Lecture notes

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1 Introduction

The goal of this minicourse will be to present a proof of the following theorem:

Theorem 1.1 ([CP12]). *Let $G = G_1 \times G_2$ where G_1 is a simple higher rank connected Lie group with trivial center, and G_2 is a simple p-adic Lie group with trivial center, and let $\Lambda < G_1 \times G_2$ be an irreducible lattice. Then for any ergodic, probability measure preserving action on non-atomic space $\Lambda \curvearrowright (X, \nu)$ is essentially free.*

As an example, consider a prime p, and $n \geq 3$. Take $G_1 = PSL_n(\mathbb{R})$, $G_2 = PSL_n(\mathbb{Q}_p)$, and $\Lambda = PSL_n(\mathbb{Z}[1/p])$ embedded diagonally in $G_1 \times G_2$.

The above theorem is a special case of the results in [CP12], however, the proof in this special case already contains most of the intricacies of the more general situation. The theorem complements results of Stuck and Zimmer from [SZ94] where they obtain the same conclusion under the assumption that both G_1 and G_2 are connected and higher rank.

The proof of the above theorem can be seen as a measurable generalization of the proof of the normal subgroup theorem from [CS12] (see also [Cre11]). The strategy of proof fits into the general framework of normal subgroup rigidity techniques developed by Margulis. We refer the reader to [Mar78, Mar79, Zim84, Mar91, SZ94], and [BS06] for other other results in a similar vein. We also refer the reader to [Tho64, Kir65, Ros89, Ovč71, Bek07, DM12] and [PT13] for some similar rigidity results in the non-commutative situation, which follow from different methods.

2 Lattices and induced representations

Let G be a second countable locally compact group. A **lattice** in G is a discrete subgroup $\Gamma < G$, such that the quotient G/Γ has a finite G-invariant measure. If $\Gamma < G$ is a lattice then there exists a finite measure Borel fundamental domain $F \subset G$, i.e., F is a Borel subset of finite measure such that $G = \sqcup_{\gamma \in \Gamma} F\gamma$. Given such a fundamental domain we may consider the map $\alpha : G \times F \to \Gamma$ uniquely defined by the condition

$$gf\alpha(g, f) \in F.$$

Note that the map $f \mapsto f\Gamma$ gives a Borel isomorphism between F and G/Γ, and that under this isomorphism the action of G on G/Γ becomes $g \cdot f = gf\alpha(g, f)^{-1}$. Note also that for $g, h \in G$ and $f \in F$ we have $ghf\alpha(h, f)\alpha(g, h \cdot f) \in F$, and from this it follows that α satisfies the cocycle identity

$$\alpha(gh, f) = \alpha(h, f)\alpha(g, h \cdot f).$$

If $\Gamma \curvearrowright (X, \nu)$ is a quasi-invariant action, then we obtain the **induced action** of G on $F \times X$ by the formula

$$g \cdot (f, x) = (gf, \alpha(g, f)^{-1}x).$$

The fact that this is an action follows easily from the cocycle relation.

Similarly, if $\pi : \Gamma \to \mathcal{U}(\mathcal{H})$ is a representation, then we obtain the **induced representation** on $L^2(F, \mathcal{H})$ by the formula

$$(\tilde{\pi}(g)\xi)(f) = \pi(\alpha(g^{-1}, f))\xi(g^{-1}f).$$

Again, the fact that this is a unitary representation is an easy exercise.

Induced actions and representations defined in this way depend on the fundamental domain F. However, it is not hard to see that taking different fundamental domains gives equivalent induced actions and representations.

3 The Howe-Moore property

Let G be a second countable locally compact group. Recall that a representation $\pi : G \to \mathcal{U}(\mathcal{H})$ is mixing if for every $\xi, \eta \in \mathcal{H}$, the matrix coefficient $g \mapsto \langle \pi(g)\xi, \eta \rangle$ is in $C_0(G)$. Equivalently, the representation is mixing if for any sequence $g_n \in G$ such that $g_n \to \infty$, we have that the unitary operators $\pi(g_n)$ converge in the weak operator topology to 0. A group G has the **Howe-Moore property** if every representation without invariant vectors is mixing.

Theorem 3.1 ([HM79]). *Let G be a simple connected Lie group, then G has the Howe-Moore property.*

We will only prove the Howe-Moore property here for $SL_m(\mathbb{R})$. Recall that any matrix $a \in SL_m(\mathbb{R})$ has a polar decomposition $a = ub$ where u is an orthogonal matrix and b is positive definite and symmetric. Since b is positive definite and symmetric, it can be diagonalized as $b = u_0 d u_0^{-1}$ where d is a positive diagonal matrix with non-increasing entries, and u_0 is an orthogonal matrix. Thus any matrix $a \in SL_m(\mathbb{R})$ has an expression $a = u_1 d u_2$ where u_1 and u_2 are orthogonal and d is diagonal with non-increasing positive entries. We thus obtain the Cartan decomposition $SL_m(\mathbb{R}) = KA_+K$, where $K = SO_m(\mathbb{R})$ and A is the semi-group of positive diagonal matrices with non-increasing entries.

Proof of the Howe-Moore property for $G = SL_2(\mathbb{R})$. Suppose $G = SL_2(\mathbb{R})$, and that $\pi : G \to \mathcal{U}(\mathcal{H})$ is a (strong operator topology) continuous representation which is not mixing, then we will show that there are G-invariant vectors. Since the representation is not mixing, there exists a sequence $\pi(g_n)$ such that $g_n \to \infty$, and $\pi(g_n)$ does not converge to 0 in the weak operator topology. By taking a subsequence we may assume that $\pi(g_n)$ converges weakly to a non-zero operator $S \in \mathcal{B}(\mathcal{H})$. Using the Cartan decomposition we may write $g_n = k_n a_n k_n'$ where $k_n, k_n' \in K$, and $a_n \in A_+$. Since K is compact we have $a_n \to \infty$, and we may take another subsequence so that $\pi(k_n)$ and $\pi(k_n')$ converge in the strong operator topology to unitaries v and

w respectively. If we set $T = v^*Sw^* \neq 0$ then we have that $\pi(a_n)$ converges in the weak operator topology to T.

Write $a_n = \begin{pmatrix} r_n & 0 \\ 0 & r_n^{-1} \end{pmatrix}$, where $r_n \to \infty$, and consider the subgroup $N \subset G$ consisting of upper triangular matrices with entries 1 on the diagonal. Note that the conjugation action of $A = \langle A_+ \rangle$ on N is given by

$$\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix} \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \begin{pmatrix} r^{-1} & 0 \\ 0 & r \end{pmatrix} = \begin{pmatrix} 1 & r^2 s \\ 0 & 1 \end{pmatrix},$$

thus, for $x \in N$ we have $a_n^{-1} x a_n \to e \in G$. Hence $\pi(a_n^{-1} x a_n) \to 1$ in the strong operator topology, and so $\pi(x a_n) = \pi(a_n)\pi(a_n^{-1} x a_n) \to T$ in the weak operator topology. But we also have that $\pi(x a_n) \to \pi(x)T$ in the weak operator topology, and so we conclude that $\pi(x)T = T$ for all $x \in N$, and hence $\pi(x)TT^* = TT^*$ for all $x \in N$. Note that $TT^* \neq 0$ since $\|TT^*\| = \|T\|^2 \neq 0$. Replacing a_n with a_n^{-1} then shows that $\pi(y)T^*T = T^*T$ for all $y \in N^t$, where N^t is the transpose of N consisting of lower triangular matrices with 1's down the diagonal.

Since T and T^* are both weak limits of unitaries from A, and since A is abelian, we have $TT^* = T^*T$, and since N and N^t generate $SL_2(\mathbb{R})$ we then have that $\pi(g)TT^* = TT^*$ for all $g \in SL_2(\mathbb{R})$, thus any non-zero vector in the range of TT^* gives a non-zero invariant vector for $SL_2(\mathbb{R})$ □

Proof of the Howe-Moore property for $SL_m(\mathbb{R})$. For the case when $G = SL_m(\mathbb{R})$, with $m > 2$ we first note that again if π is not mixing then there exists a sequence $a_n \in A_+$ such that $\pi(a_n) \to T \neq 0$ in the weak operator topology. where the upper left entry of a_n is tending to ∞, and that the lower right diagonal entry is tending to 0.

For $i \neq j$, let $N_{i,j} \subset SL_m(\mathbb{R})$ denote the subgroup with 1's down the diagonal and all other entries zero except possibly the (i,j)-th entry, then exactly as above we conclude that any non-zero vector in the range of TT^* is fixed by the copy of $SL_2(\mathbb{R})$ generated by $N_{1,m}$ and $N_{m,1}$, and in particular, is fixed by the subgroup $A_{1,m}$ consisting of those diagonal matrices with positive entries which are 1 except possibly in the first or mth diagonal entries.

If we let $\mathcal{K}$ denote the set of $A_{1,m}$-invariant vectors, then to finish the proof it is enough to show that $\mathcal{K}$ is G-invariant. Indeed, if this is the case then $A_{1,m}$ is contained in the kernel of the representation restricted to $\mathcal{K}$ and since G is simple this must then be the trivial representation.

To see that $\mathcal{K}$ is G-invariant note that $N_{i,j}$ commutes with $A_{1,m}$ whenever $\{i,j\} \cap \{1,m\} = \emptyset$, in which case $N_{i,j}$ leaves $\mathcal{K}$ invariant. On the other hand, if $\{i,j\} \cap \{1,m\} \neq \emptyset$ then $A_{1,m}$ acts on $B_{i,j}$ by conjugation, and this action is isomorphic to the action of A on N described above for $SL_2(\mathbb{R})$. Thus, as above we must have that any vector which is fixed by $A_{1,m}$ is also fixed by $B_{i,j}$ and in particular we have that $B_{i,j}$ leaves $\mathcal{K}$ invariant in this case as well.

Since G is generated by $B_{i,j}$, for $1 \le i, j \le m$ this then shows that $\mathcal{K}$ is indeed G-invariant. □

We remark that the proof above also works equally well for $SL_m(K)$ where K is any non-discrete local field.

4 Property (T)

Let G be a locally compact group and let $\pi : G \to \mathcal{U}(\mathcal{H})$ be a representation. The representation π has **almost invariant vectors** if there exists a net $\xi_n \in \mathcal{H}$, such that $\|\xi_n\| = 1$, and $\|\pi(g)\xi_n - \xi_n\| \to 0$, for all $g \in G$. If $H < G$ is a closed subgroup, then the pair (G, H) has **relative property (T)** if every representation of G which has almost invariant vectors, has H-invariant vectors. The group G has **property (T)** if the pair (G, G) has relative property (T).

Theorem 4.1. *The pair $(SL_2(\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$ has relative property (T).*

Proof. Suppose $\pi : SL_2(\mathbb{R}) \ltimes \mathbb{R}^2 \to \mathcal{U}(\mathcal{H})$ is a representation which has almost invariant vectors $\xi_n \in \mathcal{H}$. Restricting π to $\mathbb{R}^2$ we obtain a unitary representation of $\mathbb{R}^2$, and hence this extends to a representation of the C^*-algebra $C^*(\mathbb{R}^2) \cong C_0(\widehat{\mathbb{R}^2})$. The sequence $\{\xi_n\}$ then defines a sequence of states φ_n on $C_0(\widehat{\mathbb{R}^2})$ by the formula $\varphi_n(f) = \langle \pi(f)\xi_n, \xi_n \rangle$. Since ξ_n are $\mathbb{R}^2$-almost invariant it then follows that $\varphi_n(f) \to f(e)$, for all $f \in C_0(\widehat{\mathbb{R}^2})$. And since ξ_n are $SL_2(\mathbb{R})$-almost invariant, it also follows that for $g \in SL_2(\mathbb{R})$, and $f \in C_0(\widehat{\mathbb{R}^2})$ we have

$$\varphi_n(f \circ g^t) - \varphi_n(f) = \langle \pi(f)\pi(g)\xi_n, \pi(g)\xi_n \rangle - \langle \pi(f)\xi_n, \xi_n \rangle \to 0.$$

By the Reisz representation theorem we may associate φ_n to a sequence of Radon probability measures $\nu_n \in \mathrm{Prob}(\widehat{\mathbb{R}^2})$, which then satisfy $\nu_n(B) \to 1$ for any neighborhood of e, and $\nu_n(gB) - \nu_n(B) \to 0$ for any $g \in SL_2(\mathbb{R})$. Taking a weak*-accumulation point we obtain a mean (i.e., a finitely additive probability measure) m on $\widehat{\mathbb{R}^2}$ such that $m(B) = 1$ for any neighborhood of e, and $m(gB) = m(B)$ for all $g \in SL_2(\mathbb{R})$.

Identify $\widehat{\mathbb{R}^2}$ with $\mathbb{R}^2$ and set

$$A = \{(x, y) \in \mathbb{R}^2 \mid x > 0, -x < y \le x\};$$
$$B = \{(x, y) \in \mathbb{R}^2 \mid y > 0, -y \le x < y\};$$
$$C = \{(x, y) \in \mathbb{R}^2 \mid x < 0, x \le y < -x\};$$
$$D = \{(x, y) \in \mathbb{R}^2 \mid y < 0, y < x \le -y\}.$$

A simple calculation shows that for $k \ge 0$ the sets $A_k = \begin{pmatrix} 1 & 0 \\ 2^k & 1 \end{pmatrix} A$ are pairwise disjoint. Thus, we must have that $m(A) = 0$. A similar argument also shows that $m(B) = m(C) = m(D) = 0$. Hence we conclude that $m(\{(0,0)\}) = m(\mathbb{R}^2 \setminus (A \cup B \cup C \cup D)) = 1$.

Thus, we have $\sup_{f \in C_0(\widehat{\mathbb{R}^2}), \|f\| \leq 1} \{|\varphi_n(f) - f(e)|\} \to 0$, and hence $\sup_{a \in \mathbb{R}^2} |1 - \langle \pi(a)\xi_n, \xi_n \rangle| \to 0$. In particular, for some fixed n we have $\Re(\langle \pi(a)\xi_n, \xi_n \rangle) \geq 1/2$ for all $a \in \mathbb{R}^2$. If we set $K = \overline{\text{co}}\{\pi(a)\xi_n \mid a \in \mathbb{R}^2\}$, then K is a closed convex set and hence has a unique element of minimal norm $\xi_0 \in K$. Note that $\xi_0 \neq 0$ since $\Re(\langle \xi_0, \xi_n \rangle) \geq 1/2$. Also note that $\mathbb{R}^2$ acts on K and preserves the norm, hence must also preserve the unique element of minimal norm. Hence, we have produced a non-zero $\mathbb{R}^2$-invariant vector, and so $(SL_2(\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$ has relative property (T). □

Theorem 4.2 ([Kaž67])**.** *Let G be a simple higher rank Lie group. Then G has property (T).*

Proof for $SL_m(\mathbb{R})$, $m \geq 3$. We consider the group $SL_2(\mathbb{R}) < SL_m(\mathbb{R})$ embedded as matrices in the upper left corner. We also consider the group $\mathbb{R}^2 < SL_m(\mathbb{R})$ embedded as those matrices with 1's on the diagonal, and all other entries zero except possibly the $(1,n)$th, and $(2,n)$th entries. Note that the embedding of $SL_2(\mathbb{R})$ normalizes the embedding of $\mathbb{R}^2$, and these groups generate a copy of $SL_2(\mathbb{R}) \ltimes \mathbb{R}^2$.

If $\pi : SL_m(\mathbb{R})$ is a representation which has almost invariant vectors, then by Theorem 4.1 we have that the copy of $\mathbb{R}^2$ has a non-zero invariant vector. By the Howe-Moore property it then follows that π has an $SL_m(\mathbb{R})$-invariant vector. □

Theorem 4.3. *[Kaž67] Let G be a second countable locally compact group, and $\Gamma < G$ a lattice, then G has property (T) if and only if Γ has property (T).*

Proof. □

Recall that a group G is **amenable** for any compact Hausdorff space K on which G acts, there exists an invariant probability measure.

Proposition 4.4. *A locally compact group G is compact if and only if it is amenable and has property (T).*

Proof. □

5 Boundaries

5.1 Harmonic functions on the unit disk

Recall that given a Dirichlet boundary condition on the unit disk $\hat{f} \in L^\infty(\mathbb{T})$, integration with respect to the Poisson kernels

$$P_r(\theta) = \text{Re}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right), \quad 0 \leq r < 1$$

yields a bounded harmonic function f on the unit disk $D = \{e^{i\theta} \mid -\pi < \theta \leq \pi\}$, given by the Poisson integration formula

$$f(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\hat{f}(e^{it})dt.$$

Every bounded harmonic function arrises in this way and from f we can recover $\hat{f}$ by the formula

$$\hat{f}(e^{i\theta}) = \lim_{r\to 1} f(re^{i\theta}).$$

Thus, the map $\hat{f} \mapsto f$ gives a positivity preserving Banach space isomorphism between $L^\infty(\mathbb{T})$ and the space of bounded harmonic functions on D.

If G denotes the group of fractional linear transformations which preserve the disk (i.e., $G \cong \mathrm{PSL}_2(\mathbb{R})$), then G acts both on $L^\infty(\mathbb{T})$ as well as the space of bounded harmonic functions on D, and the isomorphism $\hat{f} \mapsto f$ is G-equivariant. The action of G on the unit disk is a homogeneous space G/K where K is a maximal compact subgroup, and so for a harmonic function f on D we can lift this to a function $\tilde{f}$ on G by the formula $\tilde{f}(g) = f(g(0))$ and this function will also be harmonic in the sense that it will be in the kernel of the corresponding differential operator $\tilde{\Delta}$ on G. In this setting the Poisson integration formula has a particularly nice form

$$\tilde{f}(g) = \int \hat{f}(g\zeta)dm(\zeta)$$

where m is the normalized Lebesgue measure on the circle $\mathbb{T}$.

5.2 Poisson boundaries

In order to generalize the above situation to other locally compact groups, Furstenberg introduced in [Fur63] the notion of an abstract Poisson boundary. The starting point for this construction is to note that the differential operator $\tilde{\Delta}$ generates a 1-parameter semi-group under convolution of probability measures $\mu_t \in \mathrm{Prob}(G)$ ([Hun56]), and a function $\tilde{f} \in L^\infty(G)$ is harmonic if and only if it is stationary with respect to convolution for some μ_t, i.e.,

$$\tilde{f} * \mu_t = \tilde{f}. \tag{1}$$

Consider a second countable locally compact group G, and a probability measure $\mu \in \mathrm{Prob}(G)$ which is in the same measure class as Haar measure. We define a function $f \in L^\infty(G)$ to be μ-harmonic if it satisfies equation (1), i.e., $f * \mu = f$.

Consider the Borel space $\Omega_0 = \prod_1^\infty G$, which we endow with the product probability measure $\prod_1^\infty \mu$. We define the map $T : \Omega_0 \to \Omega_0$ by

$$T(x_1, x_2, x_3, \ldots) = (x_1x_2, x_3, x_4, \ldots).$$

The group G acts on Ω_0 as

$$g(x_1, x_2, x_3, \ldots) = (gx_1, x_2, x_3, \ldots)$$

and as this action commutes with T we obtain a quasi-invariant action of G on the algebra $L^\infty(\Omega_0, \prod_1^\infty \mu)^T$ of T-invariant functions. By Mackey's point realization theorem [Mac62] we may realize this action of G as a quasi-invariant action on a probability space $G \curvearrowright (B, \eta)$. We refer to this action as the μ-boundary of G. Note that this action is μ-stationary, i.e., $\mu * \eta = \eta$.

Given a function in the μ-boundary $\hat{f} \in L^\infty(B, \eta)$ we can define a function $f \in L^\infty(G)$, the **Poisson transform** of $\hat{f}$, by the formula

$$f(g) = \int \hat{f}(gx) d\eta(x).$$

Note that we have

$$(f * \mu)(g) = \iint \hat{f}(ghx) d\eta(x) d\mu(h) = \int \hat{f}(gx) d(\mu * \eta)(x) = f(g),$$

thus f is μ-harmonic (and hence f is continuous since $L^\infty(G) * L^1(G) \subset C_b(G)$). Conversely, if we are given a μ-harmonic function $f \in L^\infty(G)$, then we can consider the sequence of functions $\hat{f}_n \in L^\infty(\Omega_0)$ given by $\hat{f}_n(x_1, x_2, \ldots) = f_n(x_1 x_2 \cdots x_n)$. Each f_n is measurable with respect to the σ-algebra generated by the first n copies of G, and if we denote E the conditional expectation onto this σ-algebra then since f is μ-harmonic we have

$$E(\hat{f}_{n+1})(x_1, x_2, \ldots) = \int \hat{f}_{n+1}(x_1, x_2, \ldots, x_{n+1}) d\mu(x_{n+1})$$

$$= \int f(x_1 x_2 \cdots x_n x_{n+1}) d\mu(x_{n+1}) = \hat{f}_n(x_1, x_2, \ldots).$$

Thus, the sequence $\{\hat{f}_n\}_n$ forms a martingale and hence by the martingale convergence theorem converges strongly to a function $\hat{f} \in L^\infty(\prod_1^\infty G)$, which is clearly T-invariant, hence $\hat{f} \in L^\infty(B, \eta)$.

We have thus constructed a positivity preserving Banach space isomorphism $\hat{f} \mapsto f$ from $L^\infty(B, \eta)$ to $\mathrm{Har}(G, \mu)$, the space of bounded μ-harmonic functions. Moreover, this isomorphism is G-invariant.

6 Contractive actions

Definition 6.1 ([Jaw94]). A quasi-invariant action $G \curvearrowright (B, \eta)$ is **contractive** if for every $E \subset B$, we have $\inf_{g \in G} \eta(gE) \in \{0, 1\}$.

Note that we could also replace the infimum above with a supremum. Also note that by considering simple functions it's easy to see that an action $G \curvearrowright (B, \eta)$ is contractive if and only if for every $f \in L^\infty(B, \eta)$ we have $\sup_{g \in G} |\int f \, dg\eta| = \|f\|_\infty$, i.e., the Poisson transform is isometric.

Proposition 6.2 ([Jaw94]). *Let G be a second countable locally compact group, and let $\mu \in \mathrm{Prob}(G)$ be a probability measure which is absolutely continuous with respect to Haar measure. Then the action of G in its μ-boundary (B, η) is contractive.*

Proof. Identifying $L^\infty(B, \eta)$ with the space of bounded μ-harmonic functions via the map $\hat{f} \mapsto f$ defined above, we see that $\int \hat{f} d\eta = f(e)$, and thus $\int \hat{f} dg\eta = f(g)$, for each $g \in G$. Therefore, $\sup_{g\in G} |\int \hat{f} dg\eta| = \|f\|_\infty = \|\hat{f}\|_\infty$. Thus, the action is contractive. $\square$

Lemma 6.3. *Let G be a second countable locally compact group, suppose G acts continuously on a compact metric space B, and $\eta \in \mathrm{Prob}(B)$ is a Radon measure such that $G \curvearrowright (B, \eta)$ is contractive. Then for each $y \in B$ there exists a sequence $g_n \in G$ such that $g_n\eta \to \delta_y$ in the weak*-topology.*

Proof. Consider $y \in B$ and take O_n a sequence of open neighborhoods of y such that $\cap_n O_n = \{y\}$. For each $n \in \mathbb{N}$ there exists $g_n \in G$ such that $\nu(g_n O_n) > 1 - \frac{1}{n}$. Thus, if $O \subset B$ is an open set which contains y, then O will contain O_n for large enough n and hence $\nu(g_n O) \to 1$. Conversely, if O does not contain y then we have that $O \cap O_n = \emptyset$ for large enough n and hence $\nu(g_n O) \to 0$. Thus, for this sequence we have that $g_n \eta \to \delta_y$ in the weak*-topology. $\square$

Note that, by a result of Varadarajan, every quasi-invariant action of G on a separable measure space has a model where the space B is a compact metric space (see, e.g., Theorem 2.1.19 in [Zim84]). Also note that the preceding proposition has a converse. That is to say if $G \curvearrowright (B, \eta)$ such that every compact model has the above property then the action is contractive, [FG10].

The next lemma follows trivially from the definitions.

Lemma 6.4. *Let G be a second countable locally compact group and suppose that $G \curvearrowright (B, \eta)$ is contractive, and $\pi : (B, \eta) \to (Z, \zeta)$ is a G-map of G-spaces, then $G \curvearrowright (Z, \zeta)$ is contractive.*

6.1 Boundary actions restricted to lattices

Proposition 6.5 (The random ergodic theorem [Kak51]). *Let G be a second countable locally compact group, with $\mu \in \mathrm{Prob}(G)$ a Radon probability measure in the same measure class as Haar measure. If $G \curvearrowright (X, \nu)$ is an ergodic, probability measure preserving action then for every $f \in L^1(X, \nu)$, and for $\Pi_{\mathbb{N}}\mu$-almost every sequence $(g_n)_n$, and ν-almost every $x \in X$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{n} f(g_n g_{n-1} \cdots g_1 x) = \int f \, d\nu.$$

The proof of the random ergodic is obtained by applying Birkhoff's ergodic theorem to the ergodic transformation $T : \Omega \times X \to \Omega \times X$ given by $T((g_n)_n, x) = ((g_{n-1})_n, g_1 x)$. We omit the details.

Proposition 6.6 ([CS12]). *Let G be a second countable locally compact group, with $\mu \in \mathrm{Prob}(G)$ a Radon probability measure in the same measure class as Haar measure, and suppose that $\Gamma < G$ is a lattice. If $G \curvearrowright (B, \eta)$ is the Poisson boundary action with respect to μ, then the restriction to Γ is again contractive.*

Proof. Fix a compact set $K \subset G$ such that $K\Gamma \subset G/\Gamma$ has positive measure. By the random ergodic theorem we then have that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{n} 1_{K\Gamma}(g_n g_{n-1} \cdots g_1 x) = \nu(K\Gamma),$$

for $\Pi_{\mathbb{N}}\mu$-almost every sequence $(g_n)_n$. In particular, for some $x \in G$, we have that for almost every sequence $(g_n)_n$, the sequence intersects $Kx\Gamma$ infinitely often.

Suppose $E \subset B$, such that $0 < \eta(E) < 1$. By the discussion in Section 5 we have that

$$\Pi_{\mathbb{N}}(\{(g_n)_n \mid \eta(g_n g_{n-1} \cdots g_1 E) \to 0\}) = 1 - \eta(E) > 0,$$

hence there exists a sequence $(g_n)_n$ such that $\eta(g_n g_{n-1} \cdots g_1 E) \to 0$, and $h_n = g_n g_{n-1} \cdots g_1$ meets $Kx\Gamma$ infinitely often. Taking a subsequence we may then write $h_n = k_n x \gamma_n$ where $k_n \in K$ converges to an element $k \in K$.

Since $\eta(h_n E) \to 0$, and $x^{-1} k_n^{-1} \to x^{-1} k^{-1}$ we have that $\eta(\gamma_n E) = \eta((x^{-1} k_n^{-1}) h_n E) \to 0$. Thus $\Gamma \curvearrowright (B, \eta)$ is contractive. □

6.2 Rigidity of contractive actions

Proposition 6.7 ([CS12]). *Let $G \curvearrowright (B, \eta)$ be a contractive action, and suppose $\pi : (B, \eta) \to (Z, \zeta)$ is a factor map. If $\pi' : (B, \eta) \to (Z, \zeta')$ is also a factor map, where $\zeta \prec \zeta'$, then we have $\pi = \pi'$.*

Proof. We may assume that B and Z are compact metric spaces such that π and π' are continuous. By Lemma 6.3 for each $y \in B$ there exists a sequence $g_n \in G$ such that $g_n \eta \to \delta_y$ in the weak*-topology. Since π is continuous we have that $\pi_* : \mathrm{Prob}(B) \to \mathrm{Prob}(Z)$ is weak*-continuous and hence $g_n \zeta = g_n \pi_* \eta = \pi_*(g_n \eta) \to \pi_*(\delta_y) = \delta_{\pi(y)}$ in the weak*-topology. Similarly we have $g_n \zeta' \to \delta_{\pi'(y)}$.

If $\pi(y) \neq \pi'(y)$ then there would exist an open set $O \subset Z$ such that $\pi(y) \in O$, and $\pi'(y) \notin \overline{O}$. Thus $\zeta(g_n O) \to 1$ while $\zeta'(g_n O) \to 0$. Taking a subsequence we may then assume that $\zeta(g_n O) \geq 1/2$, while $\zeta'(g_n O) \leq 1/2^n$, for each $n \in \mathbb{N}$, and hence

$$\lim_{N \to \infty} \zeta'(\cup_{n \geq N} g_n O) = 0,$$

while

$$\lim_{N\to\infty} \zeta(\cup_{n\geq N} g_n O) \geq 1/2.$$

Thus if we consider the G_δ-set $B = \cap_{N\in\mathbb{N}}(\cup_{n\geq N} g_n O)$ then we have $\zeta'(B) = 0$ while $\zeta(B) \geq 1/2$ contradicting the fact that $\zeta \prec \zeta'$. □

7 Amenable actions

Recall that a group G is amenable if and only if whenever E is a Banach space, $\alpha : G \to \text{Isom}(E)$ is an isometric action, and $K \subset E_1^*$ is a compact convex subset which is invariant under the dual action of G, then there exists a G-fixed point in K. Motivated by this characterization, Zimmer defined the notion of amenability for actions which we will now describe.

Suppose $G \curvearrowright (B, \eta)$ is a quasi-invariant action of G on a standard probability space. If E is a separable Banach space, then a cocycle $\alpha : G \times B \to \text{Isom}(E)$ is a Borel map, such that $\alpha(gh, b) = \alpha(g, hb)\alpha(h, b)$ for every g, h, b. A cocycle induces a adjoint cocycle α^* given by $\alpha^*(g, b) = (\alpha(g, b)^{-1})^*$. Suppose for each $b \in B$ we have a compact convex subset $K_b \subset E_1^*$, such that K_b varies in a Borel fashion, i.e., $\{(b, K_b)\} \subset B \times E_1^*$ is a Borel subset. If for all $g \in G$, and almost every $b \in B$ we have that $\alpha^*(g, b)K_b = K_{gb}$ Then $F(B, \{K_b\}) = \{\varphi : B \to E_1^* \mid \varphi(b) \in K_b, \text{ for almost every } b \in B\}$ defines a compact convex subset of $L^\infty(B, E^*)$ which is invariant under the action of G. We will call $F(B, \{K_b\})$ an **affine G-space over B**.

The action $G \curvearrowright (B, \eta)$ is amenable if for any affine G-space over B, $F(B, \{K_b\})$ there exists an invariant section, i.e., there exists a Borel map $\pi : B \to E_1^*$ such that $\pi(b) \in K_b$, and $\alpha^*(g, b)\pi(b) = \pi(gb)$ for all $g \in G$, and almost every $b \in B$.

Theorem 7.1. *Suppose $G \curvearrowright (B, \eta)$ is a quasi-invariant action. Then $G \curvearrowright (B, \eta)$ is amenable if and only if there exists a G-equivariant conditional expectation $E : L^\infty(B \times G) \to L^\infty(B)$.*

Proof. Let us first suppose that $G \curvearrowright (B, \eta)$ is amenable. Let $C_b(G)$ denote the C^*-algebra of all bounded left uniformly continuous functions on G. For each G-invariant separable C^*-subalgebra $A \subset C_b(G)$, we may consider the cocycle $\alpha : G \times B \to \text{Isom}(A)$ given by $\alpha(g, x) = \lambda_g$. Clearly α is a cocycle, and hence since $G \curvearrowright B$ is amenable there exists an invariant section $\pi : B \to S(A)$, where $S(A) \subset A^*$ denotes the state space of A.

We thus obtain a conditional expectation $E_A : L^\infty(B, \eta) \otimes A \to L^\infty(B, \eta)$ given by $E_A(f_1 \otimes f_2)(b) = \pi(b)(f_2)f_1(b)$. Since the section π is invariant we have

$$\begin{aligned} E_A(f_1 \otimes f_2)(gb) &= \pi(gb)(f_2)f_1(gb) \\ &= (\alpha(g,b)^*\pi(b))(f_2)f_1(gb) \\ &= \pi(b)(\lambda_g(f_1))f_2(gb) \\ &= E_A((f_1 \otimes f_1) \circ g)(b). \end{aligned}$$

If we let E be a cluster point of the net $\{E_A\}_A$ then we see that E is a G-invariant conditional expectation $E : L^\infty(B, \eta) \otimes C_b(G) \to L^\infty(B, \eta)$.

To produce a mean on $L^\infty(B, \eta)\overline{\otimes}L^\infty(G)$ we start by taking an approximate identity $\{\psi_n\} \subset C_c(G)$. Specifically, we want that each $\psi_n \in C_c(G)$ is a non-negative function, $\|\psi_n\|_1 = 1$, $\mathrm{supp}(\psi_n) \to \{e\}$, and $\|\psi_n * \delta_g - \delta_g * \psi_n\|_1 \to 0$ for each $g \in G$. If $f \in L^\infty(B, \eta)\overline{\otimes}L^\infty(G)$, then taking convolution pointwise we have $\psi_n * f \in L^\infty(X, \mu) \otimes UC_b(G)$ for each n, and $\|(\psi_n * f) \circ g - \psi_n * (f \circ g)\|_\infty \to 0$ for each $g \in G$, and $f \in L^\infty(B, \eta)\overline{\otimes}L^\infty(G)$. If we set $\Phi_n : L^\infty(B, \eta)\overline{\otimes}L^\infty(G) \to L^\infty(B, \eta)$, by $\Phi_n(f) = \Phi(\psi_n * f)$, then it follows that any accumulation point of $\{\Phi_n\}$ gives a G-equivariant conditional expectation.

Now suppose that there exists a G-equivariant conditional expectation $E : L^\infty(B, \eta)\overline{\otimes}L^\infty(G) \to L^\infty(B, \eta)$, and let $F(B, \{K_b\})$ be an affine G-space over B. Fix any section $\pi_0 : B \to E_1^*$ such that $\pi_0(b) \in K_b$. We set $K_b^0 = \overline{\{\alpha^*(g, g^{-1}b)\pi_0(g^{-1}b) \mid g \in G\}} \subset K_b$. Let A be the C^*-algebra consisting of all essentially bounded Borel functions $f : \sqcup_{b \in G} K_b^0 \to \mathbb{C}$ such that $k \mapsto f(b, k)$ is continous for almost every $b \in B$. Then we have a G-equivariant homomorphism $\phi : A \to L^\infty(B, \eta)\overline{\otimes}L^\infty(G)$ given by $\phi(f)(b, g) = f(\alpha^*(g, g^{-1}b)\pi_0(g^{-1}b))$. If we restrict the condition expectation E to $\phi(A)$, then we may interpret this as a G-equivariant map $p : B \to \mathrm{Prob}(K_b^0)$ such that we have the formula $E(\phi(f))(b) = \int f(b, k)\, dp(b)(k)$.

If we then let $\pi(b)$ be the barycenter $\pi(b) = \int k\, dp(b)(k)$ then a simple calculation shows that π is an invariant section for $F(B, \{K_b\})$. □

Proposition 7.2 ([Zim77]). *Let G be a second countable locally compact group, and $H < G$ a closed subgroup. If the action $G \curvearrowright (B, \eta)$ is amenable, then the action $H \curvearrowright (B, \eta)$ is also amenable.*

Proof. If we fix a right Borel fundamental domain F for H, then we obtain a H-equivariant homomorphism ϕ from $L^\infty(S, \eta)\overline{\otimes}L^\infty(H)$ to $L^\infty(S, \eta)\overline{\otimes}L^\infty(G)$, by $\phi(f)(b, hs_0) = f(b, h)$ whenever $s_0 \in F$. If $E : L^\infty(S, \eta)\overline{\otimes}L^\infty(G) \to L^\infty(S, \eta)$ is a G-equivariant conditional expectation, then $E \circ \phi$ defines a H-equivariant conditional expectation, showing that $H \curvearrowright (B, \eta)$ is amenable. □

Proposition 7.3 ([Zim77]). *Let G be a locally compact group and $\mu \in$ $\mathrm{Prob}(G)$ a probability measure which is absolutely continuous with respect to Haar measure. Then the action of G in its μ-boundary (B, η) is amenable.*

Proof. As above, we identify $L^\infty(B,\eta)$ with the space of bounded μ-harmonic functions $\mathrm{Har}(G,\mu) \subset L^\infty(G)$. Let ω be a non-principle ultrafilter on $\mathbb{N}$ and consider the G-invariant positivity preserving map $\mathcal{E} : C_b(G) \to \mathrm{Har}(G,\mu)$ given by $\mathcal{E}(f) = \lim_{n\to\omega} \frac{1}{N}\sum_{n=1}^N f * \mu^n$.

Since the action of G on $C_b(G)$ is continuous it follows that the action of G on $\mathcal{E}(C_b(G))$ is also continous and hence so is the action on the dense C^*-subalgebra A of $L^\infty(B,\eta)$ which the range of $\mathcal{E}$ generates.

Thus by replacing B with the Gelfand spectrum of A we may assume that B is a compact Hausdorff space on which G acts continuously, and $\mathcal{E} : C_b(G) \to C(B) \subset L^\infty(B,\eta)$. Hence we obtain a G-invariant map π from B to the state space Σ of $\mathrm{C_b}(G)$ given by $\pi(b)(f) = \mathcal{E}(f)(b)$.

We then have a G-equivariant conditional expectation $E : L^\infty(B,\eta) \otimes C_b(G) \to L^\infty(B,\eta)$ given by $E(f_1 \otimes f_2)(b) = f_1(b)\pi(b)(f_2)$. The proof then finishes as in Theorem 7.1. □

8 The amenability half of the Creutz-Shalom normal subgroup theorem

Theorem 8.1 ([CS12]). *Let G be a locally compact group with a lattice $\Gamma < G$, suppose $\Lambda < G$ is a countable dense subgroup which contains and commensurates Γ. Suppose that for every closed normal subgroup $N_0 \lhd G$ we have either $|N_0 \cap \Lambda| < \infty$, or $[\Gamma : N_0 \cap \Gamma] < \infty$, (e.g., if G is simple). If $N \lhd \Lambda$ such that $|N| = \infty$, then $\Gamma/(\Gamma \cap N)$ is amenable.*

Proof. Suppose $\Gamma/(\Gamma\cap N) \curvearrowright K$ is a continuous action on a compact Hausdorff space, which we view as an action of Γ such that $(\Gamma \cap N)$ acts trivially. By Propositions 6.2, 6.6, 7.3, and 7.2, there exists a quasi-invariant action $G \curvearrowright (B,\eta)$ such that the restriction of this action to Γ is amenable and contractive.

Since the action $\Gamma \curvearrowright (B,\eta)$ is amenable, there exists a Γ-invariant map $\pi : B \to \mathrm{Prob}(K)$. We may then consider this as a Γ-factor map with the push forward measure $\pi : (B,\eta) \to (\mathrm{Prob}(K), \pi_*\eta)$.

If $\lambda \in \Lambda \cap N$ then we may consider the map $\pi' : (B,\eta) \to (\mathrm{Prob}(K), \pi'_*\eta)$ defined by $\pi'(y) = \pi(\lambda y)$. Note that since $\lambda\eta \sim \eta$ we have that $\pi_*\eta \sim \pi'_*\eta$.

If $\gamma \in \Gamma \cap \lambda^{-1}\Gamma\lambda$, and we write $\gamma = \lambda^{-1}\gamma_0\lambda$ for $\gamma_0 \in \Gamma$ then we have that $\gamma^{-1}\gamma_0 = \lambda^{-1}(\gamma_0^{-1}\lambda\gamma_0) \in N \cap \Gamma$ hence $\gamma_0 k = \gamma k$ for each $k \in K$. Thus, we have

$$\pi'(\gamma y) = \pi(\lambda(\lambda^{-1}\gamma_0\lambda)y) = \gamma_0\pi'(y) = \gamma\pi'(y),$$

for each $y \in B$ and hence π' is $(\Gamma \cap \lambda^{-1}\Gamma\lambda)$-invariant. Since this group has finite index in Γ, the action $(\Gamma \cap \lambda^{-1}\Gamma\lambda) \curvearrowright (B,\eta)$ is also contractive by Proposition 6.6. Thus from Proposition 6.7 it follows that $\pi' = \pi$, i.e., $\pi(\lambda y) = \pi(y)$ for almost every $y \in B$.

But the action $G \curvearrowright (B, \eta)$ is weakly continuous and hence the map $\lambda \mapsto \pi \circ \lambda$ is also weakly continuous. Thus, for every $g \in \overline{N}$ we have that $\pi = \pi \circ g$. Since $|N| = \infty$ it follows from the hypothesis of the theorem that $[\Gamma : \overline{N} \cap \Gamma] < \infty$, thus $\gamma \circ \pi = \pi \circ \gamma = \pi$ for $\gamma \in \Gamma_0 = \overline{N} \cap \Gamma$. It then follows that Γ_0 acts almost everywhere trivially on $(\mathrm{Prob}(K), \pi_* \eta)$, but as this is a factor of a contractive space, this space is again contractive and hence must be the trivial one point space. Thus, π is almost everywhere constant, and the essential range provides a Γ-invariant probability measure on K showing that $\Gamma/(\Gamma \cap N)$ is amenable. □

Corollary 8.2 ([BS06]). *Let $G = G_1 \times G_2$ be a product of locally compact second countable simple groups, and let $\Lambda < G_1 \times G_2$ be an irreducible lattice. Suppose that G has property (T), and G_2 is totally disconnected, then any non-trivial normal subgroup of Λ has finite index.*

Proof. Suppose $\Lambda < G_1 \times G_2$ is as above, and $N \lhd \Lambda$ is a non-trivial normal subgroup. Let $K < G_2$ be a compact open subgroup, and set $\Gamma = \Lambda \cap (G_1 \times K)$. Since K is open it follows that $\Gamma < G_1 \times K$ is a lattice which is commensurated by Λ.

For $i = 1, 2$ we set Γ_i (resp. Λ_i, N_i) to be the projection of Γ (resp. Λ, N) into G_i. Note that since Λ is an irreducible lattice, and since G_1 and G_2 are simple we have that these projections are injective, so that $\Lambda_i \cong \Lambda$, and $\Gamma_i \cong \Gamma$. Moreover, Λ_i is dense in G_i, while $\Gamma_1 < G_1$ is a lattice.

Note that since Λ_i is dense in G_i, and since G_i is simple, we have that $\overline{N_i} = G_i$. In particular, N_1 is not finite. Applying the previous theorem to $\Gamma_1 < \Lambda_1 < G_1$ then gives $\Gamma_1/(\Gamma_1 \cap N_1)$ is amenable. Since G_1 has (T), so does $\Gamma_1/(\Gamma_1 \cap N_1)$. We therefore conclude that $\Gamma/(\Gamma \cap N) \cong \Gamma_1/(\Gamma_1 \cap N_1)$ is finite.

Note that we have a natural bijection between the countable sets Λ/Γ and G_2/K which is given by $\lambda\Gamma \mapsto \overline{p_2(\lambda\Gamma)} = p_2(\lambda)K$, where p_2 is the projection onto G_2. Moreover, the inverse of this bijection is given by $gK \mapsto p_2^{-1}(gK)$. Take $F \subset \Gamma$ finite so that $\Gamma \subset FN$. From this bijection we then see that $FN = \Lambda \cap p_2^{-1}(\overline{p_2(FN)}) \supset \Lambda \cap p_2^{-1}(\overline{p_2(N)}) = \Lambda$, hence $[\Lambda : N] < \infty$. □

The previous corollary actually holds in a much more general situation. In particular, the hypothesis that G_2 is totally disconnected can be dropped, and the hypothesis that G has property (T) can be significantly relaxed. We refer the reader to [BS06], and [CS12] for details.

9 Stabilizers for actions restricted to dense subgroups

Theorem 9.1 ([Zim87], Lemma 6). *Let G be a locally compact second countable connected simple group with the Howe-Moore property, and let $\Lambda < G$*

be a countable subgroup. If $G \curvearrowright (X,\nu)$ is a non-trivial ergodic probability measure preserving action, then the restriction $\Lambda \curvearrowright (X,\nu)$ is free.

Proof. Suppose $G \curvearrowright (X,\nu)$ is a non-trivial ergodic probability measure preserving action. For each $x \in X$ let K_x be the connected component of the stabilizer subgroup G_x, and let $k(x)$ denote it's dimension.

Note that for $g \in G$ we have $k(gx) = \dim(K_{gx}) = \dim(gK_xg^{-1}) = k(x)$. Thus, k is G-invariant and hence must be constant by ergodicity. Since G has the Howe-Moore property, and G acts ergodically on (X,μ) it follows that the action of G on X is mixing, and so the action of G on $X \times X$ is again ergodic.

By ergodicity, we again have that $\tilde{k}(x,y) = \dim(K_{(x,y)}) = \dim(K_x \cap K_y)$ is constant. If $\tilde{k} = k > 0$ then we would have that $K = K_x = K_y$ for almost every $(x,y) \in X \times X$. But then $gKg^{-1} = K$ for every $g \in G$, and hence K is a non-trivial normal subgroup of G contradicting that G is simple.

Thus, either $k = 0$, or else $\tilde{k} < k$. Applying induction it then follows that the action of G on X^m has discrete stabilizers almost everywhere, for some $m \geq 1$.

Suppose now that $\Lambda < G$ is a countable dense subgroup and $\lambda \in \Lambda \setminus \{e\}$ such that $E = \{x \in X \mid \lambda x = x\}$ has positive measure. We then have that $E^m = \{\tilde{x} \in X^m \mid \lambda\tilde{x} = x\}$ also has positive measure. By continuity we have $\lim_{g \to e} \nu(E^m \Delta g E^m) \to 0$, and hence there exists a sequence $g_n \in G$ such that $g_n \to e$, $\{g_n \lambda g_n^{-1}\}_n$ are pairwise distinct, and $\nu(\cap_{n\in\mathbb{N}} g_n E^m) > 0$. But then for $\tilde{x} \in \cap_{n\in\mathbb{N}} g_n E^m$ we have that $g_n\lambda g_n^{-1} \in G_{\tilde{x}}$, and $g_n \lambda g_n^{-1} \to \lambda$, contradicting the fact that $G_{\tilde{x}}$ is discrete for almost every $\tilde{x} \in X^m$. □

Corollary 9.2 ([CP12])**.** *Let G be a locally compact second countable connected simple group with the Howe-Moore property, and let $\Lambda < G$ be a countable dense subgroup. If $\Lambda \curvearrowright (X,\nu)$ is an ergodic probability measure preserving action, then either this action is essentially free, or else Λ_x is dense in G for almost every $x \in X$.*

Proof. Let $\Lambda \curvearrowright (X,\nu)$ be an ergodic probability measure preserving action, and suppose that this action is not free. Let $\mathrm{Sub}(G)$ denote the space of closed subgroups of G, which we view as a closed (and hence compact) subspace of 2^G. Note that G acts on $\mathrm{Sub}(G)$ by conjugation, and that this action is continuous. The map $\pi : X \to \mathrm{Sub}(G)$ given by $\pi(x) = \overline{\Lambda_x}$ defines a Borel map, and we have $\pi(\lambda x) = \lambda\pi(x)\lambda^{-1}$. Thus, if we consider the push-forward measure $\pi_*\nu$ on $\mathrm{Sub}(G)$ then we have that this measure is Λ-invariant.

Since G acts continuously on $\mathrm{Sub}(G)$ and since Λ is dense, we then have that $\pi_*\nu$ is also G-invariant. Note that as a Λ action $(\mathrm{Sub}(G), \pi_*\nu)$ is a factor of (X,ν), and since Λ does not act freely on (X,ν) it follows that Λ does not act freely on $(\mathrm{Sub}(G), \pi_*\nu)$ either. Therefore, by Theorem 9.1 the action of G on $(\mathrm{Sub}(G), \pi_*\nu)$ must be the trivial action, i.e., $\pi_*\nu$ is supported on the space of normal subgroups, which is $\{\{e\}, G\}$ since G is simple. Since the action of Λ is non-free and ergodic, it follows that $\{e\}$ cannot be in the support of $\pi_*\nu$, and so $\pi_*\nu = \delta_G$. Or in other words $\overline{\Lambda_x} = G$ for almost every $x \in X$. □

10 Weakly amenable actions

Let Γ be a countable group, and suppose that $\Gamma \curvearrowright (X, \nu)$ is a quasi-invariant action. An affine Γ-space $F(B, \{K_b\})$ over X is said to be orbital if $\alpha(g, b) = \mathrm{id}$ whenever $g \in \Gamma_b$. The action $\Gamma \curvearrowright (X, \nu)$ is **weakly amenable** if every orbital affine Γ-space over X has an invariant section.

Given an action of Γ we consider the orbit equivalence relation $\mathcal{R} = \mathcal{R}_{\Gamma \curvearrowright X}$ given by $x\mathcal{R}y$ if and only if $\Gamma y = \Gamma x$. If $\theta : X \to X$ is a measurable bijection such that $(\theta(x), x) \in \mathcal{R}$ for all $x \in X$, then we may obtain a map $\alpha : X \to \Gamma$ such that $\theta(x) = \alpha(x)x$ for all $x \in X$. We may assume that the map α is measurable by choosing an enumeration of Γ, and letting $\alpha(x)$ be the first element in Γ such that $\theta(x) = \alpha(x)x$. Since the Γ action is measure preserving it is then easy to check that θ is also measure preserving. The set of all such θ is the **full group** of the equivalence relation $\mathcal{R}$, and is denoted by $[\mathcal{R}]$.

Consider on $\mathcal{R}$ the measure $\tilde{\nu}$ given by $\tilde{\nu}(E) = \int |\{(x, y) \in E\}| \, d\nu(x)$. Note that we have an embedding $L^\infty(X, \nu) \subset L^\infty(\mathcal{R}, \tilde{\nu})$ as functions which are supported on the diagonal $\Delta = \{(x, x) \mid x \in X\}$. If $\theta \in [\mathcal{R}]$ then we may consider the diagonal action on $\mathcal{R}$ given by $\theta \cdot (x, y) = (\theta(x), y)$. Note that this action preserves the measure $\tilde{\nu}$.

Similar to Theorem 7.1 (and with a similar proof which we will not present here), we have the following characterization of weakly amenable actions.

Proposition 10.1 ([Zim77])**.** *If $\Gamma \curvearrowright (X, \nu)$ is a quasi-invariant action and $\mathcal{R} = \mathcal{R}_{\Gamma \curvearrowright X}$. Then $\Gamma \curvearrowright (X, \nu)$ is weakly amenable if and only if there exists a conditional expectation $E : L^\infty(\mathcal{R}, \tilde{\nu}) \to L^\infty(X, \nu)$ such that $E(f \circ \theta) = E(f) \circ \theta$ for all $f \in L^\infty(\mathcal{R}, \tilde{\nu})$ and $\theta \in [\mathcal{R}]$.*

Corollary 10.2. *Let Γ be a countable group with property (T), and suppose that $\Gamma \curvearrowright (X, \nu)$ is a measure preserving action. Then $\mathcal{R} = \mathcal{R}_{\Gamma \curvearrowright X}$ is amenable if and only if almost every Γ-orbit is finite.*

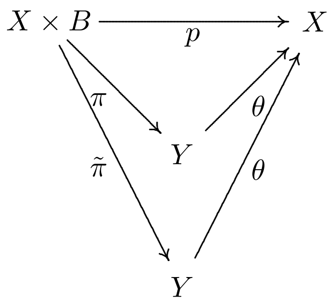
Proof. If almost every Γ-orbit is finite then it is an easy exercise to see that $\mathcal{R}$ is amenable, thus we will only focus on the converse. Suppose that $\mathcal{R}$ is amenable. Then there exists a Γ-equivariant conditional expectation $E : L^\infty(\mathcal{R}, \tilde{\nu}) \to L^\infty(X, \nu)$. Composing E with the integral on $L^\infty(X, \nu)$ then gives a Γ-invariant state φ on $L^\infty(\mathcal{R}, \tilde{\nu})$. Since $L^1(\mathcal{R}, \tilde{\nu})$ is weak* dense in $L^\infty(\mathcal{R}, \tilde{\nu})^*$, there exists a sequence $\eta_n \in L^1(\mathcal{R}, \tilde{\nu})_+$ such that $\|\eta_n\|_1 = 1$, and $\|\eta_n - \gamma\eta_n\|_1 \to 0$, for all $\gamma \in \Gamma$.

If we set $\xi_n = \sqrt{\eta_n}$ then ξ_n forms a sequence of almost invariant vectors for the representation of Γ on $L^2(\mathcal{R}, \tilde{\nu})$, as Γ has property (T) it then follows that there is a non-zero Γ-invariant vector $\xi_0 \in L^2(\mathcal{R}, \tilde{\nu})$. But as $\xi_0 \in L^2(\mathcal{R}, \tilde{\nu})$ is constant on Γ-orbits it follows that for almost every $(x, y) \in \mathcal{R}$ such that $\xi_0(x, y) \neq 0$, we have $|\Gamma x| < \infty$. Since $\xi_0 \neq 0$ it follows that there is a positive measure set $E_0 \subset X$ such that the Γ-orbit of x is finite whenever $x \in E_0$. This property also holds for $E = \Gamma E_0$, and E is Γ-invariant.

A simple maximality argument then finishes the corollary. $\square$

11 Relatively contractive actions

Theorem 11.1 ([CP12])**.** *Let $\Gamma \curvearrowright (B, \eta)$ be a contractive action, and let $\Gamma \curvearrowright (X, \nu)$ be probability measure preserving. Suppose that Z is a Borel space, on which Γ acts, and we have a Γ-equivariant Borel map $\theta : Z \to X$. Suppose also that $\pi : X \times B \to Z$, and $\tilde{\pi} : X \times B \to Z$ are two Γ-equivariant maps such that $\zeta = \pi_*(\nu \times \eta) \sim \tilde{\pi}_*(\nu \times \eta) = \tilde{\zeta}$, and the following diagram commutes:*



Then we have $\pi = \tilde{\pi}$.

Proof. We'll show that $\pi = \tilde{\pi}$ by showing that for every Borel set $E \subset Y$ we have $(\nu \times \eta)(\pi^{-1}(E) \Delta \tilde{\pi}^{-1}(E)) = 0$. So, arguing by contradiction suppose that this is not the case and $E \subset Y$ is Borel such that $(\nu \times \eta)(\pi^{-1}(E) \Delta \tilde{\pi}^{-1}(E)) \neq 0$. With out loss of generality we may assume that if $\tilde{E} = \pi^{-1}(E) \setminus \tilde{\pi}^{-1}(E)$, then we have $(\nu \times \eta)(\tilde{E}) > 0$.

Write $\tilde{E} = \cup_{b \in B} E_b \times \{b\}$, and let $c_0 = \sup_{b \in B} \nu(E_b)$ which must be positive by Fubini's theorem. Fix $\varepsilon > 0$ and set $F_0 = \{b \in B \mid \nu(E_b) > c_0 - \varepsilon\}$. If we consider the map $F_0 \ni b \mapsto E_b \subset X$, then by separability of $L^1(X, \nu)$ there exists a set $F \subset F_0$ such that $\eta(F) > 0$, and $\nu(E_b \Delta E_{b'}) < \varepsilon$ for all $b, b' \in F$. Fix one such $b_0 \in F$ and set $X_0 = E_{b_0}$.

Since $\Gamma \curvearrowright (B, \eta)$ is contractive there exists $\gamma \in \Gamma$ such that $\eta(\gamma F) > 1 - \varepsilon$, and hence $(\nu \times \eta)(\gamma(\tilde{E} \cap (X_0 \times B))) > \nu(\gamma E_0) - 2\varepsilon > c_0 - 3\varepsilon$.

It then follows that

$$\begin{aligned} \zeta(\gamma(E \cap \theta^{-1}(X_0))) &\geq \zeta(\gamma(\pi(\tilde{E}) \cap \theta^{-1}(X_0))) \\ &\geq (\nu \times \eta)(\gamma(\tilde{E} \cap (X_0 \times B))) > c_0 - 3\varepsilon, \end{aligned}$$

while

$$\begin{aligned}\tilde{\zeta}(\gamma(E \cap \theta^{-1}(X_0))) &= \nu(\gamma X_0) - \tilde{\zeta}(\gamma(E^c \cap \theta^{-1}(X_0))) \\ &< \nu(X_0) - \tilde{\zeta}(\gamma(\tilde{\pi}(E^c) \cap \theta^{-1}(X_0))) \\ &< \nu(X_0) - (\nu \times \eta)(\gamma(\tilde{E} \cap (X_0 \times B))) < 4\varepsilon.\end{aligned}$$

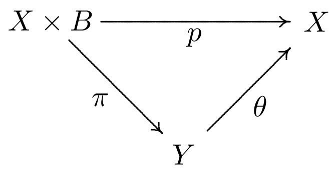
Since $\varepsilon > 0$ is arbitrary and independent of c_0 it then follows that $\zeta \not\prec \tilde{\zeta}$. $\square$

Theorem 11.2 ([CP12]). *Let G be a locally compact second countable group with the Howe-Moore property and having no compact normal subgroups, $\Gamma < G$ a lattice, and $\Lambda < G$ a countable dense subgroup which contains and commensurates Γ.*

If $\Lambda \curvearrowright (X, \nu)$ is an ergodic probability measure preserving action, then either $\Lambda \curvearrowright (X, \nu)$ is free, or else $\Gamma \curvearrowright (X, \nu)$ is weakly amenable.

Proof. Suppose $\Lambda \curvearrowright (X, \nu)$ is an ergodic probability measure preserving action which is not free. Suppose $F(X, \{K_x\})$ is an orbital affine Γ-space, with orbital cocycle action $\alpha : \Gamma \times X \to \mathrm{Isom}(E)$. By Propositions 6.2, 6.6, 7.3, and 7.2, there exists a quasi-invariant action $G \curvearrowright (B, \eta)$ such that the restriction of this action to Γ is amenable and contractive.

Consider the cocycle $\beta : B \times \Gamma \to \mathrm{Isom}(L^1(X, E))$ given by $\beta(\gamma, b)(x) = \alpha(\gamma, x)$. Since $\Gamma \curvearrowright (B, \eta)$ is amenable there exists a Γ-equivariant map $\pi_0 : B \to L^\infty(X, E_1^*)$ such that $\pi_0(b)(x) \in K_x$ for almost all $(x, b) \in X \times B$. Consider the Γ-equivariant map $\pi : X \times B \to X \times E_1^*$ given by $\pi(x, b) = (x, \pi_0(b)(x))$, and let $\theta : X \times E_1^* \to X$ be the natural projection map. Then setting $(Y, \zeta) = (X \times E_1^*, \pi_*(\eta \times \nu))$, and $p : X \times B \to X$ the natural projection map, we have a commutative diagram of Γ-equivariant maps:



Fix $\lambda \in \Lambda$, and let $E_\lambda = \{x \in X \mid x \sim \lambda x\}$. Choose a measurable map $\varphi : E_\lambda \to \Gamma$ so that $\lambda x = \varphi(x)x$ for each $x \in E_\lambda$.

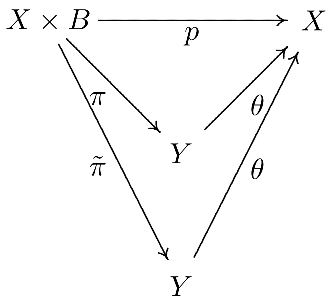
We define a new map $\tilde{\pi} : X \times B \to Y$ by $\tilde{\pi}(x, b) = \varphi(x)^{-1}\pi(\lambda(x, b))$ for $x \in E_\lambda$, and $\tilde{\pi}(x, b) = \pi(x, b)$ for $x \notin E_\lambda$.

We again have $p = \theta \circ \tilde{\pi}$, and if $\gamma \in \Gamma \cap \lambda^{-1}\Gamma\lambda$, then for $x \in E$ we have

$$\tilde{\pi}(\gamma(x, b)) = \varphi(\gamma x)^{-1}\pi(\lambda\gamma(x, b)) = \varphi(\gamma x)^{-1}\lambda\gamma\lambda^{-1}\pi(\lambda(x, b)).$$

Since $\varphi(x)x = \lambda x$ we have $(\varphi(\gamma x)^{-1}\lambda\gamma\lambda^{-1})\lambda x = \gamma\varphi(x)^{-1}\lambda x$. Hence, $\tilde{\pi}(\gamma(x, b)) = \gamma\varphi(x)^{-1}\pi(\lambda(x, b)) = \gamma\tilde{\pi}(x, b)$.

Thus, we have a commutative diagram of $(\Gamma \cap \lambda^{-1}\Gamma\lambda)$-equivariant maps:



Note that since Λ action on $X \times B$ is quasi-invariant, it follows that $\tilde{\pi}_*(\nu \times \eta) \sim \pi_*(\nu \times \eta)$. By Theorem 11.1 it then follows that $\tilde{\pi} = \pi$. In particular, for almost every $(x, b) \in X \times B$ we have $\pi(x, \lambda b) = \pi(\lambda(x, b)) = \pi(x, b)$ for all $\lambda \in \Lambda_x$.

However, for almost every $x \in X$, the map $b \mapsto \pi(x, b)$ is measurable, and since $G \curvearrowright (B, \eta)$ weakly continuously, and since almost every Λ_x is dense in G by Corollary 9.2, we then must have that $\pi(x, gb) = \pi(x, b)$ for almost every $(x, b) \in X \times B$ and every $g \in G$. Since $G \curvearrowright (B, \eta)$ is ergodic it then follows that π is independent of the second variable, i.e., π is a Γ-equivariant map from X to E_1^*, such that $\pi(x) \in K_x$ for almost every $x \in X$. Thus $\Gamma \curvearrowright (X, \nu)$ is weakly amenable. □

Corollary 11.3 ([CP12])**.** *Let G be a locally compact second countable group with the Howe-Moore property, property (T), and having no non-trivial compact normal subgroups, $\Gamma < G$ a lattice, and $\Lambda < G$ a countable dense subgroup which contains and commensurates Γ.*

If $\Lambda \curvearrowright (X, \nu)$ is an ergodic probability measure preserving action, then either $\Lambda \curvearrowright (X, \nu)$ is free, or else $[\Gamma : \Gamma_x] < \infty$ for almost every $x \in X$.

Proof. If $\Lambda \curvearrowright (X, \nu)$ is not free then from the previous theorem we have that $\Gamma \curvearrowright (X, \nu)$ is weakly amenable. Since Γ has property (T) we then have from Corollary 10.2 that the orbits of Γ are finite, and so $[\Gamma : \Gamma_x] < \infty$ for almost every $x \in X$. □

We are now in position to prove the theorem stated in the introduction.

Theorem 11.4 ([CP12])**.** *Let $G = G_1 \times G_2$ where G_1 is a simple higher rank connected Lie group with trivial center, and G_2 is a simple p-adic Lie group with trivial center, and let $\Lambda < G_1 \times G_2$ be an irreducible lattice. Then for any ergodic, probability measure preserving action on non-atomic space $\Lambda \curvearrowright (X, \nu)$ is essentially free.*

Proof. Suppose that $\Lambda \curvearrowright (X, \nu)$ is not free. If $K < G_2$ is a compact open subgroup then we may consider $\Gamma = \Lambda \cap (G_1 \times K)$, and Γ_1 the projection of Γ to G_1. Then as in Corollary 8.2 we have that $\Gamma_1 < G_1$ is a lattice and the projection of Λ to G_1 is dense and commensurates Γ.

By Corollary 11.3 we then have that $[\Gamma_1 : (\Gamma_1)_x] < \infty$ for almost every $x \in X$. Let p_2 denote the projection from $G_1 \times G_2$ to G_2 and note that $\overline{p_2(\Lambda_x)}$ contains $\overline{p_2(\Gamma_x)}$ which is compact and finite index in K, hence open. Thus $\overline{p_2(\Lambda_x)}$ is also open since it contains a compact open subgroup. Since G_2 has Howe-Moore we have that the only open subgroups are either compact or all of G_2. If $\overline{p_2(\Lambda_x)}$ were compact for a positive measure subset of X, then by ergodicity it would then follows that $\overline{p_2(\Lambda_x)}$ is compact for almost every $x \in X$. Since there are only countably many compact open subsets of G_2 it would then follow that there is a positive measure subset $E \subset X$ such that $K_0 = \overline{p_2(\Lambda_x)}$ does not depend on $x \in E$. But since $\lambda K_0 \lambda^{-1} = \overline{p_2(\Lambda_{\lambda x})}$ for $\lambda \in \Lambda$, and since $\Lambda \curvearrowright (X, \nu)$ is measure preserving , we then have that K_0 has finite conjugacy class in G_2, which implies that G_2 has a compact open normal subgroup contradicting the fact that G_2 is simple. Hence, we conclude that $\overline{p_2(\Lambda_x)} = G_2$ for almost every $x \in X$.

We have a natural bijection between Λ/Γ and G_2/K given by $\lambda\Gamma \mapsto \overline{p_2(\lambda\Gamma)} = p_2(\lambda)K$. For each $x \in X$ let $F_x \subset \Gamma$ be finite such that $\Gamma \subset F_x\Gamma_x$. From the above bijection we have $F_x\Lambda_x = \Lambda \cap p_2^{-1}(\overline{p_2(F_x\Gamma_x)}) \supset \Lambda \cap p_2^{-1}(\overline{p_2(\Lambda_x)}) = \Lambda$, hence $[\Lambda : \Lambda_x] < \infty$ for almost every $x \in X$. By ergodicity it then follows that Λ has a single orbit, and hence (X, ν) is a finite atomic probability space. $\square$

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