

Von Neumann Equivalence

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Def: (Gromov '93): Two groups Γ & Λ are measure equivalent (ME) if there is a σ -finite measure space $(\Omega, \mathcal{S})$ and a mp action $\Gamma \times \Lambda \curvearrowright (\Omega, \mathcal{S})$ st Γ & Λ have finite-measure fundamental domains.

Examples: Γ & Λ lattices in G
 $\Gamma \curvearrowright G \curvearrowleft \Lambda$ by left & right multiplication.

- $\Gamma \curvearrowright (X, \mu)$ free ergodic pmp.
- $\Lambda \curvearrowright (Y, \nu)$ free ergodic pmp.

$\Theta: (X, \mu) \longrightarrow (Y, \nu)$ st
 $\Theta(\Gamma \cdot x) = \Lambda \cdot \Theta(x)$ a.e. $x \in X$

$$\Gamma \curvearrowright X \times \Lambda \curvearrowleft \Lambda$$

$$s \cdot (x, \lambda) \cdot t = (sx, \lambda(s, x) \lambda t)$$

Ornstein-weiss '80: All infinite amenable groups are ME

Furman '99: If $\Gamma \curvearrowright_{ME} \Lambda$ then Γ and Λ have (stably) OE actions

Invariants: (Furman, Cowling, Haagerup, Zimmer, Gaborian, ...)

- Amenability.
- Property (T)
- Haagerup property
- Cowling-Haagerup Constant.
- Ratios of ℓ^2 -Betti numbers

$$\mathbb{F}_2 \not\curvearrowright_{ME} \mathbb{F}_\infty$$

Rigidity: Furman '99 - Higher-rank lattices.

Kida '10 - Mapping class groups.

$$L\Gamma := \lambda(\Gamma)'' \subset \mathcal{B}(l^2\Gamma)$$

This is a finite von Neumann alg
with trace $\gamma(x) = \langle x \delta_e, \delta_e \rangle$.

Def: $\Gamma \sim_{w^*E} \Lambda$ if $L\Gamma \simeq L\Lambda$

Invariants: (Schwartz, Connes, Jones, Choda,
cowling, Haagerup)

- Amenability
- Property (T)
- Haagerup property
- Cowling - Haagerup constant

Free group factor problem: $LF_2 \not\simeq LF_2$?

Are ratios of l^2 -Betti numbers invariants?

Connes' rigidity conjecture:

$$L(SL_3\mathbb{Z}) \simeq L\Lambda \implies \Lambda \simeq SL_3\mathbb{Z}?$$

Can one make sense of "cocycles
into linear groups"?

Chifan-Ioana '11:

$$\mathbb{Z}^*(\mathbb{Z} \times \mathbb{F}_2) \sim_{ME} \mathbb{Z}^*(S_\infty \times \mathbb{F}_2)$$

$$\mathbb{Z}^*(\mathbb{Z} \times \mathbb{F}_2) \not\sim_{w^*E} \mathbb{Z}^*(S_\infty \times \mathbb{F}_2)$$

Open Problem: (Shlyakhtenko)

Find groups Γ, Λ st
 $\Gamma \sim_{w^*E} \Lambda$ but $\Gamma \not\sim_{ME} \Lambda$.

Def: (IPR) Γ and Λ are von Neumann equivalent (vNE) if there is a semi-finite vN alg $(\mathcal{M}, \text{Tr})$ and trace-preserving action

$\Gamma \times \Lambda \curvearrowright (\mathcal{M}, \text{Tr})$ st $\Gamma \not\cong \Lambda$
 have finite-trace fundamental domains.

$\exists p \in \mathcal{P}(\mathcal{M})$ st $\sum_{t \in \Gamma} \sigma_t(p) = 1$
 (and similarly q for Λ)

Ex: $\Gamma \curvearrowright_{\text{ME}} \Lambda \Rightarrow \Gamma \curvearrowright_{\text{vNE}} \Lambda$

$\Gamma \curvearrowright_{\text{vNE}} \Lambda \Rightarrow \Gamma \curvearrowright_{\text{ME}} \Lambda$

Set $\mathcal{M} = \mathcal{B}(L^2 \Gamma)$

$\Gamma \curvearrowright \mathcal{M}$ by $\text{Ad}(\lambda(t))$ $p = q = \text{proj}_{\mathbb{C} \mathbb{1}}$
 $\Lambda \curvearrowright \mathcal{M}$ by $\text{Ad}(\rho(s))$

Ex: $\Gamma \curvearrowright (M, \tau)$ $\Lambda \curvearrowright (N, \tau)$

$\Theta: M \rtimes \Gamma \rightarrow N \rtimes \Lambda$ st $\Theta(M) = N$

set $\mathcal{M} = \langle M \rtimes \Gamma, e_M \rangle \cong \langle N \rtimes \Lambda, e_N \rangle$

$\Gamma \curvearrowright \mathcal{M}$ by $\text{Ad}(u_t)$

$\Lambda \curvearrowright \mathcal{M}$ by $\text{Ad}(su_s)$ $p = q = e_M$

Prop: Von Neumann equivalence is an ER.

Proof:

$\Gamma \curvearrowright_{\Lambda} \Lambda \curvearrowright_{\Sigma} \Sigma$ vNE couplings

Then $(\mathcal{M} \otimes \mathcal{N})^{\Lambda}$ has a natural semi-finite trace st $\Gamma \not\cong \Sigma$ have finite-trace fund domains.

Thm: Amenability, Property (T), and the Haagerup property are vNE invariants.

Def: (Boutonnet-Isaacs '18) Γ is properly proximal if $\nexists$ a left Γ -invariant state on $(\ell^\infty \Gamma / c_0 \Gamma)^{P(\Gamma)}$.

Ex: • Non-elementary hyperbolic groups.

- Convergence groups.
- Non-amenable bi-exact groups
- Lattices in S -simple Lie groups.
- Stable under extensions
- (Horbez-Huang-Lécureux '20)
 - Mapping class groups of finite-type surfaces.
 - Many $\text{Cat}(0)$ groups.

Non-Ex: • Inner-amenable groups.

Problem: Find a non-inner amenable, and non-properly proximal group.

Ex: (Cafrace '18)

$$\begin{aligned} \text{SL}_d(\mathbb{U}) &\propto (\mathbb{U}[\frac{1}{p}])^d \xrightarrow{\text{not inner-amenable}} \text{d} \geq 3 \text{ p prime} \\ &\sim_{\text{ME}} (\text{SL}_d(\mathbb{U}) \propto \mathbb{U}^d) \times \left(\frac{\mathbb{Q}_p}{\mathbb{Z}_p} \right)^d \\ &\quad \rightarrow \text{inner amenable.} \end{aligned}$$

Thm (IPR): Proper proximality is preserved under vNE.

we use the vN-coupling $\mathcal{M}$ to induce a dual Banach space action $\wedge \sim E$ to $\Gamma \curvearrowright (\mathcal{M} \bar{\otimes} E)^\Gamma$.

Note: (Changying Ding,
Srivatsav Kunnawalkam Elayavalli, P '20?)
There is a natural notion of proper
proximality for finite von Neumann algebras
so that Γ is properly proximal iff $L\Gamma$ is.

Problems: • Find more examples of
non-inner amenable & non-properly proximal
groups.

- Are there icc Property (T) groups
which are not properly proximal?
- Are $\text{Out}(F_n)$ properly proximal?
- Are acylindrically hyperbolic groups
properly proximal?

Def (IPR): • If $M \subset \mathcal{M}$ is an
inclusion of von Neumann algebras,
then a fundamental domain is an
intermediate standard form:

$$M \subset B(L^2 M) \subset \mathcal{M}$$

This fundamental domain has finite
trace if finite-rank projections
in $B(L^2 M)$ are finite in $\mathcal{M}$.

- Two vN algebras M & N are
vNE if we have inclusions
 $M, N^{op} \subset \mathcal{M}$ with $M \subset N^{op'} \cap \mathcal{M}$
st $M \subset \mathcal{M}$ and $N^{op} \subset \mathcal{M}$
have finite-trace fundamental
domains

Thm: (IPR) • $\vee NE$ is an ER

• $\Gamma \sim_{\vee NE} \Lambda$ iff $L\Gamma \sim_{\vee NE} L\Lambda$.

• If M & N are virtually isomorphic

each is isomorphic to a finite-index subfactor in an amplification of the other.

then $M \sim_{\vee NE} N$.

Problems: • Find rigidity results for $\vee NE$ of finite factors.

• Is there a finite factor M st $M \sim_{\vee NE} N \implies M \cong N$?

• Classify separable amenable $\vee N$ algs up to $\vee NE$.

• Is every finite $\vee N$ alg $\vee NE$ to a factor?

• Find a finite $\vee N$ alg M st the index group is not $\mathbb{R}_+^*$

↓
The ratios of traces of rank-one projections in fundamental domains.