and $\pi : \Lambda \to \mathcal{U}(M)$ is a representation into a finite factor such that $M = \pi(\Lambda)''$, and such that the G-algebra with respect to π is $\mathbb{C}$. If $N \subset M$ is a von Neumann subalgebra, and π is not the left regular representation then*

$$\{\sigma_\lambda^0 \otimes (JE_N(\pi(\lambda))J) \mid \lambda \in \Lambda\}' \cap L^\infty(Y, \eta) \overline{\otimes} \mathcal{B}(L^2 N) = 1 \otimes N.$$

Proof. Set $\mathcal{Q} = \{\sigma_\lambda^0 \otimes (JE_N(\pi(\lambda))J) \mid \lambda \in \Lambda\}''$. As π is not the left regular representation fix $\lambda_0 \in \Lambda \setminus \{e\}$ such that $|\tau(\pi(\lambda_0))| > 0$. For each non-empty open set $O \subset G$ we define $\mathcal{K}_O = \overline{co}\{\pi(h\lambda_0 h^{-1}) \mid h \in \Lambda \cap O\}$ where the closure is taken in the $\| \cdot \|_2$-topology (which is equal to the closure in the weak operator topology since $\mathcal{K}_O$ is convex). We let $\mathcal{K} = \cap_{O \in \mathcal{N}(e)} \mathcal{K}_O$, where $\mathcal{N}(e)$ is the space of all open neighborhoods of the identity in G.

Since $\mathcal{K}$ is a $\| \cdot \|_2$-closed convex set it has a unique element $x \in \mathcal{K}$ which minimizes $\| \cdot \|_2$, and note that $x \neq 0$ since $\tau(y) = \tau(\pi(\lambda_0)) \neq 0$ for each $y \in \mathcal{K}$. If $\{\lambda_n\} \subset \Lambda$ is a sequence such that $\lambda_n \to e$ in G, then as $\pi(\lambda_n)\mathcal{K}_{\lambda_n^{-1}O}\pi(\lambda_n^{-1}) = \mathcal{K}_O$, we have that for each $O \in \mathcal{N}(e)$, there is large enough $N \in \mathbb{N}$ such that $\pi(\lambda_n)x\pi(\lambda_n^{-1}) \in \mathcal{K}_O$ for all $n \geq N$. Consequently, if y is any weak operator topology cluster point of the sequence $\{\pi(\lambda_n)x\pi(\lambda_n^{-1})\}$, then $y \in \mathcal{K}$, and $\|y\|_2 \leq \|x\|_2$ which implies $y = x$ by uniqueness.

Therefore $\pi(\lambda_n)x\pi(\lambda_n)$ converges to x in the weak operator topology and hence

$$\|\pi(\lambda_n)x\pi(\lambda_n) - x\|_2^2 = 2\|x\|_2^2 - 2\Re(\langle \pi(\lambda_n)x\pi(\lambda_n), x\rangle) \to 0.$$

Hence x is in the G-algebra of π, which is $\mathbb{C}$ by hypothesis, and so $x = \tau(\pi(\lambda_0)) \in \mathbb{C}$.

We will now prove that $\sigma_{\lambda_0}^0 \otimes 1 \in \mathcal{Q}$. Indeed, suppose $\varepsilon > 0$, and we have vectors $\xi_1 \in L^2(Y, \eta)$, and $\xi_2 \in N \subset L^2 M$, such that $\|\xi_1\|_2, \|\xi_2\|_\infty \leq 1$. Then by continuity of the G action on (Y, η) there exists an open neighborhood $O \in \mathcal{N}(e)$ such that $\|\sigma_{g\lambda_0 g^{-1}}^0(\xi_1) - \sigma_{\lambda_0}^0(\xi_1)\|_2 < \varepsilon$ for all $g \in O$. And from above, there exists a convex combination $\sum_{j=1}^n \alpha_j \pi(\lambda_j \lambda_0 \lambda_j^{-1})$ such that $\lambda_j \in O$ for all $1 \leq j \leq n$, and $\| \sum_{j=1}^n \alpha_j \pi(\lambda_j \lambda_0 \lambda_j^{-1}) - \tau(\pi(\lambda_0))\|_2 < \varepsilon$. Hence,

$$\begin{aligned}
&\|(\sigma_{\lambda_0}^0 \otimes \tau(\pi(\lambda_0)) - \sum_{j=1}^n \alpha_j \sigma_{\lambda_j \lambda_0 \lambda_j^{-1}}^0 \otimes E_N(\pi(\lambda_j \lambda_0 \lambda_j^{-1})))(\xi_1 \otimes \xi_2)\|_2 \\
&\quad \leq \sum_{j=1}^n \alpha_j \|(\sigma_{\lambda_0}^0 - \sigma_{\lambda_j \lambda_0 \lambda_j^{-1}}^0)(\xi_1)\|_2 \|\xi_2\|_2 \\
&\qquad + \|\sigma_{\lambda_0}^0 \otimes (\tau(\pi(\lambda_0)) - \sum_{j=1}^n \alpha_j E_N(\pi(\lambda_j \lambda_0 \lambda_j^{-1})))(\xi_1 \otimes \xi_2)\|_2 \\
&\quad < \varepsilon + \|E_N(\tau(\pi(\lambda_0)) - \sum_{j=1}^n \alpha_j \pi(\lambda_j \lambda_0 \lambda_j^{-1}))\xi_2\|_2 < \varepsilon + \varepsilon \|\xi_2\|_\infty \leq 2\varepsilon.
\end{aligned}$$

As the operators above are uniformly bounded, and the span of vectors of the form $\xi_1 \otimes \xi_2$ is dense in $L^2(Y,\eta)\overline{\otimes}L^2N$, we then have that $\sigma^0_{\lambda_0} \otimes \tau(\pi(\lambda_0))$ (and hence also $\sigma^0_{\lambda_0} \otimes 1$ since $\tau(\pi(\lambda_0)) \neq 0$) is in the strong operator closure of $\{\sigma^0_\lambda \otimes (JE_N(\pi(\lambda))J) \mid \lambda \in \Lambda\}$.

We have therefore shown that $\sigma^0_g \otimes 1 \in \mathcal{Q}$ whenever $g \in \Lambda$ such that $\tau(\pi(g)) \neq 0$. As the set of such $g \in \Lambda$ is preserved under conjugation, we then have that the non-trivial subgroup $\Lambda_0 < \Lambda$ they generate is normal, and since the Λ intersection with any proper subproduct of G is trivial we then have $\overline{\Lambda_0} = G$ showing that $\sigma^0_g \otimes 1 \in \mathcal{Q}$ for all $g \in G$, and hence we have $1 \otimes (JE_N(\pi(\lambda))J) \in \mathcal{Q}$, for all $\lambda \in \Lambda$.

By ergodicity of the G action on (Y,η) it follows that $\mathcal{Q}' \cap L^\infty(Y,\eta)\overline{\otimes}\mathcal{B}(L^2N) \subset 1 \otimes \mathcal{B}(L^2N)$, since $\sigma^0_g \otimes 1 \in \mathcal{Q}$ for all $g \in G$. Also, since $\pi(\Lambda)'' = M$ we have that $\{E_N(\pi(\lambda)) \mid \lambda \in \Lambda\}$ spans a strong operator topology dense subset of N, hence $\mathcal{Q}' \cap 1 \otimes \mathcal{B}(L^2N) \subset 1 \otimes (JNJ)' = 1 \otimes N$. □

5 Finite factor representations restricted to the lattice

Proposition 5.1. *Suppose G is a second countable locally compact group, and $\Gamma < \Lambda < G$ where $\Gamma < G$ is a lattice, and $\Lambda < G$ is a countable dense subgroup which contains and commensurates Γ. Suppose also that $\pi : \Lambda \to \mathcal{U}(M)$ is a finite von Neumann algebra representation such that $\pi(\Lambda)'' = M$, and set $N = \pi(\Gamma)''$. Let $G \curvearrowright (B,\eta)$ be a quasi-invariant action which is contractive when restricted to Γ. Then we have*

$$\begin{aligned}&\{\sigma^0_\gamma \otimes (JE_N(\pi(\gamma))J) \mid \gamma \in \Gamma\}' \cap L^\infty(B,\eta)\overline{\otimes}\mathcal{B}(L^2(N,\tau))\\ =&\{\sigma^0_\lambda \otimes (JE_N(\pi(\lambda))J) \mid \lambda \in \Lambda\}' \cap L^\infty(B,\eta)\overline{\otimes}\mathcal{B}(L^2(N,\tau)).\end{aligned}$$

Proof. Fix $\lambda \in \Lambda$ and consider the polar decomposition $E_N(\pi(\lambda)) = v_\lambda |E_N(\pi(\lambda))|$. If we set $\Gamma_0 = \Gamma \cap \lambda\Gamma\lambda^{-1}$, then we have $E_N(\pi(\lambda^{-1}))E_N(\pi(\lambda)) \in \pi(\lambda^{-1}\Gamma_0\lambda)' \cap N$, and hence $|E_N(\pi(\lambda))| \in \pi(\lambda^{-1}\Gamma_0\lambda)' \cap N$. Thus, for $\gamma \in \Gamma_0$ we have $\pi(\gamma)v_\lambda = v_\lambda \pi(\lambda^{-1}\gamma\lambda)$, and taking adjoints then also gives $v_\lambda^* \pi(\gamma^{-1}) = \pi(\lambda^{-1}\gamma^{-1}\lambda)v_\lambda^*$. If we define $p_\lambda = v_\lambda v_\lambda^* \in \mathcal{P}(N)$, then we have $p_\lambda \in \pi(\Gamma_0)' \cap N$, and hence by Corollary 3.3, $Jp_\lambda J, J|E_N(\pi(\lambda))|J \in \mathcal{B}_N'$. Similarly, if we define $q_\lambda = v_\lambda^* v_\lambda$ then we also have $Jq_\lambda J \in \mathcal{B}_N'$.

Define the map $\Phi : \mathcal{B}_N \to L^\infty(B,\eta)\overline{\otimes}\mathcal{B}(L^2(N,\tau))$, by

$$\Phi(x) = \sigma_\lambda \otimes \mathrm{Ad}(Jv_\lambda J)(x) + (1 - Jp_\lambda J)x.$$

It is easy to see that Φ is N-bimodular normal unital completely positive and if $\gamma \in \Gamma_0$, and $x \in \mathcal{B}_N$ then we have

$$\begin{aligned}
&\sigma_\gamma \otimes \mathrm{Ad}(J\pi(\gamma)J)(\Phi(x)) \\
&\quad = \sigma_{\gamma\lambda} \otimes \mathrm{Ad}(J\pi(\gamma)v_\lambda J)(x) + \sigma_\gamma \otimes \mathrm{Ad}(J\pi(\gamma)J)((1 - Jp_\lambda J)x) \\
&\quad = \sigma_\lambda \otimes \mathrm{Ad}(Jv_\lambda J)(\sigma_{\lambda^{-1}\gamma\lambda} \otimes \mathrm{Ad}(J\pi(\lambda^{-1}\gamma\lambda)J)(x)) \\
&\qquad\quad + (1 - Jp_\lambda J)\sigma_\gamma \otimes \mathrm{Ad}(J\pi(\gamma)J)(x) \\
&\quad = \Phi(x).
\end{aligned}$$

Hence $\Phi : \mathcal{B}_N \to \widetilde{\mathcal{B}_N} = \{\sigma^0_\gamma \otimes (JE_N(\pi(\gamma))J) \mid \gamma \in \Gamma_0\}' \cap L^\infty(B,\eta)\overline{\otimes}\mathcal{B}(L^2(N,\tau))$, and by Theorem 3.2 we then have $\Phi = \mathrm{id}$.

Hence, for $x \in \mathcal{B}_N$ we have $\sigma^0_\lambda \otimes (Jv_\lambda J)x\sigma^0_{\lambda^{-1}} \otimes (Jv^*_\lambda J) = Jp_\lambda Jx = xJp_\lambda J$. Multiplying on the right by $\sigma^0_\lambda \otimes Jv_\lambda J$, and using that $Jq_\lambda J \in \mathcal{B}'_N$, we then have $\sigma^0_\lambda \otimes (Jv_\lambda J)x = x\sigma^0_\lambda \otimes (Jv_\lambda J)$. As, $J|E_N(\pi(\lambda))|J \in \mathcal{B}'_N$, the result then follows. □

Theorem 5.2. *Suppose G is a locally compact group which is a product of simple groups with the Howe-Moore property, and $\Lambda < G$ is a countable dense subgroup which commensurates a (strongly) irreducible lattice $\Gamma < G$, which in the case G is totally disconnected is not a torsion group. Suppose also that $\pi : \Lambda \to \mathcal{U}(M)$ is a finite factor representation such that $\pi(\Lambda)'' = M$, and set $N = \pi(\Gamma)''$. If π is not the left regular representation then N is injective.*

Proof. If we take any Poisson boundary (B,η) of G then $G \curvearrowright (B,\eta)$ is amenable and contractive, thus the restriction to Γ is again amenable and contractive. Since $\Gamma \curvearrowright (B,\eta)$ is amenable, Theorem 5.1 in [Zim77] shows that $\mathcal{B}_N$ is injective.

By Proposition 5.1, we have that $\mathcal{B}_N \subset \{\sigma^0_\lambda \otimes (JE_N(\pi(\lambda))J) \mid \lambda \in \Lambda\}'$, and so by combining Lemma 4.3, with Proposition 4.1, and Theorem 4.4, if π is not the left regular representation then $\mathcal{B}_N \subset 1 \otimes N$, and hence $1 \otimes N = \mathcal{B}_N$ is then injective. □

If, in addition, G has property (T) then the conclusion of the previous theorem can be strengthened. In the sequel we will see that using the notion of resolutions from [Cor06] this is also the case when G has one non-compact factor with property (T) (see also [Cre13]).

Corollary 5.3. *Suppose $\Gamma < \Lambda < G$ is as in the hypotheses of Theorem 5.2 and in addition G has property (T). Suppose also that $\pi : \Lambda \to \mathcal{U}(M)$ is a finite factor representation such that $\pi(\Lambda)'' = M$, and set $N = \pi(\Gamma)''$. If π is not the left regular representation then N is a direct sum of matrix algebras.*

Proof. If π is not the left regular representation then by Theorem 5.2 we have that N is injective. Since G has property (T) so does Γ, and hence it then follows from Theorem C in [Rob93] that $\pi(\Gamma) \subset \mathcal{U}(N)$ is precompact in the strong operator topology. Thus, it follows from the Peter-Weyl theorem that N is a direct sum of matrix algebras. □

6 Operator algebraic superrigidity for commensurators

Proposition 6.1. *Suppose G is a second countable locally compact group, $H \lhd G$ a closed normal subgroup, and $\Gamma < G$ a lattice such that the image of Γ is dense in G/H. Suppose also that $\pi : \Gamma \to \mathcal{U}(M)$ is a homomorphism into the unitary group of a finite factor M such that $\pi(\Gamma)'' = M$. For any compact neighborhood of the identity $U \subset G/H$ set $\Gamma_U = \{\gamma \in \Gamma \mid \gamma H \in U\}$.*

If $\pi(\Gamma_U) \subset \mathcal{U}(M)$ is precompact in the strong operator topology for some compact neighborhood of the identity $U \subset G/H$, then the G/H-algebra $M_0 \subset M$ (with respect to the quotient map $\Gamma \to G/H$) is finite index.

Proof. For each compact neighborhood of the identity $U \subset G/H$ let K_U be the strong operator topology closure of $\pi(\Gamma_U)$, and set K be the intersection of all K_U, and set $N = K''$. By hypothesis K is compact, hence $N = K''$ is completely atomic. If $\gamma \in \Gamma$ then we have $\pi(\gamma)K_U\pi(\gamma^{-1}) = K_{\gamma U \gamma^{-1}}$ and hence it follows that $\pi(\gamma)N\pi(\gamma^{-1}) = N$ for all $\gamma \in \Gamma$.

If $p \in \mathcal{P}(N)$ is a minimal central projection then we have $\vee_{\gamma\in\Gamma}\pi(\gamma)p\pi(\gamma^{-1})$ is a non-zero projection which is central since $\pi(\Gamma)'' = M$. Thus $\vee_{\gamma\in\Gamma}\pi(\gamma)p\pi(\gamma^{-1}) = 1$, and since $\pi(\gamma_1)p\pi(\gamma_1^{-1})$ and $\pi(\gamma_2)p\pi(\gamma_2^{-1})$ are either equal or orthogonal for all $\gamma_1, \gamma_2 \in \Gamma$, it then follows that $\mathcal{Z}(N)$ is finite dimensional and hence so is N.

We set $M_0 = N' \cap M$ which is then a finite index von Neumann subalgebra of M. If $\{\gamma_n\} \subset \Gamma$ is a sequence such that $\gamma_n \to e$ in G/H, then by hypothesis we have that $\{\pi(\gamma_n)\}$ is precompact in the strong operator topology and hence for any subsequence $\{\pi(\gamma_{n_k})\}$ of $\{\pi(\gamma_n)\}$ there exists a unitary $u \in \mathcal{U}(N)$ which is a strong operator topology cluster point, and hence for $x \in M_0$ we have that $x = uxu^*$ is a strong operator topology cluster point of $\{\pi(\gamma_{n_k})x\pi(\gamma_{n_k}^{-1}\}$. As the subsequence $\{\pi(\gamma_{n_k})\}$ was arbitrary it then follows that $\pi(\gamma_n)x\pi(\gamma_n^{-1}) \to x$ in the strong operator topology. Thus, we have shown that the finite index subalgebra $M_0 \subset M$ is contained in the G/H-algebra. □

Proposition 6.2. *Suppose G and H are non-compact non-discrete second countable locally compact groups such that H is a product of non-compact non-discrete groups with the Howe-Moore property, suppose also that $\Gamma < G \times H$ is a (strongly) irreducible lattice, such that in the case when G is totally disconnected there is an element $\gamma_0 \in \Gamma$ with the projection of $\langle \gamma_0 \rangle$ to G being unbounded, and $\pi : \Gamma \to \mathcal{U}(M)$ is a representation into a finite von Neumann algebra with $\pi(\Gamma)'' = M$ and such that either*

1. *G has property (T) and M has the Haagerup property, or*

2. *$\pi(\Gamma \cap (G \times U))$ is precompact for all compact neighborhoods of the identity $U \subset H$,*

then M is completely atomic.

Proof. By considering the integral decomposition of M into factors it is enough to treat the case when M is a factor [Tho64b]. For the first case, since G has property (T) it follows from Theorem 1.8 and Proposition 1.11 in [Cor06] that the subset $\Gamma_U = \Gamma \cap (G \times U) \subset \Gamma$ has relative property (T) for some (and hence all) compact neighborhood of the identity $U \subset H$. The same argument in [CJ85] for the case of property (T) groups then implies that $\pi(\Gamma_U)$ is precompact in the strong operator topology (see also [Rob93]), and hence we have reduced the problem to the second case.

For the second case we may then apply Proposition 6.1 to conclude that the G-algebra $M_0 \subset M$ is finite index. However, by Proposition 4.1 and Lemma 4.3 we must have $M_0 = \mathbb{C}$, and hence M is finite dimensional. □

Theorem 6.3. *Suppose G is a locally compact group which is a product of simple groups with the Howe-Moore property, and at least one factor having property (T), suppose also that $\Lambda < G$ is a countable dense subgroup which commensurates a (strongly) irreducible lattice $\Gamma < G$, which in the case G is totally disconnected is not a torsion group, and such that $\Lambda /\!\!/ \Gamma$ is a product of simple groups with the Howe-Moore property.*

If $\pi : \Lambda \to \mathcal{U}(M)$ is a finite factor representation such that $\pi(\Lambda)'' = M$, and if π is not the left regular representation, then M is finite dimensional, and $M = \pi(\Gamma)''$.

Proof. If we set $N = \pi(\Gamma)''$ then by Theorem 5.2 we have that N is injective and hence also has the Haagerup property. By Proposition 6.2 we then have that N is a direct sum of matrix algebras.

If we consider, as in Theorem 9.2 in [CP12], the diagonal lattice embedding $\Lambda \to G \times (\Lambda /\!\!/ \Gamma)$, then applying Proposition 6.2 a second time gives the result. □

The previous theorem easily implies Theorem B from the introduction.

Corollary 6.4. *Suppose $\Gamma < \Lambda < G$ is as in the hypotheses of Theorem 6.3. Then any probability measure-preserving ergodic action of Λ on a standard Lebesgue space is free.*

Proof. Since every representation generating a finite factor must either be the left-regular, or finite dimensional, and since there are only countably many finite dimensional representations (Theorem 10.3 in [Sha00]), this follows directly from Theorem 2.11 in [DM12] or Theorem 3.2 in [PT13]. □

Corollary 6.5. *Let G be a locally compact group which is a product of at least two simple groups with the Howe-Moore property. Suppose that at least one factor of G has property (T), at least one factor is totally disconnected, and if there exists connected factors then at least one should have property (T). Moreover, assume that Γ is not a torsion group in the case when G is totally disconnected. Let $\Gamma < G$ be a (strongly) irreducible lattice. If $\pi : \Gamma \to \mathcal{U}(M)$ is a finite factor representation such that $\pi(\Gamma)'' = M$, and if π is not the left regular representation, then M is finite dimensional. Moreover, Γ has at most countably* *many finite dimensional irreducible representations, and any probability measure-preserving ergodic action of* Γ *on a standard Lebesgue space is free.*

Proof. If we write $G = G_1 \times G_2$ where G_i are non-trivial, with G_1 having property (T), and G_2 being totally disconnected, then we can consider a compact open subgroup $K < G_2$, and if we set $\Gamma_0 = \Gamma \cap G_1 \times K$, then Γ_0 projects down to a lattice in G_1, which is commensurated by the projection of Γ. As in Section 10 of [CP12], the result then follows from Theorem 6.3 and Corollary 6.4 by considering the inclusion $\Gamma_0 < \Gamma < G_1$. □

The previous corollaries easily imply Theorem A and Corollary E from the introduction.

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Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240, U.S.A.