Lecture notes on “Completely positive semigroups and applications to II_1 factors”*

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1 Completely positive maps

An **operator system** E is a closed self adjoint subspace of a unital C^*-algebra A such that $1 \in E$. We denote by $\mathbb{M}_n(E)$ the space of $n \times n$ matrices over E. If A is a C^*-algebra, then

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$\mathbb{M}_n(A) \cong A \otimes \mathbb{M}_n(\mathbb{C})$ has a unique norm for which it is again a C^*-algebra, where the adjoint given by $[a_{i,j}]^* = [a_{j,i}^*]$. This can be seen easily for C^*-subalgebras of $\mathcal{B}(\mathcal{H})$, and the general case then follows since every C^*-algebra is isomorphic to a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ by the GNS-construction. In particular, if $E \subset A$ is an operator system then $\mathbb{M}_n(E)$ is again an operator system when viewed as a subspace of the C^*-algebra $\mathbb{M}_n(A)$.

If $\phi : E \to F$ is a linear map between operator systems, then we denote by $\phi^{(n)} : \mathbb{M}_n(E) \to \mathbb{M}_n(F)$ the map defined by $\phi^{(n)}([a_{i,j}]) = [\phi(a_{i,j})]$. We say that ϕ is **positive** if $\phi(a) \geq 0$, whenever $a \geq 0$. If $\phi^{(n)}$ is positive then we say that ϕ is ***n*-positive** and if ϕ is n-positive for every $n \in \mathbb{N}$ then we say that ϕ is **completely positive**. We'll say that ϕ is **unital** if $\phi(1) = 1$. Throughout these notes we will use the abbreviation u.c.p. for unital completely positive.

If $\phi : E \to F$ is positive then it follows easily that ϕ is Hermitian, i.e., $\phi(x^*) = \phi(x)^*$, for all $x \in E$. Also note that positive maps are continuous. Indeed, if $\phi : E \to F$ is positive and $\{x_n\}_n$ is a sequence which converges to 0 in E, such that $\lim_{n\to\infty} \phi(x_n) = y$, then since $\omega \circ \phi$ is positive (and hence continuous by the Cauchy-Schwarz inequality) for any state $\omega \in S(B)$, we then have $\omega(y) = 0$. Since ω was arbitrary it then follows that $y = 0$ and hence ϕ is bounded by the closed graph theorem.

We also remark that it follows form the Hahn-Banach theorem that any positive linear functional on E extends to a positive linear functional on A which has the same norm.

Lemma 1.1. *If A and B are unital C^*-algebras and $\phi : A \to B$ is a unital contraction then ϕ is positive.*

Proof. We first show that ϕ is Hermitian. Suppose $x = x^* \in A$ such that $\phi(x) = a + ib$ where $a, b \in B$ are self-adjoint. Assume $\|x\| \leq 1$. If $\lambda \in \sigma(b)$ then for all $t \in \mathbb{R}$ we have

$$(\lambda + t)^2 \leq \|b + t\|^2 \leq \|\phi(x + it)\|^2 \leq \|x + it\|^2 \leq 1 + t^2.$$

Hence $\lambda^2 + t\lambda \leq 1$, and as this is true for all t we must then have $\lambda = 0$, and hence $b = 0$.

Thus, $\omega \circ \phi$ is a state, for any state $\omega \in S(B)$. Hence if $x \geq 0$ then for any state ω we have $\omega(a) = \omega \circ \phi(x) \geq 0$. We must then have $a \geq 0$, and hence ϕ is positive. $\square$

1.1 Stinespring's Dilation Theorem

If $\pi : A \to \mathcal{B}(\mathcal{K})$ is a representation of a C^*-algebra A and $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, then the map $\phi : A \to \mathcal{B}(\mathcal{H})$ given by $\phi(x) = V^*\pi(x)V$ is completely positive. Indeed, if we consider the operator $V^{(n)} \in \mathcal{B}(\mathcal{H}^{\oplus n}, \mathcal{K}^{\oplus n})$ given by $V^{(n)}((\xi_i)_i) = (V\xi_i)_i$ then for all $x \in \mathbb{M}_n(A)$ we have

$$\begin{aligned}\phi^{(n)}(x^*x) &= {V^{(n)}}^*\pi^{(n)}(x^*x)V^{(n)} \\ &= (\pi^{(n)}(x)V^{(n)})^*(\pi^{(n)}(x)V^{(n)}) \geq 0.\end{aligned}$$

Generalizing the GNS construction, Stinespring showed that every completely positive map from A to $\mathcal{B}(\mathcal{H})$ arises in this way.

Theorem 1.2 (Stinespring [Sti55])**.** *Let A be a unital C^*-algebra, and suppose $\phi : A \to \mathcal{B}(\mathcal{H})$, then ϕ is completely positive if and only if there exists a representation $\pi : A \to \mathcal{B}(\mathcal{K})$ and a bounded operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ such that $\phi(x) = V^*\pi(x)V$. We also have $\|\phi\| = \|V\|^2$, and if ϕ is unital then V is an isometry. Moreover, if A is a von Neumann algebra and ϕ is a normal completely positive map, then π is a normal representation.*

Proof. Consider the sesquilinear form on $A \otimes \mathcal{H}$ given by $\langle a \otimes \xi, b \otimes \eta \rangle_\phi = \langle \phi(b^* a)\xi, \eta \rangle$, for $a, b \in A$, $\xi, \eta \in \mathcal{H}$. If $(a_i)_i \in A^{\oplus n}$, and $(\xi_i)_i \in \mathcal{H}^{\oplus n}$, then we have

$$\begin{aligned}\langle \sum_i a_i \otimes \xi_i, \sum_j a_j \otimes \xi_j \rangle_\phi &= \sum_{i,j} \langle \phi(a_j^* a_i)\xi_i, \xi_j \rangle \\ &= \langle \phi((a_i)_i^* (a_i)_i)(\xi_i)_i, (\xi_i)_i \rangle \geq 0.\end{aligned}$$

Thus, this form is non-negative definite and we can consider N_ϕ the kernel of this form so that $\langle \cdot, \cdot \rangle_\phi$ is positive definite on $\mathcal{K}_0 = (A \otimes \mathcal{H})/N_\phi$. Hence, we can take the Hilbert space completion $\mathcal{K} = \overline{\mathcal{K}_0}$.

As in the case of the GNS construction, we define a representation $\pi : A \to \mathcal{B}(\mathcal{K})$ by first setting $\pi_0(x)(a \otimes \xi) = (xa) \otimes \xi$ for $a \otimes \xi \in A \otimes \mathcal{H}$. Note that since ϕ is positive we have $\phi(a^* x^* x a) \leq \|x\|^2 \phi(a^* a)$, applying this to $\phi^{(n)}$ we see that $\|\pi_0(x) \sum_i a_i \otimes \xi_i\|_\phi^2 \leq \|x\|^2 \| \sum_i a_i \otimes \xi_i\|_\phi^2$. Thus, $\pi_0(x)$ descends to a well defined bounded map on $\mathcal{K}_0$ and then extends to a bounded operator $\pi(x) \in \mathcal{B}(\mathcal{K})$.

If we define $V_0 : \mathcal{H} \to \mathcal{K}_0$ by $V_0(\xi) = 1 \otimes \xi$, then we see that V_0 is bounded by $\|\phi(1)\|$ and hence extends to a bounded operator $V \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. For any $x \in A$, $\xi, \eta \in \mathcal{H}$ we then check that

$$\begin{aligned}\langle V^* \pi(x) V \xi, \eta \rangle &= \langle \pi(x)(1 \otimes \xi), 1 \otimes \eta \rangle_\phi \\ &= \langle x \otimes \xi, 1 \otimes \eta \rangle_\phi = \langle \phi(x)\xi, \eta \rangle.\end{aligned}$$

Thus, $\phi(x) = V^* \pi(x) V$ as claimed. □

Corollary 1.3 (Kadison [Kad52])**.** *If A and B are unital C^*-algebras, and $\phi : A \to B$ is u.c.p. then for all $x \in A$ we have $\phi(x)^* \phi(x) \leq \phi(x^* x)$*

Proof. We may assume that $B \subset \mathcal{B}(\mathcal{H})$. If we consider the Stinespring dilation $\phi(x) = V^* \pi(x) V$, then since ϕ is unital we have that V is an isometry. Hence $1 - VV^* \geq 0$ and so we have

$$\begin{aligned}\phi(x^* x) - \phi(x)^* \phi(x) &= V^* \pi(x^* x) V - V^* \pi(x)^* V V^* \pi(x) V \\ &= V^* \pi(x^*)(1 - VV^*)\pi(x) V \geq 0.\end{aligned}$$

□

Lemma 1.4. *A matrix $a = [a_{i,j}] \in \mathbb{M}_n(A)$ is positive if and only if*

$$\sum_{i,j=1}^n x_i^* a_{i,j} x_j \geq 0,$$

for all $x_1, \ldots, x_n \in A$.

Proof. For all $x_1, \ldots, x_n \in A$ we have that $\sum_{i,j=1}^n x_i^* a_{i,j} x_j$ is the conjugation of a by the $1 \times n$ column matrix with entries $x_1, \ldots, x_n$, hence if a is positive then so is $\sum_{i,j=1}^n x_i^* a_{i,j} x_j$.

Conversely, if $\sum_{i,j=1}^n x_i^* a_{i,j} x_j \geq 0$, for all $x_1, \ldots, x_n$ then for any representation $\pi : A \to \mathcal{B}(\mathcal{H})$, and $\xi \in \mathcal{H}$ we have

$$\left\langle (\mathrm{id} \otimes \pi)(a) \begin{pmatrix} \pi(x_1)\xi \\ \vdots \\ \pi(x_n)\xi \end{pmatrix}, \begin{pmatrix} \pi(x_1)\xi \\ \vdots \\ \pi(x_n)\xi \end{pmatrix} \right\rangle = \sum_{i,j=1}^n \langle \pi(a_{i,j})\pi(x_j)\xi, \pi(x_i)\xi \rangle$$
$$= \left\langle \pi\left(\sum_{i,j=1}^n x_i^* a_{i,j} x_j \right) \xi, \xi \right\rangle \geq 0.$$

Thus, if $\mathcal{H}$ has a cyclic vector, then $(\mathrm{id} \otimes \pi)(a) \geq 0$. But since every representation is decomposed into a direct sum of cyclic representations it then follows that $(\mathrm{id} \otimes \pi)(a) \geq 0$ for any representation, and hence $a \geq 0$ by considering a faithful representation. □

Proposition 1.5. *Let E be an operator system, and let B be an abelian C^*-algebras. If $\phi : E \to B$ is positive, then ϕ is completely positive.*

Proof. Since B is commutative we may assume $B = C_0(X)$ for some locally compact Hausdorff space X. If $a = [a_{i,j}] \in \mathbb{M}_n(E)$ such that $a \geq 0$, then for all $x_1, \ldots, x_n \in B$, and $\omega \in X$ we have

$$\left(\sum_{i,j} x_i^* \phi(a_{i,j}) x_j \right)(\omega) = \left(\sum_{i,j} \phi(\overline{x_i(\omega)} x_j(\omega) a_{i,j}) \right)(\omega)$$
$$= \phi\left(\begin{pmatrix} x_1(\omega) \\ \vdots \\ x_n(\omega) \end{pmatrix}^* a \begin{pmatrix} x_1(\omega) \\ \vdots \\ x_n(\omega) \end{pmatrix} \right)(\omega) \geq 0.$$

By Lemma 1.4, and since n was arbitrary, we then have that ϕ is completely positive. □

Proposition 1.6. *Let A and B be unital C^*-algebras such that A is abelian. If $\phi : A \to B$ is positive, then ϕ is completely positive.*

Proof. We may identify A with $C(K)$ for some compact Hausdorff space K, hence for $n \in \mathbb{N}$ we may identify $\mathbb{M}_n(A)$ with $C(K, \mathbb{M}_n(\mathbb{C}))$ where the norm is given by $\|f\| = \sup_{k \in K} \|f(k)\|$.

Suppose $f \in C(K, \mathbb{M}_n(\mathbb{C}))$ is positive, with $\|f\| \leq 1$, and let $\varepsilon > 0$ be given. Since K is compact f is uniformly continuous and hence there exists a finite open cover $\{U_1, U_2, \ldots, U_m\}$ of K, and $a_1, a_2, \ldots, a_m \in \mathbb{M}_n(\mathbb{C})_+$ such that $\|f(k) - a_j\| \leq \varepsilon$ for all $k \in U_j$.

For each $j \leq m$ chose $g_j \in C(K)$ such that $0 \leq g_j \leq 1$, $\sum_{j=1}^m g_j = 1$, and $g_{j|U_j^c} = 0$. If we consider $f_0 = \sum_{j=1}^m g_j a_j$ then we have $\|f - f_0\| \leq \varepsilon$. Therefore we have $\|\phi^{(n)}(f) - \phi^{(n)}(f_0)\| \leq \|\phi^{(n)}\|\varepsilon$.

Since $\phi^{(n)}(g_j a_j) = \phi(g_j) a_j \geq 0$, for all $1 \leq j \leq m$, we have that $\phi^{(n)}(f_0) \geq 0$, and hence since $\varepsilon > 0$ was arbitrary it follows that $\phi^{(n)}(f) \geq 0$, and it follows that ϕ is completely positive. □

The previous proposition gives us a strengthening of Kadison's inequality.

Corollary 1.7 (Kadison [Kad52]). *Let A and B be unital C^*-algebras, and $\phi : A \to B$ a unital positive map. Then for all $x \in A$ normal we have $\phi(x)^*\phi(x) \leq \phi(x^*x)$.*

Proof. Restricting ϕ to the abelian unital C^*-algebra generated by x we may then assume, by the previous proposition, that ϕ is completely positive. Hence this follows from Kadison's inequality for completely positive maps. □

Lemma 1.8. *Let A be a C^*-algebra, if $\begin{pmatrix} 0 & x^* \\ x & y \end{pmatrix} \in \mathbb{M}_2(A)$ is positive, then $x = 0$, and $y \geq 0$.*

Proof. We may assume A is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, hence if $\xi, \eta \in \mathcal{H}$ we have

$$2\mathrm{Re}(\langle x^*\eta, \xi\rangle) + \langle y\eta, \eta\rangle = \left\langle \begin{pmatrix} 0 & x^* \\ x & y \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right\rangle \geq 0.$$

The result then follows easily. □

Theorem 1.9 (Choi [Cho74]). *If $\phi : A \to B$ is a unital 2-positive map between C^*-algebras, then for $a \in A$ we have $\phi(a^*a) = \phi(a^*)\phi(a)$ if and only if $\phi(xa) = \phi(x)\phi(a)$, and $\phi(a^*x) = \phi(a^*)\phi(x)$, for all $x \in A$.*

Proof. Applying Kadison's inequality to $\phi^{(2)}$ it follows that for all $x \in A$ we have

$$\begin{pmatrix} \phi(a^*a) & \phi(a^*x) \\ \phi(x^*a) & \phi(aa^* + x^*x) \end{pmatrix} = \phi^{(2)}\left(\left| \begin{pmatrix} 0 & a^* \\ a & x \end{pmatrix} \right|^2 \right) \geq \left| \phi^{(2)}\left(\begin{pmatrix} 0 & a^* \\ a & x \end{pmatrix} \right) \right|^2$$
$$= \begin{pmatrix} \phi(a^*)\phi(a) & \phi(a^*)\phi(x) \\ \phi(x^*)\phi(a) & \phi(a)\phi(a^*) + \phi(x^*)\phi(x) \end{pmatrix}.$$

Since $\phi(a^*a) = \phi(a)^*\phi(a)$ it follows from the previous lemma that $\phi(x^*a) = \phi(x^*)\phi(a)$, and $\phi(a^*x) = \phi(a)^*\phi(x)$. □

If $\phi : A \to B$ is u.c.p., then the **multiplicative domain** of ϕ is

$$\{a \in A \mid \phi(a^*a) = \phi(a^*)\phi(a) \text{ and } \phi(aa^*) = \phi(a)\phi(a^*)\}.$$

Note that by Theorem 1.9 the multiplicative domain is a C^*-subalgebra of A, and ϕ restricted to the multiplicative domain is a homomorphism.

Corollary 1.10. *If A is a unital C^*-algebra, $\phi : A \to A$ is unital and 2-positive, and $B \subset A$ is a C^*-subalgebra such that $\phi(b) = b$ for all $b \in B$ then ϕ is B-bimodular, i.e., for all $x \in A$, $b_1, b_2 \in B$ we have $\phi(b_1 x b_2) = b_1\phi(x)b_2$.*

Theorem 1.11 (Choi [Cho72]). *If A and B are unital C^*-algebras, and $\phi : A \to B$ is a unital 2-positive isometry onto B, then ϕ is an isomorphism.*

Proof. Since a self-adjoint element x of norm at most 1 in a unital C^*-algebra is positive if and only if $\|1 - x\| \leq 1$ it follows that ϕ^{-1} is positive.

Fix $a \in A$ self adjoint, and assume $\|a\| \leq 1$. Since ϕ is onto there exists $b \in A$ such that $\phi(b) = \phi(a)^2 \leq \phi(a^2)$.

Thus $b \leq a^2$, and since ϕ^{-1} is also positive we may apply the previous corollary to the map ϕ^{-1} to conclude that

$$a^2 = \phi^{-1}(\phi(a))\phi^{-1}(\phi(a)) \leq \phi^{-1}(\phi(a)^2) = b.$$

Hence, $\phi(a)^2 = \phi(b) = \phi(a^2)$.

Since a was an arbitrary self adjoint element, and since A is generated by its self adjoint elements, Theorem 1.9 then shows that ϕ is an isomorphism. □

Exercise 1.12. Show that a C^*-algebra A is abelian if and only if for any C^*-algebra B, every positive map from B to A is completely positive.

1.2 Bhat's Dilation Theorem

Lemma 1.13. *If $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, and $V : \mathcal{H} \to \mathcal{K}$ is a partial isometry, then for $A \subset \mathcal{B}(\mathcal{H})$, $B \subset \mathcal{B}(\mathcal{K})$, we have that $V^* *\text{-alg}(VBV^*, A)V = *\text{-alg}(B, V^*AV)$.*

Proof. Using the fact that $V^*V = 1$, this follows easily by induction on the length of alternating products for monomials in VBV^*, and A. □

If $A_0 \subset \mathcal{B}(\mathcal{H}_0)$ is a C^*-algebra, and $\phi : A_0 \to A_0$ is a unital completely positive map, then one can iterate Stinespring's dilation as follows:

Lemma 1.14. *Suppose $A_0 \subset \mathcal{B}(\mathcal{H}_0)$ is a unital C^*-algebra, and $\phi_0 : A_0 \to A_0$ is a unital completely positive map. Then there exists a sequence whose entries consist of:*

(*1*) *a Hilbert space $\mathcal{H}_n$;*

(*2*) *an isometry $V_n : \mathcal{H}_{n-1} \to \mathcal{H}_n$;*

(*3*) *a unital C^*-algebra $A_n \subset \mathcal{B}(\mathcal{H}_n)$;*

(*4*) *a unital representation $\pi_n : A_{n-1} \to \mathcal{B}(\mathcal{H}_n)$, such that $\pi_n(A_{n-1})$, and $V_n A_{n-1} V_n^*$ generate A_n;*

(*5*) *a unital completely positive map $\phi_n : A_n \to A_n$;*

such that the following relationships are satisfied for each $n \in \mathbb{N}$, $x \in A_{n-1}$:

$$V_n^* \pi_n(x) V_n = \phi_{n-1}(x); \tag{1}$$

$$V_n^* A_n V_n = A_{n-1}; \tag{2}$$

$$\phi_n(\pi_n(x)) = \pi_n(\phi_{n-1}(x)); \tag{3}$$

$$\pi_{n+1}(V_n x V_n^*) = V_{n+1} \pi_n(x) V_{n+1}^*. \tag{4}$$

Moreover, for each $n \in \mathbb{N}$ we have that the central support of $V_n V_n^*$ in A_n'' is 1. Also, if A_0 is a von Neumann algebra and ϕ_0 is normal then A_n will also be a von Neumann algebra and π_n and ϕ_n will be normal for each $n \in \mathbb{N}$.

Proof. We will first construct the objects and show the relationships (1), (2), and (3) by induction, with the base case being vacuous, and we will then show that (4) also holds for all $n \in \mathbb{N}$. So suppose $n \in \mathbb{N}$ and that (1), (2), and (3) hold for all $m < n$, (we leave V_0 undefined).

Recall from the proof of Stinespring's Dilation Theorem that we may construct a Hilbert space $\mathcal{H}_n$ by completing the vector space $A_{n-1} \otimes \mathcal{H}_{n-1}$ with respect to the non-negative definite sesquilinear form satisfying

$$\langle a \otimes \xi, b \otimes \eta \rangle = \langle \phi_{n-1}(b^* a)\xi, \eta \rangle,$$

for all $a, b \in A_{n-1}$, $\xi, \eta \in \mathcal{H}_{n-1}$.

We also obtain a partial isometry $V_n : \mathcal{H}_{n-1} \to \mathcal{H}_n$ from the formula

$$V_n(\xi) = 1 \otimes \xi,$$

for $\xi \in \mathcal{H}_{n-1}$.

We obtain a representation $\pi_n : A_{n-1} \to \mathcal{B}(\mathcal{H}_n)$ (which is normal when A_0 is a von Neumann algebra and ϕ_0 is normal) from the formula

$$\pi_n(x)(a \otimes \xi) = (xa) \otimes \xi,$$

for $x, a \in A_{n-1}$, $\xi \in \mathcal{H}_{n-1}$. And recall the fundamental relationship $V_n^* \pi_n(x) V_n = \phi_{n-1}(x)$ for all $x \in A_{n-1}$, which establishes (1).

If we let A_n be the C^*-algebra generated by $\pi_n(A_{n-1})$ and $V_n A_{n-1} V_n^*$, then $\pi_n : A_{n-1} \to A_n$, and from Lemma 1.13 we have that $V_n^* A_n V_n$ is generated by $V_n^* \pi_n(A_{n-1}) V_n$ and A_{n-1}. However, $V_n^* \pi_n(A_{n-1}) V_n = \phi_{n-1}(A_{n-1}) \subset A_{n-1}$, hence $V_n^* A_n V_n = A_{n-1}$, establishing (2). Also, when A_0 is a von Neumann algebra and π_n is normal it then follows easily that A_n is then also a von Neumann algebra.

Also note that $\pi_n(A_{n-1}) V_n V_n^* \mathcal{H}_n$ is dense in $\mathcal{H}_n$, and so since $\pi_n(A_{n-1}) \subset A_n$ we have that the central support of $V_n V_n^*$ in A_n'' is 1.

We then define $\phi_n : A_n \to A_n$ by $\phi_n(x) = \pi_n(V_n^* x V_n)$, for $x \in A_n$. This is well defined since $V_n^* A_n V_n = A_{n-1}$, unital, and completely positive. Note that for $x \in A_{n-1}$ we have $\phi_n(\pi_n(x)) = \pi_n(V_n^* \pi_n(x) V_n) = \pi_n(\phi_{n-1}(x))$, establishing (3).

Having established (1), (2), and (3) for all $n \in \mathbb{N}$, we now show that (4) holds as well. For this, notice first that for $a, b \in A_n$, $x \in A_{n-1}$, and $\xi, \eta \in \mathcal{H}_n$ we have

$$\begin{aligned}
\langle \pi_{n+1}(V_n x V_n^*)(a \otimes \xi), b \otimes \eta \rangle &= \langle V_n x V_n^* a \otimes \xi, b \otimes \eta \rangle \\
&= \langle \phi_n(b^* V_n x V_n^* a)\xi, \eta \rangle \\
&= \langle \pi_n(V_n^* b^* V_n x V_n^* a V_n)\xi, \eta \rangle \\
&= \langle 1 \otimes \pi_n(x V_n^* a V_n)\xi, b \otimes \eta \rangle.
\end{aligned}$$

Setting $x = 1$ and using that $V_{n+1}^*(1 \otimes \zeta) = \zeta$ for each $\zeta \in \mathcal{H}_n$, we see that

$$\begin{aligned}
(V_{n+1} V_{n+1}^*) \pi_{n+1}(V_n V_n^*)(a \otimes \xi) &= (V_{n+1} V_{n+1}^*)(1 \otimes \pi_n(V_n^* a V_n)\xi) \\
&= 1 \otimes \pi_n(V_n^* a V_n)\xi \\
&= \pi_{n+1}(V_n V_n^*)(a \otimes \xi),
\end{aligned}$$

and hence $\pi_{n+1}(V_nV_n^*) \leq V_{n+1}V_{n+1}^*$. If instead we set $a = 1$ then we have

$$V_{n+1}\pi_n(x)\xi = 1 \otimes \pi_n(x)\xi = \pi_{n+1}(V_nxV_n^*)V_{n+1}\xi,$$

and so $V_{n+1}\pi_n(x) = \pi_{n+1}(V_nxV_n^*)V_{n+1}$. Multiplying on the right by V_{n+1}^* and using that $\pi_n(V_nV_n^*) \leq V_{n+1}V_{n+1}^*$ then gives $V_{n+1}\pi_n(x)V_{n+1}^* = \pi_{n+1}(V_nxV_n^*)$. □

The following theorem was originally proved by Bhat in [Bha99] in the setting of completely positive semigroups, building on work from [Bha96], [BP94], and [BP95]. Other proofs appear in [BS00], [MS02], and Chapter 8 of [Arv03]. We include an elementary proof based on the idea of iterating Stinespring's dilation [Sti55].

Theorem 1.15 (Bhat [Bha99])**.** *Let $A_0 \subset \mathcal{B}(\mathcal{H}_0)$ be a unital C^*-algebra, and $\phi_0 : A_0 \to A_0$ a unital completely positive map. Then there exists*

(1) a Hilbert space $\mathcal{K}$;

(2) an isometry $W : \mathcal{H}_0 \to \mathcal{K}$;

(3) a C^*-algebra $B \subset \mathcal{B}(\mathcal{K})$;

(4) a unital $*$-endomorphism $\alpha : B \to B$;

such that $W^*BW = A_0$, and for all $x \in A_0$ we have

$$\phi_0^k(x) = W^*\alpha^k(WxW^*)W.$$

Moreover, we have that the central support of P_0 in B'' is 1, and for $y \in \mathcal{B}(\mathcal{K})$ we have $y \in B$ if and only if $\alpha^k(WW^*)y\alpha^k(WW^*) \in \alpha^k(WA_0W^*)$ for all $k \geq 0$. Also, if A_0 is a von Neumann algebra and ϕ_0 is normal then B will also be a von Neumann algebra, and α will also be normal.

Proof. Using the notation from the previous lemma, we may define a Hilbert space $\mathcal{K}$ as the directed limit of the Hilbert spaces $\mathcal{H}_n$ with respect to the inclusions $V_{n+1} : \mathcal{H}_n \to \mathcal{H}_{n+1}$. We denote by $W_n : \mathcal{H}_n \to \mathcal{K}$ the associated sequence of isometries satisfying $W_{n+1}^*W_n = V_{n+1}$, for $n \in \mathbb{N}$, and we set $P_n = W_nW_n^*$, an increasing sequence of projections.

From (2) we have that $P_{n-1}W_nA_nW_n^*P_{n-1} = W_{n-1}A_{n-1}W_{n-1}^*$, and hence if we define the C^*-algebra $B = \{x \in \mathcal{B}(\mathcal{K}) \mid W_n^*xW_n \in A_n, n \geq 0\}$, then we have $W_n^*BW_n = A_n$, for all $n \geq 0$. Also, if A_0 is a von Neumann algebra, then so is A_n for each $n \in \mathbb{N}$ and from this it follows easily that B is also a von Neumann algebra.

We define the unital $*$-endomorphism $\alpha : B \to B$ (which is normal when A_0 is a von Neumann algebra and ϕ_0 is normal) by the formula

$$\alpha(x) = \lim_{n\to\infty} W_{n+1}\pi_{n+1}(W_n^*xW_n)W_{n+1},$$

where the limit is taken in the strong operator topology. Note that $\alpha(P_n) = P_{n+1} \geq P_n$. From (4) we see that in general, the strong operator topology limit exists in B, and that for $x \in A_n \cong P_nA_\infty P_n$ the limit stabilizes as $\alpha(W_nxW_n^*) = W_{n+1}\pi_{n+1}(x)W_{n+1}^*$.

From (1) we see that for $n \geq 0$, and $x \in A_n$ we have

$$\begin{aligned} P_n \alpha(W_n x W_n^*) P_n &= W_n W_n^* W_{n+1} \pi_{n+1}(x) W_{n+1}^* W_n W_n^* \\ &= W_n V_{n+1}^* \pi_{n+1}(x) V_{n+1} W_n^* \\ &= W_n \phi_n(x) W_n^*. \end{aligned}$$

By induction we then see that also for $k > 1$, and $x \in A_0$ we have

$$\begin{aligned} P_0 \alpha^k(W_0 x W_0^*) P_0 &= P_0 \alpha^{k-1}(P_0 \alpha(W_0 x W_0^*) P_0) P_0 \\ &= P_0 \alpha^{k-1}(W_0 \phi_0(x) W_0^*) P_0 \\ &= W_0 \phi_0^k(x) W_0^*. \end{aligned}$$

By the previous lemma we have that the central support of P_n in $W_n A_n'' W_n^*$ is P_{n+1}. Hence it follows that the central support of P_0 in B is 1. ☐

Corollary 1.16 (Connes [Con80]). *Let M be a countably decomposable properly infinite von Neumann algebra and suppose $\phi : M \to M$ is a normal unital completely positive map, then there exists a (possibly non-unital) $*$-endomorphism $\alpha : M \to M$, and an isometry $v \in M$ such that $\phi(x) = v^* \alpha(x) v$, for all $x \in M$. Also, if ϕ is normal then α is normal as well.*

Proof. Bhat's dilation provides a von Neumann algebra $\tilde{M}$, a projection $p \in \tilde{M}$ with central support 1 such that $p\tilde{M}p = M$, and a normal (unital) $*$-endomorphism $\alpha_0 : \tilde{M} \to \tilde{M}$, such that $\phi(x) = p\alpha_0(x)p$, for all $x \in M$. Since $M = p\tilde{M}p$ is property infinite and since p has central support 1 it follows that p and 1 are equivalent in $\tilde{M}$. Thus there exists a partial isometry $v_0 \in \tilde{M}$ such that $v_0 v_0^* = 1$, and $v_0^* v_0 = p$.

If we then consider the possibly non-unital normal $*$-endomorphism $\alpha : M \to M$ given by $\alpha(x) = v_0^* \alpha_0(x) v_0$ then setting $v = v_0^* p$ we have $\phi(x) = p\alpha_0(x)p = v^* \alpha(x) v$, for all $x \in M = p\tilde{M}p$. ☐

1.3 Poisson boundaries

Poisson boundaries of completely positive maps were first defined by Izumi in [Izu02] using the Choi-Effros product from [CE77]. Izumi further developed the theory in [Izu04], and in [Izu12] he credits Arveson with the description of Poisson boundaries as the fixed point algebra of Bhat's dilation, and this is the perspective we take here.

If $A \subset \mathcal{B}(\mathcal{H})$ is a unital C^*-algebra, and $\phi : A \to A$ a unital completely positive map, then a projection $p \in A$ is said to be **coinvariant**, if $\{\phi^n(p)\}$ defines an increasing sequence of projections which strongly converge to 1 in $\mathcal{B}(\mathcal{H})$, and such that for $y \in \mathcal{B}(\mathcal{H})$ we have $y \in A$ if and only if $\phi^n(p) y \phi^n(p) \in A$ for all $n \geq 0$. Note that for $n \geq 0$, $\phi^n(p)$ is in the multiplicative domain for ϕ, and is again coinvariant. We define $\phi_p : pAp \to pAp$ to be the map $\phi_p(x) = p\phi(x)p$, then ϕ_p is normal unital completely positive. Moreover, we have that $\phi_p^k(x) = p\phi^k(x)p$ for all $x \in pAp$, which can be seen by induction from

$$p\phi^k(x)p = p\phi^{k-1}(p)\phi^k(x)\phi^{k-1}(p)p = p\phi^{k-1}(\phi_p(x))p.$$

Theorem 1.17 (Izumi [Izu12]). *Let $A \subset \mathcal{B}(\mathcal{H})$ be a unital C^*-algebra, $\phi : A \to A$ a unital completely positive map, and $p \in A$ a coinvariant projection. Then the map $\theta : \mathrm{Har}(A, \phi) \to \mathrm{Har}(pAp, \phi_p)$ given by $\theta(x) = pxp$ defines a completely positive isometric surjection, between $\mathrm{Har}(A, \phi)$ and $\mathrm{Har}(pAp, \phi_p)$.*

Moreover, if A is a von Neumann algebra and ϕ is normal then θ is also normal.

Proof. First note that θ is well-defined since if $x \in \mathrm{Har}(A, \phi)$ we have $\phi_p(pxp) = p\phi(p)x\phi(p)p = pxp$. Clearly θ is completely positive (and normal in the case when A is a von Neumann algebra and ϕ is normal).

To see that it is surjective, if $x \in \mathrm{Har}(pAp, \phi_p)$ then consider the sequence $\phi^n(x)$. For each $m, n \geq 0$, we have

$$\phi^m(p)\phi^{m+n}(x)\phi^m(p) = \phi^m(p\phi^n(x)p) = \phi^m(\phi_p^n(x)) = \phi^m(x).$$

It follows that $\{\phi^n(x)\}$ converges in the strong operator topology to an element $y \in \mathcal{B}(\mathcal{H})$ such that $\phi^m(p)y\phi^m(p) = \phi^m(x)$ for each $m \geq 0$, consequently we have $y \in A$.

In particular, for $m = 0$ we have $pyp = x$. To see that $y \in \mathrm{Har}(A, \phi)$ we use that for all $z \in A$ we have the strong operator topology limit

$$\lim_{n\to\infty} \phi(\phi^n(p)z\phi^n(p)) = \phi^{n+1}(p)\phi(z)\phi^{n+1}(p) = \phi(z),$$

and hence

$$\phi(y) = \lim_{m\to\infty} \phi(\phi^m(p)y\phi^m(p)) = \lim_{m\to\infty} \phi^{m+1}(x) = y.$$

Thus θ is surjective, and since $\phi^n(p)$ converges strongly to 1, and each $\phi^n(p)$ is in the multiplicative domain of ϕ, it follows that if $x \in \mathrm{Har}(A, \phi)$ then $\phi^n(pxp)$ converges strongly to x and hence

$$\|x\| = \lim_{n\to\infty} \|\phi^n(pxp)\| \leq \|pxp\| \leq \|x\|.$$

Thus, θ is also isometric. □

Corollary 1.18 (Izumi [Izu02]). *Let A be a unital C^*-algebra, and $\phi : A \to A$ a unital completely positive map. Then there exists a C^*-algebra B and a completely positive isometric surjection $\theta : B \to \mathrm{Har}(A, \phi)$.*

Moreover B and θ are unique in the sense that if $\tilde{B}$ is another C^*-algebra, and $\tilde{\theta} : \tilde{B} \to \mathrm{Har}(A, \phi)$ is a completely positive isometric surjection, then $\theta^{-1} \circ \tilde{\theta}$ is an isomorphism.

Also, if A is a von Neumann algebra and ϕ is normal, then B is also a von Neumann algebra and θ is normal.

Proof. Note that we may assume $A \subset \mathcal{B}(\mathcal{H})$. Existence then follows by applying the previous theorem to Bhat's dilation. Uniqueness follows from Theorem 1.11. □

We refer to the C^*-algebra B from the previous corollary as the **Poisson boundary of** ϕ, and we refer to the map θ as the **Poisson transform**.

Corollary 1.19 (Choi-Effros [CE77]). *Let A be a unital C^*-algebra and $F \subset A$ an operator system. If $E : A \to F$ is a completely positive map such that $E_{|F} = \mathrm{id}$, then F has a unique C^*-algebraic structure which is given by $x \cdot y = E(xy)$. Moreover, if A is a von Neumann algebra and F is weakly closed then this gives a von Neumann algebraic structure on F.*

Proof. When A is a C^*-algebra this follows from Corollary 1.18 since $\mathrm{Har}(A, E) = F$. Also note that since $E^n = E$ it follows from the proof of Theorem 1.17 that the product structure coming from the Poisson boundary is given by $x \cdot y = E(xy)$.

If A is a von Neumann algebra and F is weakly closed then F has a predual $F_\perp = \{\varphi \in A_* \mid \varphi(x) = 0, \text{ for all } x \in F\}$ and hence A is isomorphic to a von Neumann algebraic by Sakai's theorem. □

Note that if A is a C^*-algebra, $F \subset A$ an operator system, and $E : A \to F$ completely positive with $E_{|F} = \mathrm{id}$, then we still have a form of bimodularity for E when we endow F with the Choi-Effros product from Corollary 1.19. In this case though the bimodularity is with respect to two different product structures, i.e., we have

$$E(xay) = x \cdot E(a) \cdot y,$$

for all $x, y \in F$, $a \in A$.

Proposition 1.20. *Let A be an abelian C^*-algebra and $\phi : A \to A$ a normal unital completely positive map. Then the Poisson boundary of ϕ is also abelian.*

Proof. Let B be the Poisson boundary of ϕ, and let $\theta : B \to \mathrm{Har}(A, \phi)$ be the Poisson transform. If C is a C^*-algebra and $\psi : C \to B$ is a positive map then $\theta \circ \psi : C \to \mathrm{Har}(A, \phi) \subset A$ is positive, and since A is abelian it is then completely positive by Proposition 1.5. Hence, ψ is also completely positive. Since every positive map from a C^*-algebra to B is completely positive it then follows that B is abelian. □

Example 1.21. Let Γ be a discrete group and $\mu \in \mathrm{Prob}(\Gamma)$ a probability measure on Γ such that the support of μ generates Γ. Then on $\ell^\infty\Gamma$ we may consider the normal unital (completely) positive map ϕ_μ given by $\phi_\mu(f) = \mu * f$, where $\mu * f$ is the convolution $(\mu * f)(x) = \int f(g^{-1}x)\, d\mu(g)$. Then $\mathrm{Har}(\mu) = \mathrm{Har}(\ell^\infty\Gamma, \phi_\mu)$ has a unique von Neumann algebraic structure which is abelian by the previous proposition. Notice that Γ acts on $\mathrm{Har}(\mu)$ by right translation, and since this action preserves positivity it follows from Theorem 1.11 that Γ preserves the multiplication structure as well.

Since the support of μ generates Γ, for a non-negative function $f \in \mathrm{Har}(\mu)_+$, we have $f(e) = 0$ if and only if $f = 0$. Thus we obtain a natural normal faithful state φ on $\mathrm{Har}(\mu)$ which is given by $\varphi(f) = f(e)$.

Since φ is Γ-equivariant, this extends to a normal u.c.p. map $\tilde{\varphi} : \ell^\infty\Gamma \rtimes \Gamma \to \ell^\infty\Gamma \rtimes \Gamma$ such that $\tilde{\varphi}_{L\Gamma} = \mathrm{id}$.[1] It is an easy exercise to see that the Poisson boundary of $\tilde{\phi}$ is nothing but the crossed product $\mathrm{Har}(\mu) \rtimes \Gamma$.

Example 1.22. Let X be a set. A random walk on X is given by transition probabilities $\mu : X \to \mathrm{Prob}(X)$. If $f \in \ell^\infty X$ we define the convolution $\mu * f \in \ell^\infty X$ by

$$\mu * f(x) = \int f(y)\, d\mu(x, y).$$

[1] Note that $\ell^\infty\Gamma \rtimes \Gamma \cong \mathcal{B}(\ell^2\Gamma)$.

Clearly, convolution gives a normal u.c.p. map on $\ell^\infty X$, and hence we may consider the Poisson boundary of this map, which coincides with the boundary of the random walk. If Γ is a discrete group and $\mu \in \text{Prob}(\Gamma)$ then we recover the previous example by consider the transition probabilities $\tilde{\mu}(gx, x) = \mu(g)$.

A particular useful example to consider is when X is a connected locally finite graph and the transition probabilities for the random walk are given by $\mu(y, x) = 0$, if there is no edge from x to y, and $\mu(y, x) = \frac{1}{d(x)}$ otherwise, where $d(x)$ denotes the degree of x, i.e., the number of edges starting from x.

Example 1.23. Consider the one sided shift operator $s : \ell^2\mathbb{N} \to \ell^2\mathbb{N}$ given by $s(\delta_n) = \delta_{n+1}$. We may then consider the normal u.c.p. map $\phi : \mathcal{B}(\ell^2\mathbb{N}) \to \mathcal{B}(\ell^2\mathbb{N})$ given by $\phi(T) = sTs^*$. Thinking of operators $\mathcal{B}(\ell^2\mathbb{N})$ as matrices, we then have that $\text{Har}(\phi)$ consist of all Toeplitz matrices, i.e., those matrices whose entries $\langle T\delta_n, \delta_m\rangle$ only depend on $n - m$.

If we also consider the two sided shift operator $\tilde{s} : \ell^2\mathbb{Z} \to \ell^2\mathbb{Z}$, then $\tilde{s}$ is a unitary and so induces an automorphism $\alpha : \mathcal{B}(\ell^2\mathbb{N}) \to \mathcal{B}(\ell^2\mathbb{N})$ by $\alpha(T) = \tilde{s}T\tilde{s}^*$. Moreover, if we consider the usual embedding $\ell^2\mathbb{N} \subset \ell^2\mathbb{Z}$, and denote by p the projection onto $\ell^2\mathbb{N}$, then for $T \in \mathcal{B}(\ell^2\mathbb{N}) = p\mathcal{B}(\ell^2\mathbb{Z})p$ we have $p\alpha(T)p = \phi(T)$. Thus, in this case we have an explicit description of Bhat's dilation.

In particular, we have an identification of the Poisson boundary of ϕ with $\text{Har}(\alpha) = \{\tilde{s}\}' = L\mathbb{Z} \subset \mathcal{B}(\ell^2\mathbb{Z})$.

2 The Poisson boundary of a finite von Neumann algebra

Theorem 2.1 (Connes [Con80]). *Let N be a finite von Neumann algebra with a normal faithful trace τ. If $\phi : N \to N$ is a normal u.c.p. map which preserves τ, then there is a normal Hilbert N-bimodule $\mathcal{H}$, together with a unit vector $\xi_0 \in \mathcal{H}$ such that $\tau(\phi(x)y) = \langle x\xi_0 y, \xi_0\rangle$ for all $x, y \in N$.*

Proof. This is very similar to Stinespring's theorem and we only sketch the proof. First, consider on $N \otimes_{\text{alg}} N$ the sesquilinear form given by $\langle x \otimes a, y \otimes b\rangle = \tau(b^*\phi(y^*x)a)$. Since ϕ is completely positive it follows easily that this is non-negative definite, hence after separation and completion we obtain a Hilbert space $\mathcal{H}$.

Since ϕ is unital and preserves the trace it is then easy to see that the representations $x_0 \cdot (x \otimes a) = x_0 x \otimes a$, and $(x \otimes a) \cdot x_0 = x \otimes a x_0$, extend to commuting normal (anti-)representations of N on $\mathcal{H}$, so that $\mathcal{H}$ is an N-bimodule.

Finally, if we set $\xi_0 = 1 \otimes 1$, then for all $x, y \in N$ we have

$$\langle x\xi_0 y, \xi_0\rangle = \langle x \otimes y, 1 \otimes 1\rangle = \tau(\phi(x)y).$$

□

Corollary 2.2. *Let N be a finite von Neumann algebra with a normal faithful trace τ. If $\phi : N \to N$ is a normal u.c.p. map which preserves τ, then* $\text{Har}(\phi, N)$ *is a von Neumann subalgebra of N.*

Proof. If we let $\mathcal{H}$ be the normal Hilbert N-bimodule from the previous theorem, and if we take

$\xi_0 \in \mathcal{H}$ such that $\tau(\phi(x)y) = \langle x\xi_0 y, \xi_0\rangle$, for all $x, y \in N$. Then for $x \in N$ we have

$$\begin{aligned}\|x - \phi(x)\|_2^2 &= \|x\|_2^2 + \|\phi(x)\|_2^2 + 2\mathrm{Re}(\tau(\phi(x)x^*)) \\ &\leq 2\|x\|_2^2 + 2\mathrm{Re}(\tau(\phi(x)x^*)) \\ &= \|x\xi_0\|^2 + \|\xi_0 x\|^2 + 2\mathrm{Re}(\langle x\xi_0 x^*, \xi_0\rangle) \\ &= \|x\xi_0 - \xi_0 x\|^2. \end{aligned} \tag{5}$$

And similarly, we have $\|x\xi_0 - \xi_0 x\|^2 = 2\mathrm{Re}(\tau((x - \phi(x))x^*)) \leq \|x - \phi(x)\|_2\|x\|$.

Thus we see that $\mathrm{Har}(\phi, N)$ coincides with $\{x \in N \mid x\xi_0 - \xi_0 x\}$, and since the latter is clearly a von Neumann subalgebra this then finishes the proof. □

To find interesting examples of boundaries coming from a finite von Neumann algebra N, we see that we should not be looking at u.c.p. maps on N. However, by looking at u.c.p. maps on $\mathcal{B}(L^2(N, \tau))$ instead we can indeed find interesting examples.

Let M be a von Neumann algebra, and let $N \subset M$ be a finite von Neumann subalgebra with a normal faithful trace τ. Given a state $\varphi \in M^*$ we will say that φ is a **hyperstate** if it extends τ. To such a hyperstate we obtain a natural inclusion $L^2(N, \tau) \subset L^2(M, \varphi)$ induced from the map $x1_\tau \mapsto x1_\varphi$ for $x \in N$. Let $e_N \in \mathcal{B}(L^2(M, \varphi))$ denote the orthogonal projection onto $L^2(N, \tau)$. We may then consider the u.c.p. map $\psi_\varphi : M \to \mathcal{B}(L^2(N, \tau))$, defined by $\psi_\varphi(x) = e_N x e_N$. Note that if $x \in N \subset M$ then we have $\psi_\varphi(x) = x$. We shall refer to the map ψ_φ as the **Poisson transform** (with respect to φ) of the inclusion $N \subset M$.

Theorem 2.3. *Let M be a von Neumann algebra, and let $N \subset M$ be a finite von Neumann subalgebra with a normal faithful trace τ. The correspondence $\varphi \mapsto \psi_\varphi$ defined above gives a bijective correspondence between hyperstates on M, and u.c.p., N-bimodular maps from M to $\mathcal{B}(L^2(N, \tau))$. Moreover, ψ_φ is normal if and only if φ is normal.*

Also, this corresondence is a homeomorphism when we consider the space of hyperstates with the weak*-topology, and the space of u.c.p., N-bimodular maps with the topology of pointwise weak operator topology convergence.

Proof. First note that if φ is a hyperstate on M, then for all $x \in M$ we have

$$\varphi(x) = \langle x, 1\rangle_\tau = \langle \psi_\varphi(x)1, 1\rangle_\tau.$$

From this it follows that the correspondence $\varphi \mapsto \psi_\varphi$ is one-to-one. To see that it is onto, suppose that $\psi : M \to \mathcal{B}(L^2(N, \tau))$ is u.c.p. and N-bimodular. We define a state φ on M by $\varphi(x) = \tau \circ \psi(x)$. For all $y \in N$ we then have $\varphi(y) = \tau \circ \psi(y) = \tau(y)$, hence φ is a hyperstate. Moreover, if $y, z \in N$, and $x \in M$ then we have

$$\begin{aligned}\langle \psi_\varphi(x)y, z\rangle_\tau &= \tau \circ \psi_\varphi(z^* x y) \qquad (6)\\ &= \varphi(z^* x y) = \langle \psi(x)y, z\rangle_\tau,\end{aligned}$$

hence, $\psi_\varphi = \psi$.

It is also easy to check that ψ_φ is normal if and only if φ is.

To see that this correspondence is a homeomorphism when given the topologies above, suppose that φ is a hyperstate, and φ_α is a net of hyperstates. If we set $y = z = 1$ in Equation 6, then it follows easily that if ψ_{φ_α} converges in the pointwise weak operator topology to ψ_φ then we have that φ_α converges weak* to φ. Conversely, if φ_α converges weak* to φ then again using Equation 6 we see that for $x \in M$, and $y, z \in N$ we have $\langle \psi_{\varphi_\alpha}(x)y, z\rangle_\tau \to \langle \psi_\varphi(x)y, z\rangle_\tau$. Since N is dense in $L^2(N,\tau)$, and since $\|\psi_{\varphi_\alpha}(x) - \psi_\varphi(x)\| \leq 2\|x\|$, for all α it then follows that ψ_{φ_α} converges to ψ_φ in the pointwise weak operator topology. □

When $M = \mathcal{B}(L^2(N,\tau))$ in the previous theorem then to each hyperstate on $\mathcal{B}(L^2(N,\tau))$ we obtain a u.c.p. N-bimodular map on $\mathcal{B}(L^2(N,\tau))$. In particular, composing such maps gives a convolution operation on the space of hyperstates. More generally, if M is a von Neumann algebra, and $N \subset M$ is a finite von Neumann subalgebra with a normal faithful trace τ, then for hyperstates $\varphi_1 \in M^*$, and $\varphi_2 \in \mathcal{B}(L^2(N,\tau))^*$ we define the **convolution** $\varphi_1 * \varphi_2$ to be the unique hyperstate on M such that $\psi_{\varphi_1 * \varphi_2} = \psi_{\varphi_1} \circ \psi_{\varphi_2}$.

Lemma 2.4. *Using the same notation as above, if* $\mathrm{Hyp}(N \subset M)$ *denotes the space of hyperstates on* M*, then the mapping*

$$\mathrm{Hyp}(N \subset M) \ni \varphi_0 \mapsto \varphi * \varphi_0 \in \mathrm{Hyp}(N \subset M)$$

is continuous in the weak-topology.*

Moreover, if $\varphi_0 \in \mathcal{B}(L^2(N,\tau))_*$ *is a normal hyperstate, then the mapping*

$$\mathrm{Hyp}(N \subset \mathcal{B}(L^2(N,\tau))) \ni \varphi_0 \mapsto \varphi_0 * \varphi \in \mathrm{Hyp}(N \subset M)$$

is also continuous.

Proof. Since $\varphi \mapsto \psi_\varphi$ is a homeomorphism this lemma then follows easily. □

If $\varphi \in \mathcal{B}(L^2(N,\tau))^*$ then we define the **Poisson boundary** B of N with respect to φ to be the Poisson boundary of the u.c.p. map ψ_φ. From Example 1.21 we see that if Γ is a countable group and $\Gamma \curvearrowright (Y,\eta)$ is the action of Γ on a Poisson boundary, then $L^\infty(Y,\eta) \rtimes \Gamma$ is a Poisson boundary for $L\Gamma$. Note that we always have a natural inclusion $N \subset B$, since ψ_φ is N-bimodular.

Restricting the hyperstate φ to $\mathrm{Har}(\mathcal{B}(L^2(N,\tau)), \psi_\varphi)$ we obtain a natural state on B (which we again denote by φ), and note that with respect to this state the Poisson transform of the inclusion $N \subset B$ agrees with the Poisson transform $B \to \mathrm{Har}(\mathcal{B}(L^2(N,\tau)), \psi_\varphi)$ defined above. Thus, our terminology is consistent in this setting.

Theorem 2.5. *Let M be a von Neumann algebra, and let $N \subset M$ be a finite von Neumann subalgebra with a normal faithful trace τ. Let $\varphi \in \mathcal{B}(L^2(N,\tau))_*$ be a normal hyper state, and let B be the Poisson boundary of N with respect to φ. Then there exists a conditional expectation $E : \mathcal{B}(L^2(N,\tau)) \to \mathrm{Har}(B(L^2(N,\tau)), \psi_\varphi)$.*

Proof. Let φ_0 be a weak*-cluster point of $\{\frac{1}{N}\sum_{k=1}^N \varphi^{*k}\}$ where φ^{*k} denotes the convolution of φ with itself k-times. Then $E = \psi_{\varphi_0}$ is a cluster point of $\{\psi_N\}$ in the topology of pointwise weak operator topology convergence, where $\psi_N = \frac{1}{N}\sum_{k=1}^N \psi_\varphi^k$.

If $x \in \mathcal{B}(L^2(N,\tau))$ then we have $\|\psi_\varphi(\psi_N(x)) - \psi_N(x)\| \leq \frac{2}{N}\|x\|$. Hence it follows that $E : B(L^2(N,\tau)) \to \mathrm{Har}(B(L^2(N,\tau)), \psi_\varphi)$. Since, E is the identity on $\mathrm{Har}(B(L^2(N,\tau))$ we then have $E^2 = E$. □

The trivial case is when $\varphi_e(x) = \langle x1, 1\rangle_\tau$ in which case we have that $\psi_{\varphi_e} = \mathrm{id}$, and the Poisson boundary is nothing but $\mathcal{B}(L^2(N,\tau))$. Note that φ_e gives an identity with respect to convolution. Also note that if $\varphi \in \mathcal{B}(L^2(N,\tau))^*$ is a hyperstate, then we have a description of the space of harmonic operators as:

$$\mathrm{Har}(\mathcal{B}(L^2(N,\tau)), \psi_\varphi) = \{x \in \mathcal{B}(L^2(N,\tau)) \mid \varphi(axb) = \varphi_e(axb) \text{ for all } a, b \in N\}.$$

We'll say that a normal hyperstate $\varphi \in \mathcal{B}(L^2(N,\tau))_*$ is **admissible** if N is the largest $*$-subalgebra of $\mathcal{B}(L^2(N,\tau))$ which is contained in $\mathrm{Har}(\psi_\varphi)$. We'll say that φ is **symmetric** if $\varphi(JxJ) = \tau(x)$ for all $x \in N$.

Examples of symmetric admissible hyperstates are easy to find. For example, if N is generated by a countable set of unitaries $S \subset \mathcal{U}(N)$ such that $1 \in S$, and $S = S$, then for a fully supported probability measure $\nu \in \mathrm{Prob}(S)$ such that $\nu(u) = \nu(u^*)$ we may consider the normal u.c.p. map ψ on $\mathcal{B}(L^2(N,\tau))$ given by

$$\psi(x) = \int (JuJ)x(Ju^*J)\, d\nu(u).$$

Since S generate N, a convexity argument then shows that N is the largest $*$-subalgebra of $\mathcal{B}(L^2(N,\tau))$ which is contained in $\mathrm{Har}(\psi)$. Indeed, if $x, x^*x \in \mathrm{Har}(\psi)$, then for each $a \in N$ we have

$$\begin{aligned}\|\int (JuJ)x(Ju^*J)a\, d\nu(u)\|_2^2 &= \|\psi(x)a\|_2^2 = \langle \psi(x^*x)a, a\rangle \\ &= \int \|x(Ju^*J)a\|_2^2\, d\nu(u) = \int \|(JuJ)x(Ju^*J)a\|_2^2\, d\nu(u).\end{aligned}$$

Hence we must have $(JuJ)x(Ju^*J) = x$ for each $u \in S$, and so $x \in (JNJ)' = N$.

Proposition 2.6. *Let N be a finite von Neumann algebra with a normal faithful trace τ. Let $\varphi \in \mathcal{B}(L^2(N,\tau))_*$ be a symmetric admissible hyperstate, and let B be the corresponding Poisson boundary. Then $N' \cap B = \mathcal{Z}(N)$, and in particular, when N is a factor then so is B.*

Proof. Let $\theta : B \to \mathcal{B}(L^2(N,\tau))$ denote the Poisson transform. If $x \in N' \cap B$, then $\theta(x) \in N' \cap \mathcal{B}(L^2(N,\tau)) = JNJ$. Since φ is symmetric, ψ_φ preserves the trace when restricted to JNJ. Thus $\mathrm{Har}(\psi_\varphi, JNJ)$ is a von Neumann subalgebra of JNJ by Corollary 2.2. Since φ is admissible we then have $x \in \mathrm{Har}(\psi_\varphi, JNJ) = JNJ \cap N = \mathcal{Z}(N)$. □

3 Derivations

Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is Lipschitz if $\|f\|_{Lip} = \sup_{s,t\in\mathbb{R}} \frac{|f(s)-f(t)|}{|s-t|} < \infty$. We denote by $\mathrm{Lip}_0(\mathbb{R})$ the set of Lipschitz functions f such that $f(0) = 0$.

Lemma 3.1. *Suppose $f \in \mathrm{Lip}_0(\mathbb{R})$ and $\xi, \eta \in L^2(N,\tau)_{s.a.}$, then $f(\xi), f(\eta) \in L^2(N,\tau)$ and*

$$\|f(\xi) - f(\eta)\|_2 \le \|f\|_{\mathrm{Lip}} \|\xi - \eta\|_2.$$

Proof. First, let $\xi = \int s dP(s)$ be the spectral decomposition for ξ. Then we have

$$\int |f(s)|^2 d(\tau \circ P)(s) \leq \|f\|_{\mathrm{Lip}}^2 \int s^2 d(\tau \circ P)(s) = \|f\|_{\mathrm{Lip}}^2 \|\xi\|_2^2,$$

hence $f(\xi) = \int f(s) dP(s) \in L^2(N, \tau)$.

To show the above inequality we use Connes' joint distribution trick [Con76]. We have a state ϕ on $C_0(\mathbb{R} \times \mathbb{R})$ defined by $\Sigma_i g_i \otimes h_i \mapsto \tau(g_i(\xi) h_i(\eta))$ and hence by the Riesz representation theorem there exists a Radon measure μ on $\mathbb{R} \times \mathbb{R}$ such that $\phi(g \otimes h) = \int g(s) h(t) d\mu(s,t)$ for all $g, h \in C_0(\mathbb{R})$. Since

$$\lim_{S,T \to \infty} \int_{s>S, t>T} (s^2 + t^2) d\mu = 0,$$

it follows that we also have $\phi(g \otimes h) = \int g(s) h(t) d\mu(s,t)$ for $g, h \in Lip_0(\mathbb{R})$. Hence

$$\|f(\xi) - f(\eta)\|_2^2 = \tau(f^2(\xi) - 2f(\xi)f(\eta) + f^2(\eta))$$

$$= \int f^2(s) - 2f(s)f(t) + f^2(t) d\mu(s,t) = \int |f(s) - f(t)|^2 d\mu(s,t)$$

$$\leq \|f\|_{\mathrm{Lip}}^2 \int |s - t|^2 d\mu(s,t) = \|f\|_{\mathrm{Lip}}^2 \|\xi - \eta\|_2^2.$$

□

A Hilbert N-bimodule is a Hilbert space $\mathcal{H}$ together with a pair of commuting representations $\pi : N \to \mathcal{B}(\mathcal{H})$, $\pi^{\mathrm{op}} : N^{\mathrm{op}} \to \mathcal{B}(\mathcal{H})$. Given a Hilbert N-bimodule we will denote by $x\xi y$ the vector $\pi(x)\pi^{\mathrm{op}}(y^{\mathrm{op}})\xi$ whenever $x, y \in N$ and $\xi \in \mathcal{H}$. We'll say that the bimodule is normal if the representations π and π^{op} are normal. We'll say that the bimodule is symmetric if there exists an antilinear involution $\mathcal{J}$ on $\mathcal{H}$ such that $\mathcal{J}(x\xi y) = y^* \mathcal{J}(\xi) x^*$ for all $x, y \in N$ and $\xi \in \mathcal{H}$.

The trivial bimodule is $L^2(N, \tau)$ where the bimodule structure is given by left or right multiplication. We will also call $L^2(N, \tau) \overline{\otimes} L^2(N, \tau)$ the coarse bimodule, where the bimodule structure is given by $\pi(x)\xi = (x \otimes 1)\xi$, and $\pi^{\mathrm{op}}(y^{\mathrm{op}})\xi = \xi(1 \otimes y)$.

If $\mathcal{H}$ is a Hilbert N-bimodule, then a closable derivation is a densely defined closable operator $\delta : L^2(N, \tau) \to \mathcal{H}$ such that $D(\delta) \subset \mathfrak{n}_\tau$ is a $*$-algebra and $\delta(xy) = x\delta(y) + \delta(x)y$ for all $x, y \in D(\delta)$. We'll say that a derivation is symmetric if $\mathcal{H}$ is symmetric and we have $\mathcal{J}(\delta(x)) = \delta(x^*)$ for all $x \in D(\delta)$.

Recall that if $T : \mathcal{K} \to \mathcal{K}$ is a closable operator and $\xi_k \in D(T)$ such that $\xi_k \to \xi$ and $K = \limsup_k \|T(\xi_k)\| < \infty$ then $\xi \in D(\overline{T})$, and $\|\overline{T}(\xi)\| \leq K$. Indeed, if T is closable then $\overline{\mathrm{Graph}(T)} \subset \mathcal{K} \oplus \mathcal{K}$ is a subspace, and since $(\xi_k, T(\xi_k)) \in \mathrm{Graph}(T)$ is bounded, it has a weak limit cluster point.

Theorem 3.2 (Sauvageot [Sau90], Davies-Lindsay [DL92]). *Let $\mathcal{H}$ be a Hilbert N-bimodule, let $\delta : L^2(N, \tau) \to \mathcal{H}$ be a closable symmetric derivation. If $\xi \in D(\overline{\delta})_{s.a.}$, and $f \in \mathrm{Lip}_0(\mathbb{R})$, then $f(\xi) \in D(\overline{\delta})$ and $\|\overline{\delta}(f(\xi))\| \leq \|f\|_{\mathrm{Lip}} \|\overline{\delta}(\xi)\|$.*

Moreover, If δ is closed as an operator from N with the uniform topology to $\mathcal{H}$, $x \in D(\delta)_{s.a.}$, and $f \in \mathrm{Lip}_0(\mathbb{R})$ then $f(x) \in D(\delta)$.

Proof. First consider the case $\xi = x \in D(\delta) \subset N$. If we let $K = \sigma(x)$ then the representations of N in $\mathcal{B}(\mathcal{H})$ gives rise to a representation $\tilde{\pi}$ of $C(K \times K) \cong C^*(x) \otimes_{\max} C^*(x)$ in $\mathcal{B}(\mathcal{H})$ such that $\tilde{\pi}(\Sigma_i g_i \otimes h_i)\xi = \Sigma_i g_i(x)\xi h_i(x)$.

Given $f \in C^1(K)$ denote by $\tilde{f} \in C(K \times K)$ the function

$$\tilde{f}(s,t) = \begin{cases} \frac{f(s)-f(t)}{s-t} & \text{if } s \neq t; \\ f'(s) & \text{otherwise.} \end{cases}$$

If we are given the polynomial $p(t) = t^n$ then we have

$$\delta(p(x)) = \Sigma_{k=0}^{n-1} x^k \delta(x) x^{n-1-k}$$

$$= \Sigma_{k=0}^{n-1} \pi(s^k t^{n-1-k}) \cdot \delta(x) = \tilde{\pi}(p) \cdot \delta(x).$$

By linearity we therefore have that $\delta(p(a)) = \pi(\tilde{p}) \cdot \delta(a)$ for all polynomials p with vanishing scalar coefficient. Taking a polynomial approximation of $f \in C^1(K)$ it follows that $f(x) \in D(\overline{\delta})$ and $\overline{\delta}(f(x)) = \pi(\tilde{f}) \cdot \delta(x)$. In particular we have that $\|\overline{\delta}(f(a))\| \leq \|\tilde{f}\|_\infty \|\delta(a)\| = \|f\|_{\text{Lip}} \|\delta(a)\|$ for all $f \in C^1(\mathbb{R})$.

If now $f \in \text{Lip}_0(\mathbb{R})$ then by taking a sequence of functions $\varphi_n \geq 0$ with support $[-1/n, 1/n]$ such that $\varphi_n \in C^1(\mathbb{R})$, and $\int \varphi_n d\mu = 1$, it is then easy to show that $f * \varphi_n \in C^1(\mathbb{R})$, $f * \varphi_n \to f$ uniformly on compact sets, and $\|f * \varphi_n\|_{\text{Lip}} \leq \|f\|_{\text{Lip}}$. Hence it follows that $f(x) \in D(\overline{\delta})$ and

$$\|\overline{\delta}(f(x))\| \leq \limsup_n \|\overline{\delta}((f * \varphi_n)(x))\|$$

$$= \limsup_n \|\tilde{\pi}(\widetilde{f * \varphi_n}) \cdot \delta(x)\|$$

$$\leq \limsup_n \|f * \varphi_n\|_{\text{Lip}} \|\delta(x)\| \leq \|f\|_{\text{Lip}} \|\delta(x)\|.$$

Since $f * \varphi_n \to f$ uniformly on K it follows that $\|(f * \varphi_n)(x) - f(x)\|_\infty \to 0$ and since each $(f * \varphi_n)$ can be uniformly approximated in $C^1(K)$ by polynomials it follows that if δ is closed as an operator from N with the uniform topology to $\mathcal{H}$ then we have $f(x) \in D(\delta)$.

If we consider now $\xi \in D(\overline{\delta})_{s.a.}$ then there exists a sequence $x_n \in D(\delta)_{s.a.}$ such that $\|\xi - x_n\|_2 \to 0$ and $\|\overline{\delta}(\xi) - \delta(x_n)\| \to 0$. By Lemma 3.1 $\|f(\xi) - f(x_n)\|_2 \to 0$ and it therefore follows that $f(\xi) \in D(\overline{\delta})$ and

$$\|\overline{\delta}(f(\xi))\| \leq \limsup_n \|\overline{\delta}(f(x_n))\|$$

$$\leq \limsup_n \|f\|_{\text{Lip}} \|\delta(x_n)\| = \|f\|_{\text{Lip}} \|\overline{\delta}(\xi)\|.$$

□

Notation 3.3. Let $\mathcal{H}$ be a Hilbert N-bimodule, let $\delta : L^2(N, \tau) \to \mathcal{H}$ be a closable symmetric derivation. We introduce the following operators on $L^2(N, \tau)$:

$$\Delta := \delta^* \overline{\delta};$$

$$\phi_t := \exp -t\Delta;$$

$$\rho_\alpha := \frac{\alpha}{\alpha + \Delta};$$

It follows from general theory that the above maps also give corresponding operators on N for which we will use the same notation.

Using the spectral theorem for unbounded operators we have the following relationships between these maps (all limits are in the sense of pointwise convergence):

$$\phi_t \circ \phi_s = \phi_{t+s};$$

$$(\beta - \alpha)\rho_\alpha \circ \rho_\beta = \beta\rho_\alpha - \alpha\rho_\beta;$$

$$\Delta = \lim_{t\to 0} \frac{1}{t}(\mathrm{id} - \phi_t) = \lim_{\alpha\to\infty} \alpha(\mathrm{id} - \rho_\alpha);$$

$$\phi_t = \exp(-t\Delta) = \lim_{\alpha\to\infty} \exp(-t\alpha(\mathrm{id} - \rho_\alpha)),$$

$$\rho_\alpha = \alpha(\alpha + \Delta)^{-1} = \alpha \int_0^\infty e^{-\alpha t}\phi_t dt.$$

Theorem 3.4 (Sauvageot [Sau90])**.** *Let $\mathcal{H}$ be a Hilbert N-bimodule, let $\delta : L^2(N,\tau) \to \mathcal{H}$ be a closable symmetric derivation. Then the maps ϕ_t, and ρ_α are both contractive, τ-symmetric, and completely positive.*

Proof. Since $\lim_{\alpha\to\infty} \exp(-t\alpha(\mathrm{id} - \rho_\alpha)) = \lim_{\alpha\to\infty} \Sigma_{n=0}^\infty \frac{(t\alpha)^n}{n!}\rho_\alpha^n$ it is enough to show that the ρ_α's are completely positive. Also, by scaling δ it is enough to show that ρ_1 is completely positive.

Moreover, if we consider $\delta^{(n)} : L^2(\mathbb{M}_n(\mathbb{C})\overline{\otimes}, \mathrm{tr}\otimes\tau) \to L^2(\mathbb{M}_n(\mathbb{C}), \mathrm{tr})\overline{\otimes}\mathcal{H}$ such that $D(\delta^{(n)}) = \mathbb{M}_n(\mathbb{C}) \otimes D(\delta)$, and $\delta^{(n)}(A \otimes x) = A \otimes \delta(x)$. Then it is not hard to see that $\rho_1^{(n)} = \frac{1}{1+\overline{\delta^{(n)*}\delta^{(n)}}}$. Hence, it is enough to show that ρ_1 is positive.

Suppose $\eta \in L^2(N,\tau)$ such that $\eta \geq 0$, and denote $\xi = \rho_1(\eta) \in D(\overline{\Delta}) \subset D(\overline{\delta})$ so that $\xi + \Delta(\xi) = \eta$. Since δ is symmetric it is easy to see that ξ is self-adjoint.

Denote by $I : D(\overline{\delta}) \to L^2(N,\tau)$ the identity map, and note that if we endow $D(\overline{\delta})$ with the graph norm $\|x\|_{\mathcal{G}}^2 = \|x\|_2^2 + \|\overline{\delta}(x)\|^2$ then we have that $D(\overline{\delta})$ is a Hilbert space, I is a bounded operator and $\rho_1 = I^*$.

By hypothesis we have that $\|\overline{\delta}(|\xi|)\| \leq \|\overline{\delta}(\xi)\|$ and hence $\||\xi|\|_{\mathcal{G}} \leq \|\xi\|_{\mathcal{G}}$. Also, note that since $\eta \geq 0$ we have $\eta^{1/2}\xi\eta^{1/2} \leq \eta^{1/2}|\xi|\eta^{1/2}$ and hence $|\langle\xi,\eta\rangle| \leq \langle|\xi|,\eta\rangle$. Therefore we have

$$\|\xi\|_{\mathcal{G}}^2 = |\langle\xi, I^*\eta\rangle_{\mathcal{G}}| = |\langle\xi,\eta\rangle| \leq \langle|\xi|,\eta\rangle$$

$$= \langle|\xi|, I^*\eta\rangle_{\mathcal{G}} = \langle|\xi|,\xi\rangle_{\mathcal{G}} \leq \||\xi|\|_{\mathcal{G}}\|\xi\|_{\mathcal{G}} \leq \|\xi\|_{\mathcal{G}}^2.$$

Thus $\||\xi|\|_{\mathcal{G}} = \langle|\xi|,\xi\rangle_{\mathcal{G}} = \|\xi\|_{\mathcal{G}}^2$ and hence $\xi = |\xi|$. □

Lemma 3.5. *Let $\mathcal{H}$ be a Hilbert N-bimodule, let $\delta : L^2(N,\tau) \to \mathcal{H}$ be a closable symmetric derivation. If $a \in D(\overline{\delta})$, then $|a| \in D(\overline{\delta})$ and $\|\overline{\delta}(|a|)\|^2 + \|\overline{\delta}(|a^*|)\|^2 \leq \|\delta(a)\|^2 + \|\delta(a^*)\|^2$.*

Proof. Consider $\delta^{(2)} : \mathbb{M}_2(L^2(N,\tau)) \to \mathbb{M}_2(\mathcal{H}) \cong \mathcal{H}^{\oplus 4}$. Then $\delta^{(2)}$ is a closable symmetric derivation with $\overline{\delta^{(2)}} = \overline{\delta}^{(2)}$. If $a \in D(\overline{\delta})$ then $\begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix} \in D(\overline{\delta}^{(2)})$ and hence by Theorem 3.2 we have that

$$\left|\begin{pmatrix} 0 & a \\ a^* & 0 \end{pmatrix}\right| = \begin{pmatrix} |a| & 0 \\ 0 & |a^*| \end{pmatrix} \in D(\overline{\delta}^{(2)}),$$

and $\|\overline{\delta}(|a|)\|^2 + \|\overline{\delta}(|a^*|)\|^2 \leq \|\delta(a)\|^2 + \|\delta(a^*)\|^2$. □

Lemma 3.6. *Let $\mathcal{H}$ be a Hilbert N-bimodule, let $\delta : L^2(N,\tau) \to \mathcal{H}$ be a closable symmetric derivation. Suppose that δ is closed as an operator from N with the uniform topology to $\mathcal{H}$. If $x \in D(\overline{\delta})_{s.a.} \cap N$ then there exists a sequence $x_n \in D(\delta)_{s.a.}$ such that $\|x_n - x\|_2 \to 0$, $\|\delta(x_n) - \delta(x)\|_2 \to 0$, and $\|x_n\|_\infty \leq \|x\|_\infty$ for all $n \in \mathbb{N}$.*

Proof. Note that from Theorem 3.2 we have that $D(\delta)_{s.a.}$ is closed under Lipschitz functional calculus. Let $f(t) = t \wedge \|x\|_\infty \vee (-\|x\|_\infty)$, and suppose $y_n \in D(\delta)_{s.a.}$ such that $\|y_n - x\|_2$, and $\|\delta(y_n) - \delta(x)\| \to 0$.

By Lemma 3.1 we have that $\|f(y_n) - x\|_2 = \|f(y_n) - f(x)\|_2 \to 0$. Also,

$$\limsup_n \|\overline{\delta}(f(y_n))\| = \|\overline{\delta}(x)\| < \infty.$$

Hence a subsequence of $f(y_n)$ converges weakly to x in the graph norm on $D(\overline{\delta})$, and so by taking convex combinations of $f(y_n)$ we obtain the desired sequence. □

Theorem 3.7 (Davies-Lindsay [DL92])**.** *Let $\mathcal{H}$ be a Hilbert N-bimodule, and let $\delta : L^2(N,\tau) \to \mathcal{H}$ be a closable symmetric derivation. Then $D(\overline{\delta}) \cap \mathfrak{n}_\tau$ is a $*$-subalgebra. If $\mathcal{H}$ is normal then we have that $\overline{\delta}_{|D(\overline{\delta})\cap\mathfrak{n}_\tau}$ is again a derivation.*

Proof. If we consider first the closure of δ as an operator from N with the uniform topology to $\mathcal{H}$, then it follows easily from the triangle inequality that the domain of this closure is again a $*$-algebra and the extension of δ to this domain is again a derivation. Thus we may assume that δ is closed as an operator from N to $\mathcal{H}$.

If $a \in D(\overline{\delta}) \cap \mathfrak{n}_\tau$ then since $t \mapsto t^2 \wedge \|a\|^2$ is Lipschitz, it from Lemma 3.5 together with Lemma 3.1 that $|a|^2 \in D(\overline{\delta})$. Now if $x, y \in D(\overline{\delta}) \cap \mathfrak{n}_\tau$ then we may apply the polarization identity to conclude

$$x^* y = \frac{1}{4}\Sigma_{j=0}^3 \sqrt{-1}^j |x + \sqrt{-1}^j y|^2 \in D(\overline{\delta}).$$

The fact that $\overline{\delta}_{|D(\overline{\delta})\cap\mathfrak{n}_\tau}$ is again a derivation if $\mathcal{H}$ is a normal bimodule follows by applying Lemma 3.6 together with the triangle inequality. □

Remark 3.8. In the previous theorem in order to show that $D(\overline{\delta}) \cap \mathfrak{n}_\tau$ was again a $*$-subalgebra, we used only the facts that the quadradic form $q(x) = \|\overline{\delta}(x)\|^2$ and its amplification $q^{(2)}(x) = \|\overline{\delta}^{(2)}(x)\|^2$ satisfy the properties that the domain $D(\overline{\delta})$ is self-adjoint, closed under taking absolute values and Lipschitz functional calculus on self-adjoint elements, and satisfies

$$q^{(n)}(x^*) = q^{(n)}(x);$$

$$q^{(n)}(|x|) + q^{(n)}(|x^*|) \leq q^{(n)}(x) + q^{(n)}(x^*);$$
$$q^{(n)}(f(y)) \leq \|f\|^2_{\mathrm{Lip}_0} q^{(n)}(y),$$

where $n = 1, 2$, $x, y \in D(\overline{\delta})$ with $y = y^*$, and $f \in \mathrm{Lip}_0(\mathbb{R})$.

Theorem 3.9 (Sauvageot [Sau89])**.** *If $\phi_t : N \to N$ is a strongly continuous semigroup of τ-symmetric, unital, completely positive maps then there is a Hilbert N-bimodule $\mathcal{H}$ and a densely defined closable symmetric derivation $\delta : L^2(N,\tau) \to \mathcal{H}$ such that $\phi_t = \exp(-t\delta^*\overline{\delta})$, for all $t \geq 0$.*

Proof. By the Hille-Yoshida theorem there is a positive operator Δ on $L^2(N,\tau)$ such that $\phi_t = e^{-t\Delta}$ for $t > 0$, where

$$D(\Delta) = \{\xi \in L^2(N,\tau) \mid \lim_{t\to 0} \|\xi - \phi_t(\xi)\|_2 < \infty\},$$

and $\|\Delta\xi\|_2 = \lim_{t\to 0} \|\xi - \phi_t(\xi)\|_2$ for $\xi \in D(\Delta)$. We will show that Δ is of the form $\delta^*\delta$ for a closable symmetric derivation δ.

By Theorem 2.1 to each ϕ_t we can associate a bimodule $\mathcal{H}_t$, and a vector $\xi_t \in \mathcal{H}_t$ such that $\tau(\phi_t(x)y) = \langle x\xi_t y, \xi_t\rangle$ for each $x, y \in N$. Also, note that since ϕ_t is τ-symmetric, there exist an anti-linear isometric involution J on $\mathcal{H}_t$ given by $J(x\xi_t y) = y^*\xi_t x^*$. Consider the derivation, $\delta_t : L^2(N,\tau) \to \mathcal{H}_t$ given by $D(\delta_t) = \mathfrak{n}_\tau$, and

$$\delta_t(x) = x\xi_t - \xi_t x.$$

It is then easy to check that δ_t is a closable symmetric derivation for each $t > 0$.

Note that

$$D(\Delta^{1/2}) = \{\xi \in L^2(N,\tau) \mid \lim_{t\to 0} \frac{1}{t}\|(\mathrm{id} - \phi_t)^{1/2}(\xi)\|_2^2 < \infty\},$$

and we have

$$\|\Delta^{1/2}\xi\|_2^2 = \lim_{t\to 0} \frac{1}{t}\|(\mathrm{id} - \phi_t)^{1/2}(\xi)\|_2^2 = \lim_{t\to 0} \frac{1}{t}\|\delta_t(\xi)\|^2,$$

for $\xi \in D(\Delta^{1/2})$.

By taking a limit, it then follows from Lemma 3.5, and Theorem 3.2, that for $x \in D(\Delta^{1/2})$ we have $|x| \in D(\Delta^{1/2})$ with

$$\|\Delta^{1/2}(|x|)\|_2^2 + \|\Delta^{1/2}(|x^*|)\|_2^2 \leq \|\Delta^{1/2}(x)\|_2^2 + \|\Delta^{1/2}(x^*)\|_2^2,$$

and if $x = x^* \in D(\Delta^{1/2})$, and $f \in \mathrm{Lip}_0(\mathbb{R})$, then $f(x) \in D(\Delta^{1/2})$ and

$$\|\Delta^{1/2}(f(x))\|_2 \leq \|f\|_{\mathrm{Lip}}\|\Delta^{1/2}(x)\|_2.$$

By Remark 3.8 we then have that $D(\Delta^{1/2})$ is a $*$-subalgebra.

On $D(\Delta^{1/2}) \otimes_{\mathrm{alg}} D(\Delta^{1/2})$ we may then define the sesquilinear form

$$\begin{aligned}\langle y \otimes b, x \otimes a\rangle &= \lim_{t\to 0} \frac{1}{t}\tau(a^*(\phi_t(x^*y) - \phi_t(x^*)\phi_t(y))b) \\ &= -\tau(a^*(\Delta(x^*y) - \Delta(x^*)y - x^*\Delta(y)).\end{aligned}$$

This is well defined since $x^*y \in D(\Delta^{1/2}) \subset D(\Delta)$. Also, this is non-negative definite which can be seen by applying Kadison's inequality for the u.c.p. maps $\phi_t^{(n)}$, and then taking a limit. We then obtain a Hilbert space $\mathcal{H}$ by separation and completion. Moreover, we have an anti-linear isometric involution J on $\mathcal{H}$ given by $J(x \otimes a) = a^* \otimes x^* - a^*x^* \otimes 1$.

We may define a right N-module structure to $\mathcal{H}$ by the formula $(x \otimes a) \cdot z = x \otimes az$, and then using J we may define a right N-module structure by the formula $z \cdot (x \otimes a) = zx \otimes a - z \otimes xa$. A simple check shows that this gives a well defined bimodule structure to $\mathcal{H}$.

Finally, we define the derivation $\delta : D(\Delta^{1/2}) \cap \mathfrak{n}_\tau \to \mathcal{H}$ by the formula

$$\delta(x) = \frac{1}{\sqrt{2}}(x \otimes 1).$$

It is easy to check that this is indeed a symmetric derivation and we have

$$\begin{aligned}\langle \delta(x), \delta(y) \rangle &= \frac{1}{2}\langle x \otimes 1, y \otimes 1 \rangle \\ &= \frac{1}{2}\tau(\Delta(y^*)x + y^*\Delta(x) - \Delta(y^*x)) \\ &= \langle \Delta^{1/2}(x), \Delta^{1/2}(y) \rangle,\end{aligned}$$

which shows that δ is closable and $\Delta = \delta^*\overline{\delta}$. □

3.1 Examples

We now list some examples of closable derivations which appear in von Neumann algebras. Many of these examples overlap. For many of these examples the corresponding semigroups ϕ_t can be computed explicitly. It is a good exercise to try to do so when possible.

Note that for verifying closability we will use repeatedly the well known fact that an operator $D : \mathcal{H} \to \mathcal{H}$ is closable if and only if the adjoint D^* is densely defined.

3.1.1 Inner derivations

Let $\mathcal{H}$ be a normal symmetric Hilbert N-bimodule and let $\xi \in \mathcal{H}$ be given such that $\mathcal{J}(\xi) = \xi$, then one can construct a inner derivation whose domain is $\mathfrak{n}_\tau \subset L^2(N, \tau)$ given by

$$\delta(x) = x\xi - \xi x.$$

A vector $\eta \in \mathcal{H}$ is said to be left (resp. right) bounded if there is a constant $C > 0$ such that $\|x\eta\| \leq C\|x\|_2\|\eta\|$ (resp. $\|\eta x\| \leq C\|x\|_2\|\eta\|$). Given any $\eta \in \mathcal{H}$ if we consider the normal positive linear functional ψ on N given by $\psi(x) = \langle x\eta, \eta \rangle$ then by duality ψ corresponds to a positive element $\eta' \in L^1(N, \tau)$. Thus there exists a sequence of projections $p_n \in N$ which converge σ-weakly to the identity such that $p_n\eta' \in N$ for all n and hence $p_n\eta$ is left bounded for all $n \in \mathbb{N}$. The same argument holds for right boundedness and it therefore follows that the set of vectors in $\mathcal{H}$ which are both left and right bounded is dense in $\mathcal{H}$.

Since $D(\delta^*)$ contains the set of left and right bounded vectors it follows that δ is closable.

3.1.2 Approximately inner derivations

Suppose that N is separable and that we have a countable generating set of self-adjoint elements $x_1, x_2, \ldots$ Also, suppose $\mathcal{H}_j$ is a sequence of symmetric normal Hilbert N-bimodules and $\xi_i \in \mathcal{H}_i$ are given such that $\mathcal{J}(\xi_i) = \xi_i$ for all $i \in \mathbb{N}$, and $\|x\xi_i - \xi_i x\| \to 0$ for all $x \in N$. Then by taking a subsequence we may assume that $\|x_j\xi_i - \xi_i x_j\|^2 < 1/2^i$ for all $1 \leq j \leq i$. We may therefore define a derivation δ on the algebra A generated by $x_1, x_2, \ldots$ given by

$$\delta(a) = \oplus_{i\in\mathbb{N}}(a\xi_i - \xi_i a),$$

for all $a \in A$.

If $\mathcal{H}_i^0$ denotes the set of vectors in $\mathcal{H}_i$ which are both left and right bounded, then it is easy to see that $D(\delta^*)$ contains the dense subset $\cup_{j\in\mathbb{N}}(\oplus_{i=1}^j \mathcal{H}_i^0) \subset \oplus_{i\in N}\mathcal{H}_i$ and hence δ is closable.

If we consider P_i the projection from $\mathcal{H}_i$ to the space of N central vectors in $\mathcal{H}_i$ then it is easy to see that δ will be inner if and only if $\Sigma_{i\in\mathbb{N}}\|\xi_i - P_i(\xi_i)\|^2 < \infty$.

A finite von neumann algebra B has property (T) if and only if no such sequence ξ_i exists with $\Sigma_{i\in\mathbb{N}}\|\xi_i - P_i(\xi_i)\|^2 = \infty$, thus we see that if B is separable II_1 factor which does not have property (T) then for any weakly dense, countably generated unital $*$-subalgebra $D \subset B$ there exists a non-inner symmetric closable derivation $\delta : L^2(N,\tau) \to \mathcal{H}$ into some normal Hilbert B-bimodule $\mathcal{H}$ such that $D \subset D(\delta)$.

The converse of the previous paragraph also holds but is a bit more subtle to establish.

3.1.3 The difference quotient

If $N = L^\infty(\mathbb{R},\mu)$ where μ is a diffuse Radon probability measure on $\mathbb{R}$. Then $L^2(\mathbb{R}\times\mathbb{R},\mu\times\mu)$ is a normal Hilbert $L^\infty(\mathbb{R},\mu)$-bimodule where the bimodule structure is given by

$$(f \cdot g \cdot h)(s,t) = f(s)g(s,t)h(t).$$

We obtain a derivation $\delta : L^2(\mathbb{R},\mu) \to L^2(\mathbb{R}\times\mathbb{R},\mu\times\mu)$ by

$$\delta(f)(s,t) = \frac{f(s) - f(t)}{s - t}.$$

Note that we do not need to define $\delta(f)$ when $s = t$ since this set has measure 0. The domain of δ is defined to be $\{f \in L^\infty(\mathbb{R},\mu) \mid \frac{f(s)-f(t)}{s-t} \in L^2(\mathbb{R}\times\mathbb{R},\mu\times\mu)$.

Note that if $F \in L^2(\mathbb{R}\times\mathbb{R},\mu\times\mu)$ such that $\frac{F(s,t)}{s-t} \in L^2(\mathbb{R}\times\mathbb{R},\mu\times\mu)$ then $F \in D(\delta^*)$. It is easy to see that such functions are dense and hence δ is closable.

For a specific example if we consider the case when $\mu(x) = \frac{1}{\sqrt{2\pi}}\exp(-\frac{1}{2}x^2)dx$ is the Gaussian measure. Then the function $f(t) = t$ is in $L^2(\mathbb{R},\mu)$ and it is easy to see that $f \in D(\overline{\delta})$ and $\overline{\delta}(f)(s,t) = 1$.

3.1.4 Group algebras and group-measure space constructions

Let Γ be a group and let $\pi : \Gamma \to \mathcal{O}(\mathcal{K})$ be an orthogonal representation. We have a pair of commuting representations of Γ on $\mathcal{H} = \mathcal{K}\overline{\otimes}_{\mathbb{R}}\ell^2\Gamma$ given by $\pi\otimes\lambda$ and $1\otimes\rho$. The representation $1\otimes\rho$ can also be viewed as a representation of Γ^{op} via the isomorphism $g^{\mathrm{op}} \mapsto g^{-1}$. By Fell's absorption lemma both of these representations are equivalent to a multiple of the left regular representation and hence extend to normal representations of the group von Neumann algebras $L\Gamma$ and $L\Gamma^{\mathrm{op}}$. Thus $\mathcal{H}$ is a normal Hilbert $L\Gamma$-bimodule, which is symmetric via the involution $\mathcal{J}(\xi \otimes_{\mathbb{R}} \Sigma_{g\in\Gamma}\alpha_g\delta_g) = \xi \otimes_{\mathbb{R}} \Sigma_{g\in\Gamma}\overline{\alpha_g}\delta_{g^{-1}}$.

A 1-cocycle is a map $c : \Gamma \to \mathcal{K}$ such that $c(gh) = c(g) + \pi(g)c(h)$ for all $g,h \in \Gamma$. Associated to this 1-cocycle is a derivation $\delta : L^2(L\Gamma,\tau) \to \mathcal{H}$ which is defined on the group algebra $\mathbb{C}\Gamma \subset L\Gamma$ by $\delta(\Sigma_{g\in\Gamma}\alpha_g u_g) = \Sigma_{g\in\Gamma}\alpha_g c(g)\otimes\delta_g$.

The derivation property as well as the symmetry is readily verified and it is easy to see that the adjoint of δ contains the dense subspace $\mathcal{K}\otimes_{\mathrm{alg}}\mathbb{C}\Gamma$, hence δ is closable.

More generally, if we also have a measure preserving action $\Gamma \curvearrowright (X,\mu)$, and we denote by $\sigma : \Gamma \to \mathcal{U}(L^2(X,\mu))$ the Koopman representation, then we again have commuting representations on

$L^2(X,\mu)\overline{\otimes}\mathcal{K}\overline{\otimes}\ell^2\Gamma$ given by $\sigma\otimes\pi\otimes\lambda$ and $1\otimes 1\otimes\rho$. Moreover, these representations extend to turn $L^2(X,\mu)\overline{\otimes}\mathcal{K}\overline{\otimes}\ell^2\Gamma$ into a normal Hilbert $L^\infty(X,\mu)\rtimes\Gamma$-bimodule.

We can then define a derivation $\delta : L^2(L^\infty(X,\mu)\rtimes\Gamma)\to L^2(X,\mu)\overline{\otimes}\mathcal{K}\overline{\otimes}\ell^2\Gamma$ which is defined by letting $\delta_{|L^\infty(X,\mu)} = 0$, and $\delta(u_g) = 1\otimes c(g)\otimes\delta_g$, and then extending to the algebra generated by $L^\infty(X,\mu)$ and $\mathbb{C}\Gamma$.

3.1.5 Equivalence relations

Let $\mathcal{R}$ be a countable discrete measure preserving equivalence relation. Let $\tilde{\mu}$ be the infinite measure on $\mathcal{R}$ given by $\tilde{\mu}(E) = \int_X |\{(x,y)\in E\}| d\mu(x)$.

Given $f,g\in L^2(\mathcal{R},\tilde{\mu})$ we define the convolution of f and g as $(fg)(x,y) = \Sigma_{z\sim x} f(x,z)g(z,y)$. In general, the convolution of f and g will no longer live in $L^2(\mathcal{R},\tilde{\mu})$, although by the Cauchy-Schwarz inequality it is well defined as a measurable function on $\mathcal{R}$.

If f is the characteristic function on the diagonal of $X\times X$ then $fg = gf = g$ for all $g\in L^2(\mathcal{R},\tilde{\mu})$. Also, we have an involution on $L^2(\mathcal{R},\tilde{\mu})$ given by $f^*(x,y) = \overline{f(y,x)}$.

If $fg\in L^2(\mathcal{R},\tilde{\mu})$ for all $g\in L^2(\mathcal{R},\tilde{\mu})$ then it can be shown using the closed graph theorem that f is a bounded operator. We may thus consider

$$L\mathcal{R} = \{f\in L^2(\mathcal{R},\tilde{\mu}) \mid D(f) = L^2(\mathcal{R},\tilde{\mu})\}\subset\mathcal{B}(L^2(\mathcal{R},\tilde{\mu})).$$

This turns out to be a von Neumann algebra, and is in fact finite since we have a trace given by

$$\tau(f) = \int f(x,x)d\mu(x).$$

$L^\infty(X,\mu)$ is naturally a von Neumann subalgebra of $L\mathcal{R}$ by considering the functions whose support is on the diagonal of $X\times X$.

An orthogonal representation of $\mathcal{R}$ consists of a Borel field of real Hilbert spaces $\mathcal{K} = ({}_x\mathcal{K}, x\in X)$ together with a Borel map $\pi:\mathcal{R}\to\mathcal{O}(\mathcal{K})$ such that $\pi(x,y)\in\mathcal{O}({}_y\mathcal{K},{}_x\mathcal{K})$ and $\pi(x,y)\pi(y,z) = \pi(x,z)$ for all $x\sim y\sim z$.

Given a representation $\mathcal{K} = ({}_x\mathcal{K}, x\in X)$ of $\mathcal{R}$ consider the Hilbert space $\mathcal{H} = L^2(\mathcal{R},\mathcal{K},\tilde{\mu})$ of $\tilde{\mu}$-square integrable functions from $\mathcal{R}$ to $\mathcal{K}\otimes_{\mathbb{R}}\mathbb{C}$ such that $\xi(x,y)\in {}_x\mathcal{K}\otimes_{\mathbb{R}}\mathbb{C}$.

We can then consider a normal action of $L\mathcal{R}$ on $\mathcal{H}$ given by

$$(f\xi)(x,y) = \Sigma_{z\sim x} f(x,z)\pi(x,z)\xi(z,y).$$

We also have a commuting normal right action of $L\mathcal{R}$ on $\mathcal{H}$ given by

$$(\xi f)(x,y) = \Sigma_{z\sim x}\xi(x,z)f(z,y) = \Sigma_{z\sim x} f(z,y)\xi(x,z).$$

A cocycle of $\mathcal{R}$ is a Borel map $c:\mathcal{R}\to\mathcal{K}$ such that $c(x,y)\in {}_x\mathcal{K}$ for all $x\sim y$ and

$$c(x,z) = c(x,y) + \pi(x,y)c(y,z),$$

for all $x\sim y\sim z$.

Given a cocycle we may consider the Borel map $\delta(f):\mathcal{R}\to\mathcal{H}$ by

$$\delta(f)(x,y) = f(x,y)c(x,y)$$

This map will not in general be in $\mathcal{H}$ since it need not be square integrable, however this does map into $\mathcal{H}$ for a weakly dense subalgebra $D(\delta)$ of $L\mathcal{R}$.

If $f, g \in D(\delta)$ then we have

$$\delta(fg)(x,y) = \Sigma_{z\sim x} f(x,z)c(x,y)g(z,y)$$

$$= \Sigma_{z\sim x} f(x,z)c(x,y)g(z,y) = \Sigma_{z\sim x} f(x,z)(c(x,z) + \pi(x,z)c(z,y))g(z,y)$$
$$= \delta(f)g + f\delta(g).$$

Hence, δ gives a derivation. It is also not too difficult to see that δ is both symmetric and closable.

Note also that $\delta_{|L^\infty(X,\mu)} = 0$.

3.1.6 Free products

If M_j are finite von Neumann algebras for $j = 1, 2$ and $B \subset M_j$ is a von Neumann subalgebra such that the traces on M_1 and M_2 agree on B, then $B \subset M_1 *_B M_2$ and we have a derivation $\delta : L^2(M_1 *_B M_2) \to L^2\langle M_1 *_B M_2, e_B\rangle$ which is given by

$$\delta(x) = \begin{cases} xe_B - e_B x & \text{if } x \in M_1; \\ 0 & \text{if } x \in M_2. \end{cases}$$

Since $be_B = e_B b$ for all $b \in B$, we can use freeness to extend δ to the algebraic free product, and it's not hard to see that this gives a closable symmetric derivation.

3.1.7 Free Brownian motion

Let M be a finite von Neumann algebra generated by self-adjoint elements $x_1, \ldots, x_m$. Let $s_1, \ldots, s_m$ be freely independent semi-circular operators which are also freely independent of $x_1, \ldots, x_m$. Let $t > 0$ and define N_t to be the von Neumann algebra generated by $x_1 + ts_1, \ldots, x_m + ts_m$. Such operators are algebraically free and hence for each $1 \leq k \leq m$ we can define a unique derivation $\delta_k : L^2(N_t, \tau) \to L^2(N_t \overline{\otimes} N_t, \tau \otimes \tau)$ whose domain is $* - \text{alg}(x_1 + ts_1, \ldots, x_m + ts_m)$ and such that $\delta_k(x_j + ts_j) = \delta_{j,k} 1 \otimes 1$.

With this definition Voiculescu showed that we always have $D(\delta_k) \otimes_{\text{alg}} D(\delta_k) \subset D(\delta_k^*)$ and thus δ_k is a closable derivation.

3.1.8 Tensor products

Let N_i, be a finite von Neumann algebra for $i \in \mathbb{N}$ and suppose that $\delta_i : L^2(N_i, \tau) \to \mathcal{H}_i$ is a closable symmetric derivation in to a normal Hilbert N_i-bimodule for each $i \in \mathbb{N}$. Let $N = \overline{\otimes}_{i\in\mathbb{N}} N_i$, and for each $k \in \mathbb{N}$ define the normal Hilbert N-bimodule

$$\tilde{\mathcal{H}}_k = (\overline{\otimes}_{i\in\mathbb{N}, i\neq k} L^2(N_i, \tau)) \overline{\otimes} \mathcal{H}_i.$$

We can then define a derivation $\tilde{\delta}_k : L^2(N, \tau) \to \tilde{\mathcal{H}}_k$ whose domain is the algebraic tensor product $\otimes_{i\in\mathbb{N}} D(\delta_i)$ by

$$\tilde{\delta}_k(\otimes_{i\in\mathbb{N}} x_i) = (\otimes_{i\in\mathbb{N}, i\neq k} x_i) \otimes \delta_k(x_k).$$

Taking the direct sum we can consider $\mathcal{H} = \oplus_{k\in\mathbb{N}}\tilde{\mathcal{H}}_k$ and the derivation $\delta : L^2(N,\tau) \to \mathcal{H}$ given by

$$\delta(x) = \oplus_{k\in\mathbb{N}}\tilde{\delta}_k(x).$$

Then δ is a symmetric closable derivation. Moreover, if ϕ_t^i is the semigroup of completely positive maps associated to δ_i then we can see that the semigroup of completely positive maps associated to δ is $\otimes_{i\in\mathbb{N}}\phi_t^i$.

3.1.9 q-Gaussians

Let $\mathcal{K}_0$ be a real Hilbert space and let $\mathcal{K}$ be its complexification. If $q \in [-1,1]$ is given, then we can define a sesquilinear form $\langle\cdot,\cdot\rangle_q$ on the sum of the algebraic tensor products

$$\mathbb{C}\Omega \oplus \mathcal{K} \oplus \mathcal{K}^{\otimes 2} \oplus \cdots ,$$

such that

$$\langle \xi_1 \otimes \cdots \otimes \xi_m, \eta_1 \otimes \cdots \otimes \eta_n\rangle_q = \delta_{m,n}\Sigma_{\pi\in S_n} q^{\iota(\pi)}\Pi_{k=1}^n \langle \xi_k, \eta_{\pi(k)}\rangle,$$

where

$$\iota(\pi) = |\{(i,j) \mid 1 \le i < j \le n, \pi(i) > \pi(j)\}|.$$

This is a non-negative definite form (it is even positive definite if $q \neq -1, 1$) and the corresponding Hilbert space $\mathcal{F}_q(\mathcal{K})$ is called the q-Fock space.

For each $\xi \in \mathcal{K}_0$ we obtain the creation and annihilation operators $a(\xi)^*, a(\xi) : F_q(\mathcal{K}) \to F_q(\mathcal{K})$ given by

$$a(\xi)^*(\Omega) = \xi;$$
$$a(\xi)^*(\eta_1 \otimes \cdots \otimes \eta_n) = \xi \otimes \eta_1 \otimes \cdots \otimes \eta_n;$$
$$a(\xi)(\Omega) = 0;$$
$$a(\xi)(\eta_1 \otimes \cdots \otimes \eta_n) = \Sigma_{i=1}^n q^{i-1}\langle \xi, \eta_i\rangle \eta_1 \otimes \cdots \otimes \hat{\eta_i} \otimes \cdots \eta_n,$$

where $\hat{\eta_i}$ denotes omission of η_i.

The operators $a(\xi)$ and $a(\xi)^*$ are bounded when $q < 1$, and we define the q-Gaussian operator $x(\xi) = a(\xi) + a(\xi)^*$. Note that when $q = 1$ we can still interpret the above operators as unbounded operators and in this case all the operators commute and $x(\xi)$ has Gaussian distribution with mean 0 and variance $\|\xi\|^2$.

The set of q-Gaussian operators generate a finite von Neumann algebra $\Gamma_q(\mathcal{K}_0)$ which has a trace given by $\tau(x) = \langle x\Omega, \Omega\rangle$, and in this way we can identify $L^2(\Gamma_q(\mathcal{K}_0))$ with $\mathcal{F}_q(\mathcal{K})$.

By embedding $\mathcal{K}_0$ as $\mathcal{K}_0 \oplus 0 \subset \mathcal{K}_0 \oplus \mathcal{K}_0$, we have that $L^2(\Gamma_q(\mathcal{K}_0 \oplus \mathcal{K}_0))$ is a normal Hilbert $\Gamma_q(\mathcal{K}_0)$-bimodule. We obtain a derivation

$$\delta : L^2(\Gamma_q(\mathcal{K}_0)) \to L^2(\Gamma_q(\mathcal{K}_0 \oplus \mathcal{K}_0))$$

by defining $\delta(x(\xi)) = 0 \oplus x(\xi)$ for al $\xi \in \mathcal{K}_0$, and then using the derivation property to extend to the $*$-algebra generated by the $x(\xi)$'s. If we view $L^2(\Gamma_q(\mathcal{K}_0))$ as $\mathcal{F}_q(\mathcal{K})$ via the embedding $x \mapsto x\Omega$, then we can compute δ directly as

$$\delta(\xi_1 \otimes \cdots \otimes \xi_n) = \Sigma_{k=1}^n (\xi_1 \oplus 0) \otimes \cdots (\xi_{k-1} \oplus 0) \otimes (0 \oplus \xi_k) \otimes (\xi_{k+1} \oplus 0) \otimes \cdots \otimes (\xi_n \oplus 0).$$

Actually, the $\Gamma_q(\mathcal{K}_0)$-bimodule that δ maps into is the sub-bimodule $\mathcal{H}$ of $L^2(\Gamma_q(\mathcal{K}_0 \oplus \mathcal{K}_0))$ which is generated by vectors of the form $(0 \oplus \xi)$ for $\xi \in \mathcal{K}$. Depending on q and $\dim(\mathcal{K}_0)$ the bimodule $\mathcal{H}$ will have different properties. For example, when $q = 0$ then it is not hard to check that $\mathcal{H}$, as a bimodule is isomorphic to a direct sum of coarse bimodules.

3.1.10 Conditionally negative definite kernels

Let S be a set, let $\mathcal{K}$ be a real Hilbert space and let $q : S \to \mathcal{K}$. The function on $S \times S$ given by $(s,t) \mapsto \|q(s) - q(t)\|^2$ is called a conditionally negative definite kernel.

We denote by N the semifinite von Neumann algebra $\mathcal{B}(\ell^2 S)$. We also identify N with $\mathcal{B}(\ell^2 S) \otimes 1 \subset \mathcal{B}(\ell^2 S \overline{\otimes}_{\mathbb{R}} \mathcal{K})$, in this way the Hilbert-Schmidt operators $\mathrm{HS}(\ell^2 S, \ell^2 S \overline{\otimes} \mathcal{K})$ gives a Hilbert N-bimodule, where the bimodule structure is given by composition of operators.

Associated with q is the operator $M_q : \ell^2 S \to \ell^2 S \overline{\otimes} \mathcal{K}$ whose domain is the algebraic sum $\mathbb{C}S$, and on this it is given by $M_q(\delta_s) = \delta_s \otimes q(s)$. Note that this operator is closable since it is easy to see that $\mathbb{C}S \otimes_{\mathrm{alg}} \mathcal{K} \subset D(M_q^*)$ and hence M_q^* is densely defined. It is also self-adjoint since $\mathcal{K}$ is a real Hilbert space. We also remark that if one can think about operators from $\ell^2 S$ to $\ell^2 S \overline{\otimes} \mathcal{K}$ as "$\mathcal{K}$ valued $S \times S$ matrices", and from this perspective we see that M_q is just the diagonal matrix with diagonal entries $q(s)$ (this perspective also gives an anti-linear isometry $\mathcal{J}$ by taking the adjoint of a matrix).

We can then define a derivation $\delta : L^2(N, \tau) = \mathrm{HS}(\ell^2 S) \to \mathrm{HS}(\ell^2 S, \ell^2 S \overline{\otimes} \mathcal{K})$ whose domain is the finite rank operators, and such that

$$\delta(A) = A M_q - M_q A.$$

Note that if $A \in \mathrm{FR}(\ell^2 S)$, and $B \in \mathrm{FR}(\ell^2 S, \ell^2 S \overline{\otimes} \mathcal{K})$ then we have

$$|\langle \delta(A), B \rangle| = |\tau_{\mathcal{B}(\ell^2 S \overline{\otimes} \mathcal{K})}((A M_q - M_q A) B^*)|$$

$$= |\tau_{\mathcal{B}(\ell^2 S \overline{\otimes} \mathcal{K})}(A M_q B^*) + \tau_{\mathcal{B}(\ell^2 S)}(B^* M_q A)| \leq \|A\|_{\mathrm{HS}}(\|M_q B^*\|_{\mathrm{HS}} + \|B^* M_q\|_{\mathrm{HS}}).$$

This shows that $\mathrm{FR}(\ell^2 S, \ell^2 S \overline{\otimes} \mathcal{K}) \subset D(\delta^*)$ and so in particular δ is closable.

4 Approximate bimodularity

For the sequel N will be a finite von Neumann algebra with trace τ. We will also assume that $1 \in D(\delta)$, and hence ϕ_t and ρ_α will be unital completely positive maps.

Notation 4.1. We will need still more notation. Given a closable symmetric derivation $\delta : L^2(N, \tau) \to \mathcal{H}$ we continue to denote by $\rho_\alpha = \frac{\alpha}{\alpha + \delta^* \bar{\delta}}$, we will additionally define

$$\zeta_\alpha := \sqrt{\rho_\alpha};$$

$$\theta_\alpha := \mathrm{id} - \alpha^{-1/2} \Delta^{1/2} \circ \zeta_\alpha = \mathrm{id} - (\frac{\Delta}{\alpha + \Delta})^{1/2}.$$

Lemma 4.2. *We have the following formulas:*

$$\zeta_\alpha = \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{t}(t+1)}\rho_{\alpha(t+1)/t}\,dt;$$

$$\theta_\alpha = \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{t}(t+1)}\rho_{\alpha t/(t+1)}\,dt.$$

In particular the maps ζ_α and θ_α are unital τ-symmetric, completely positive.

Proof. Since we have that

$$\sqrt{s} = \int_0^\infty \frac{s}{\sqrt{t}(t+s)}\,dt,$$

for $s > 0$, we deduce

$$\zeta_\alpha = \int_0^\infty \frac{\rho_\alpha}{\sqrt{t}(t+\rho_\alpha)}\,dt.$$

However since $\rho_\alpha = \alpha(\alpha+\Delta)^{-1}$ we conclude

$$\frac{\rho_\alpha}{t+\rho_\alpha} = \frac{\alpha}{\alpha(t+1)+t\Delta} = \frac{1}{t+1}\rho_{\alpha(t+1)/t}.$$

The formula for θ_α follows similarly. □

Lemma 4.3 (Sauvageot [Sau99])**.** *The maps $\psi_t = \exp(-t\Delta^{1/2})$ are unital, completely positive for all $t > 0$.*

Proof. We have that $\Delta^{1/2} = \lim_{\alpha\to\infty} \Delta^{1/2}\circ\zeta_\alpha = \lim_{\alpha\to\infty}\alpha(\mathrm{id}-\theta_\alpha)$, hence

$$\psi_t = \lim_{\alpha\to\infty} e^{-\alpha t}\exp(\alpha t\theta_\alpha).$$

Since θ_α is completely positive so is $\exp(\alpha t\theta_\alpha)$ and hence by taking a limit so is ψ_t. □

Note that we have $D(\Delta^{1/2}) = D(\overline{\delta})$.

Lemma 4.4 (Ozawa-Popa [OP10])**.** *Consider $x, y \in D(\overline{\delta})\cap N$, then*

$$\|\Delta^{1/2}(x^*)y + x^*\Delta^{1/2}(y) - \Delta^{1/2}(x^*y)\|_2^2 \le 16\|x\|_\infty\|\delta(x)\|\|y\|_\infty\|\delta(y)\|.$$

Proof. Note that by Theorem 3.7 we have that $x^*y \in D(\Delta^{1/2}) = D(\overline{\delta})$ for all $x, y \in D(\overline{\delta})\cap N$.

For $x, y \in D(\overline{\delta})\cap N$ we have the carré du champ:

$$\begin{aligned}\Gamma(x^*, y) &:= \Delta^{1/2}(x^*)y + x^*\Delta^{1/2}(y) - \Delta^{1/2}(x^*y)\\ &= \frac{d}{dt}|_{t=0}(\psi_t(x^*y) - \psi_t(x^*)\psi_t(y))\\ &= \lim_{t\to 0}\frac{\psi_t(x^*y) - \psi_t(x^*)\psi_t(y)}{t}.\end{aligned}$$

Hence if we define a sesquilinear form on $N \otimes D(\overline{\delta})$ by $\langle b \otimes y, a \otimes x \rangle = \tau(a^*\Gamma(x^*, y)b)$ then this form is non-negative definite (apply Kadison's inequality to the maps $\psi_t^{(n)}$). Thus we may use the Cauchy-Schwarz inequality to conclude

$$\|\Gamma(x^*, y)\|_2^2 = \sup_{\|aa^*\|_2, \|bb^*\|_2 \leq 1} |\tau(a^*\Gamma(x^*, y)b)|$$

$$\leq \sup_{\|aa^*\|_2, \|bb^*\|_2 \leq 1} \tau(a^*\Gamma(x^*, x)a)\tau(b^*\Gamma(y^*, y)b) \leq \|\Gamma(x^*, x)\|_2 \|\Gamma(y^*, y)\|_2.$$

Applying the triangle inequality together with the fact that $\|\Delta^{1/2}(x^*x)\|_2 = \|\delta(x^*x)\|$ then gives the result ◻

Lemma 4.5. *Let $\theta : N \to N$ be a unital τ-symmetric completely positive map. If $x, a \in N$ then*

$$\|\theta(ax) - \theta(a)\theta(x)\|_2 \leq 2\|x\|_\infty \|a\|_2^{1/2} \|a - \theta(a)\|_2^{1/2}.$$

Proof. If we consider the Stinespring dilation $\theta(x) = V\pi(x)V^*$ then we have

$$\|\theta(ax) - \theta(a)\theta(x)\|_2 = \|V^*\pi(x^*)(1 - VV^*)\pi(a^*)V\|_2$$

$$\leq \|x\|_\infty \|(1 - VV^*)^{1/2}\pi(a^*)V\|_2 = \|x\|_\infty \tau(\theta(aa^*) - \theta(a)\theta(a^*))^{1/2}$$
$$= \|x\|_\infty \tau(aa^* - \theta(a)\theta(a^*))^{1/2} \leq 2\|x\|_\infty \|a\|_2^{1/2} \|a - \theta(a)\|_2^{1/2}.$$

◻

Theorem 4.6 (Peterson [Pet09])**.** *Let $\mathcal{H}$ be a Hilbert N-bimodule, let $\delta : L^2(N, \tau) \to \mathcal{H}$ be a closable symmetric derivation. Denote by $\tilde{\delta}_\alpha(x) = \alpha^{-1/2}\delta(\zeta_\alpha(x))$. If $x, a \in N$ then we have*

$$\|\zeta_\alpha(a)\tilde{\delta}_\alpha(x) - \tilde{\delta}_\alpha(ax))\| \leq 10\|x\|_\infty^{1/2} \|a\|_2^{1/2} \|\tilde{\delta}_\alpha(a)\|^{1/2};$$

$$\|\tilde{\delta}_\alpha(x)\zeta_\alpha(a) - \tilde{\delta}_\alpha(xa)\| \leq 10\|x\|_\infty^{1/2} \|a\|_2^{1/2} \|\tilde{\delta}_\alpha(a)\|^{1/2}.$$

Proof. First note that for all $y \in N$ we have

$$\|\tilde{\delta}_\alpha(y)\|^2 = \alpha^{-1}\langle \Delta \circ \rho_\alpha(y), y \rangle = \tau((y - \rho_\alpha(y))y^*).$$

Hence, since $\mathrm{id} - \rho_\alpha \geq 0$ as an operator on $L^2(N, \tau)$ we have $\|y - \rho_\alpha(y)\|_2^2 \leq \|\tilde{\delta}_\alpha(y)\| \leq \|y\|_\infty \|y - \rho_\alpha(y)\|_2$, and $\|\tilde{\delta}_\alpha(y)\| \leq \|y\|_2$.

Also note that as bounded operators on $L^2(N, \tau)$ it follows from functional calculus together with the inequality $1 - \sqrt{t} \leq \sqrt{1 - t}$ for $t \geq 0$ that we have $(\theta_\alpha - \zeta_\alpha)^2 \leq (\mathrm{id} - \zeta_\alpha)^2 \leq (\mathrm{id} - \rho_\alpha)$. Thus, for all $y \in N$ we have

$$\|\theta_\alpha(y) - \zeta_\alpha(y)\|_2 \leq \|y - \zeta_\alpha(y)\|_2 \leq \|\tilde{\delta}_\alpha(y)\|.$$

Suppose $a, x \in N$, then by using the product rule for δ we see that

$$\|\zeta_\alpha(a)\tilde{\delta}_\alpha(x) - \alpha^{-1/2}\delta(\zeta_\alpha(a)\zeta_\alpha(x))\| \tag{7}$$

$$\leq \|x\|_\infty \|\tilde{\delta}_\alpha(a)\| \leq \|x\|_\infty \|a\|_\infty^{1/2} \|\tilde{\delta}_\alpha(a)\|^{1/2}.$$

From Lemma 4.4 we have that

$$\|\alpha^{-1/2}\Delta^{1/2}(\zeta_\alpha(a)\zeta_\alpha(x)) - \alpha^{-1/2}\zeta_\alpha(a)\Delta^{1/2}(\zeta_\alpha(x))\|_2 \tag{8}$$
$$\leq 5\|x\|_\infty \|a\|_\infty^{1/2} \|\tilde{\delta}_\alpha(a)\|^{1/2}.$$

Also recall that $\alpha^{-1/2}\Delta^{1/2} \circ \zeta_\alpha = \mathrm{id} - \theta_\alpha$ and hence from Lemma 4.5 we have

$$\|\alpha^{-1/2}\zeta_\alpha(a)\Delta^{1/2}(\zeta_\alpha(x)) - \alpha^{-1/2}\Delta^{1/2}(\zeta_\alpha(ax))\|_2 \tag{9}$$
$$= \|\zeta_\alpha(a)x - \zeta_\alpha(a)\theta_\alpha(x) - ax + \theta_\alpha(ax)\|_2$$
$$\leq \|a - \zeta_\alpha(a)\|_2 \|x\|_\infty + \|\theta_\alpha(a) - \zeta_\alpha(a)\|_2 \|x\|_\infty + \|\theta_\alpha(a)\theta_\alpha(x) - \theta_\alpha(ax)\|_2$$
$$\leq 4\|x\|_\infty \|a\|_\infty^{1/2} \|\tilde{\delta}_\alpha(a)\|^{1/2}$$

By considering the polar decomposition of $\delta = V\Delta^{1/2}$, if we apply V to (8) and (9) above we have

$$\|\alpha^{-1/2}\delta(\zeta_\alpha(a)\zeta_\alpha(x)) - \tilde{\delta}_\alpha(ax)\|_2 \leq 9\|x\|_\infty \|a\|_\infty^{1/2} \|\tilde{\delta}_\alpha(a)\|^{1/2} \tag{10}$$

Combining (7) and (10) then gives the result.

The second inequality follows from the first by symmetry. □

The presence of the term $\zeta_\alpha(a)$ in Theorem 4.6 can be undesirable but this is necessary since $\tilde{\delta}_\alpha(x)$ may not be a left or right bounded vector. This can be overcome however by changing our bimodule $\mathcal{H}$ slightly.

Specifically, since ζ_α is unital, τ-symmetric, and completely positive it follows from Stinespring's theorem that there exists a normal pointed Hilbert N-bimodule $(\mathcal{H}^0_\alpha, \xi_\alpha)$ such that $\tau(\zeta_\alpha(x)y) = \langle x\xi_\alpha y, \xi_\alpha\rangle$ for all $x, y \in N$. Denote by $\mathcal{H}_\alpha$ the Hilbert N-bimodule generated by vectors of the form $\xi_\alpha \otimes_N \overline{\delta}(x) \otimes_N \xi_\alpha$ for $x \in D(\overline{\delta})$. This is a subbimodule of $\mathcal{H}^0_\alpha \otimes_N \mathcal{H} \otimes_N \mathcal{H}^0_\alpha$.

Denote by $\delta_\alpha : L^2(N, \tau) \to \mathcal{H}_\alpha$ the bounded operator given by

$$\delta_\alpha(x) = \frac{1}{\sqrt{\alpha}} \xi_\alpha \otimes_N \delta(x) \otimes_N \xi_\alpha.$$

Note that $\delta_\alpha(x)$ is well defined since ξ_α is a left and right tracial vector. Moreover, by the definition of the relative tensor product we have that

$$\langle a\delta_\alpha(x)b, \delta_\alpha(y)\rangle = \langle \zeta_\alpha(a)\tilde{\delta}_\alpha(x)\zeta_\alpha(b), \tilde{\delta}_\alpha(y)\rangle,$$

for all $a, b, x, y \in N$. In particular this shows that $\delta_\alpha(x)$ is a left and right bounded vector for all $x \in N$ and we can use the triangle inequality to deduce a similar approximate bimodularity property which we state now along with a number of other properties of δ_α which we have shown in this section.

Theorem 4.7. *Let $\mathcal{H}$ be a normal Hilbert N-bimodule, let $\delta : L^2(N,\tau) \to \mathcal{H}$ be a closable symmetric derivation. Let $\mathcal{H}_\alpha \subset \mathcal{H}^0_\alpha \otimes_N \mathcal{H} \otimes_N \mathcal{H}^0_\alpha$ and δ_α be defined as above. Then for all $a, x \in N$ we have that $\delta_\alpha : L^2(N,\tau) \to \mathcal{H}_\alpha$ is a contraction and*

$$\|x - \rho_\alpha(x)\|_2^2 \leq \|\delta_\alpha(x)\|^2 = \tau((x - \rho_\alpha(x))x^*) \leq \|x\|_2 \|x - \rho_\alpha(x)\|_2;$$

$$\ker(\delta_\alpha) \cap N = \ker(\delta) \cap N \text{ is a von neumann subalgebra of } N;$$

$$\alpha \mapsto \|\delta_\alpha(x)\|^2 \text{ is monotonically decreasing;}$$

$$\|\delta_\alpha(x)a - \delta_\alpha(xa)\| \leq 50\|x\|_\infty^{1/2}\|a\|_2^{1/2}\|\delta_\alpha(a)\|^{1/2};$$

$$\|a\delta_\alpha(x) - \delta_\alpha(ax)\| \leq 50\|x\|_\infty^{1/2}\|a\|_2^{1/2}\|\delta_\alpha(a)\|^{1/2}.$$

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