Notes on Veech's Theorem

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The aim of these notes is to prove the main result from [Vee79].

The following lemma is adapted from [Kec95, Theorem 8.51]. It is similar to [Nam74, Theorem 1.2], which has a slightly stronger conclusion when Y is compact.

Lemma 1. *Let X be a topological space, Y and Z metric spaces, and suppose $f : X \times Y \to Z$ is separately continuous, i.e., $x \mapsto f(x, y)$ is continuous for each $y \in Y$, and $y \mapsto f(x, y)$ is continuous for each $x \in X$. Then for each $y \in Y$, there exists a subset $E_y \subset X \times \{y\}$, which is comeager in $X \times \{y\}$, such that f is continuous at each point in E_y.*

Proof. For each $n, k \in \mathbb{N}$, set

$$F_{n,k} = \{(x, y) \in X \times Y \mid d_Z(f(x, u), f(x, v)) \leq 2^{-n} \text{ for all } u, v \in B(y, 2^{-k})\}.$$

Since $y \mapsto f(x, y)$ is continuous for each $x \in X$ we have that $X \times Y = \cap_n \cup_k F_{n,k}$.

If $\{(x_i, y_i)\} \subset F_{n,k}$ is a net such that $x_i \to x$, and $y_i \to y$, and if $u, v \in B(y, 2^{-k})$, then we may choose i_0 such that $u, v \in B(y_i, 2^{-k})$ for all $i \geq i_0$, and hence $d_Z(f(x_i, u), f(x_i, v)) \leq 2^{-n}$ for all $i \geq i_0$. As $x \mapsto f(x, u)$ and $x \mapsto f(x, v)$ are continuous we then have $d_Z(f(x, u), f(x, v)) \leq 2^{-n}$ and hence $(x, y) \in F_{n,k}$. Thus, we have shown that $F_{n,k}$ is closed for each $n, k \in \mathbb{N}$.

Fix $y \in Y$ and set $D_y = \cup_n \cup_k \partial(F_{n,k} \cap (X \times \{y\}))$. Then D_y is meager in $X \times \{y\}$, and we will show that if $(x, y) \in E_y = (X \times \{y\}) \setminus D_y$, then f is continuous at (x, y). Indeed, let $\varepsilon > 0$, and take $k, n \in \mathbb{N}$ so that $2^{-n} \leq \varepsilon$, and $(x, y) \in F_{n,k}$. Since $x \notin D_y$ we have that x is an interior point in $F_{n,k} \cap (X \times \{y\})$, and since $x \mapsto f(x, y)$ is continuous there then exists an open neighborhood V of x such that $V \times \{y\} \subset F_{n,k}$, and $d_Z(f(x, y), f(s, y)) \leq \varepsilon$ for all $s \in V$. If $s \in V$, and $t \in B(y, 2^{-k})$ we then have

$$d_Z(f(x, y), f(s, t)) \leq d_Z(f(x, y), f(s, y)) + d_Z(f(s, y), f(s, t)) \leq 2\varepsilon.$$

□

Corollary 2 (Fort [For55])**.** *Let G be group with a Baire topology such that for each $h \in G$ the function $g \mapsto hg$ is continuous. Let X be a metric space, and suppose that $G \curvearrowright X$ is an action such that $(g, x) \mapsto gx$ is separately continuous. Then the action is jointly continuous.*

Proof. Fix $(g_0, x_0) \in G \times X$. Since G is Baire, the previous lemma shows that there exists $h_0 \in G$ such that the map $(g, x) \mapsto gx$ is continuous at (h_0, x_0). If we first apply the continuous map $(g, x) \mapsto ((h_0 g_0^{-1})g, x)$, then we see that $(g, x) \mapsto gx$ is continuous at (g_0, x_0). □

Corollary 3. *Let G be a group with a topology which is metrizable and Baire (e.g., if the topology is Polish). Suppose that $(g, h) \mapsto gh$ is separately continuous, then this map is jointly continuous.*

Proof. We just consider the action of the group on itself given by left multiplication and then apply the previous corollary. □

Theorem 4. *Let G be a group with a Baire topology such that multiplication is jointly continuous. Then G is a topological group, i.e., inversion is continuous.*

Proof. It is enough to show that inversion is continuous on a comeager set (and hence at some point since G is Baire). Indeed, if inversion is continuous at $a_0 \in G$ and if $a \in G$ is arbitrary, then if $a_n \to a$, then $a_n(a^{-1}a_0) \to a_0$, hence $a_0^{-1}aa_n^{-1} \to a_0^{-1}$, and we then have $a_n^{-1} \to a^{-1}$ which shows that inversion is continuous at a.

□

Lemma 5. *Let G be a Polish group, X a Banach space, and $\pi : G \to \mathrm{Isom}(X)$ a SOT-continuous representation. If $k_n, \tilde{k}_n, g_n \in G$ such that $k_n \to k$, $\tilde{k}_n \to \tilde{k}$, and WOT-$\lim_{n\to\infty} \pi(g_n) = T$, then WOT-$\lim_{n\to\infty} \pi(k_n g_n \tilde{k}_n) = \pi(k)T\pi(\tilde{k})$.*

Proof. It is enough to consider the case when X is separable. We first note that it is easy to see that WOT-$\lim_{n\to\infty} \pi(g_n\tilde{k}_n) = T\pi(\tilde{k})$. Thus, replacing g_n with $g_n\tilde{k}_n$, and T with $T\pi(\tilde{k})$ we may assume that $\tilde{k}_n = \tilde{k} = e$.

Since X is separable, $\mathcal{B}(X)$ is metrizable in the weak operator topology. It is easy to see that the action of G on $\mathcal{B}(X)$ by left multiplication is separately continuous. Fort's joint continuity theorem then shows that the action of G on $\mathcal{B}(X)$ is jointly continuous. Hence, as $k_n \to k$, and WOT-$\lim_{n\to\infty} \pi(g_n) = T$ it follows that WOT-$\lim_{n\to\infty} \pi(k_n g_n) = \pi(k)T$. □

Lemma 6 (Mautner [Mau57])**.** *Let G, X, and π be as above. If $g, a_n \in G$, such that WOT-$\lim_{n\to\infty} \pi(a_n) = T$, and $a_n^{-1}ga_n \to e$, then $\pi(g)T = T$.*

Proof. We have $\pi(g)T =$WOT-$\lim_{n\to\infty} \pi(ga_n) =$WOT-$\lim_{n\to\infty} \pi(a_n(a_n^{-1}g_n a_n)) = T$. □

Theorem 7 (Veech [Vee79])**.** *Let G be a simple Lie group, X a Banach space and $\pi : G \to \mathrm{Isom}(X)$ a SOT-continuous representation such that $\pi(G)$ is WOT-precompact, then any WOT-cluster point of $\pi(G)$ is a projection onto the space of G-invariant vectors.*

Proof. We prove only the case $G = SL_2(R)$, leaving the general case to the reader. We let A_+ denote the group of diagonal matrices in G which have positive diagonal entries and we let $K = SO(2) < G$. We recall the Cartan decomposition $G = KA_+K$.

Suppose $g_n \in G$ is a sequence converging to infinity in G and let T be a WOT-cluster point. We write $g_n = k_n a_n \tilde{k}_n$ in the Cartan decomposition and taking a subsequence we will assume that for $k, \tilde{k} \in K$ we have $k_n \to k$, $\tilde{k}_n \to \tilde{k}$, and WOT-$\lim_{n\to\infty} \pi(g_n) = T$. By Lemma 5 we then have $S =$ WOT-$\lim_{n\to\infty} \pi(a_n) = \pi(k)T\pi(\tilde{k})$.

Without loss of generality we assume that the first entry in the matrices a_n are tending to infinity so that $a_n^{-1}xa_n \to e$ for all $x \in N_+$. Hence, by Mautner's lemma we have that $\pi(x)S = S$ for all $x \in N_+$. Since, N_+ is non-compact we may again use the Cartan decomposition to conclude that there is a sequence $b_n \in A_+$, $h_n, \tilde{h}_n \in K$, and $h, \tilde{h} \in K$, such that $b_n \to \infty$, $h_n \to h$, $\tilde{h}_n \to \tilde{h}$, and $\pi(h_n b_n \tilde{h}_n)S = S$ for all $n \in \mathbb{N}$. Therefore we have SOT-$\lim_{n\to\infty} \pi(b_n)\pi(\tilde{h})S = \pi(h)S$.

Now fix $x \in X$ in the range of $\pi(\tilde{h})S = \pi(k\tilde{h})T\pi(\tilde{k})$. Since $b_n \to \infty$, and $\pi(b_n)x$ converges it follows that for each $\varepsilon > 0$, $\{b \in A_+ \mid \|\pi(b)x - x\| < \varepsilon\}$ is non-compact. Thus, there exists a sequence $c_n \in A_+$ such that SOT-$\lim_{n\to\infty} \pi(c_n)x = x$. We then have also that SOT-$\lim_{n\to\infty} \pi(c_n^{-1})x = x$, and hence taking a subsequence and replacing c_n with c_n^{-1} if necessary we may assume that the first diagonal entries of c_n are tending to infinity (and hence the first diagonal entries of c_n^{-1} are tending to 0). Since WOT-$\lim_{n\to\infty}\pi(c_n)x = x$ we then have from Mautner's lemma that $\pi(g)x = x$ for all $g \in N_+$. Since we also have WOT-$\lim_{n\to\infty}\pi(c_n^{-1})x = x$, Mautner's lemma also shows that $\pi(g)x = x$ for all $g \in N_-$.

Finally, since $\langle N_+, N_-\rangle = G$ we conclude that x is G-invariant. Since x was an arbitrary vector in the range of $\pi(k\tilde{h})T\pi(\tilde{k})$, and since $\pi(k\tilde{h}), \pi(\tilde{k}) \in \mathrm{Isom}(X)$ it then follows that every vector in the range of T is G-invariant. Thus, T is a projection onto the space of G-invariant vectors. □

Restricting Veech's result to Hilbert spaces we obtain:

Corollary 8 (Howe-Moore [HM79])**.** *Let G be a simple Lie group, and $\pi : G \to \mathcal{U}(\mathcal{H})$ a SOT-continuous representation without G-invariant vectors, then WOT$-\lim_{n\to\infty}\pi(g_n) = 0$, whenever $g_n \to \infty$.*

Corollary 9 (Veech [Vee79])**.** *Let G be a simple Lie group, then $W(G) = \mathbb{C} + C_0(G)$.*

Proof. We consider the isometric representation $L : G \to \mathrm{Isom}(W(G))$ given by $L_g(f)(x) = f(g^{-1}x)$. Then it is easy to see that this representation satisfies the hypotheses of the previous theorem and hence for each $f \in W(G)$ there exists $\varphi(f) \in \mathbb{C}$ such that WOT-$\lim_{n\to\infty} L_{g_n}(f) = \varphi(f)$ whenever $g_n \to \infty$. Since point evaluation at e is weakly continuous on $W(G)$ we then conclude that $f(g_n^{-1}) \to \varphi(f)$, and hence $f - \varphi(f) \in C_0(G)$. □

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