Some Results for a Class of Generalized Polynomials

by

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Abstract. A class of generalized polynomials is considered consisting of the null spaces of certain differential operators with constant coefficients. This class strictly contains ordinary polynomials and appropriately scaled trigonometric polynomials. An analog of the classical Bernstein operator is introduced and it is shown that generalized Bernstein polynomials of a continuous function converge to this function. A convergence result is also proved for degree elevation of the generalized polynomials. Moreover, the geometric nature of these functions is discussed and a connection with certain rational parametric curves is established.

1. Introduction

A number of recent papers in the area of constructive approximation theory deal with properties of trigonometric polynomials and their application in computer-aided geometric design and data fitting, see [1,3,11,12,16,20,21,23]. In this paper we consider a larger class of functions which strictly contains ordinary polynomials and certain appropriately scaled trigonometric polynomials, defined as

$$T_n := \begin{cases} \text{span } \{1, \sin(2x/n), \cos(2x/n), \sin(4x/n), \cos(4x/n), \ldots, \sin(x), \cos(x)\}, & n \text{ even}, \\ \text{span } \{\sin(x/n), \cos(x/n), \sin(3x/n), \cos(3x/n), \ldots, \sin(x), \cos(x)\}, & n \text{ odd}, \end{cases}$$

where $n$ is the degree. Similar functions have been studied in [10]. In fact, the spaces considered in this paper are just scaled versions of the spaces introduced in

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While at first glance the scaling by a factor of $1/n$ may seem artificial, it makes it possible to establish several results which do not hold for the unscaled spaces. The main reason for this is that the usual spaces of trigonometric polynomials are nested in the sense that trigonometric polynomials of degree $n$ are contained in the space of degree $n + 2$. In contrast, the spaces $\mathcal{T}_n$ satisfy $\mathcal{T}_m \subset \mathcal{T}_n$, whenever $n$ is a multiple of $m$.

The first problem we address is the convergence of a generalization of the classical Bernstein operator. The given convergence result seems to be new even in the (scaled) trigonometric case. The second problem we consider is degree elevation for the generalized polynomials. The main result is that the coefficients of a degree-elevated generalized polynomial converge to this polynomial. Note that this is not true for degree elevation [1] of the unscaled trigonometric functions. Lastly, functions in the spaces studied here are closely related with certain rational parametric curves. For the above scaled trigonometric functions this fact is established in [23] and is elaborated on in detail in [20]. Here we shall discuss this issue for the mentioned more general spaces of functions.

In the remainder of this section we introduce the spaces of interest. Throughout the paper the symbol $\mathcal{P}$ will denote a two-dimensional translation-invariant space of continuous real-valued functions. It is well known (see e.g., [9]) that every such space is necessarily the null space of a differential operator of the form

$$L := D^2 + \gamma D + \delta, \quad \gamma, \delta \in \mathbb{R}, \quad D := d/dx.$$  \hfill (1.1)

The space $\mathcal{P}$ plays the same role as linear functions in the space of ordinary polynomials.

By $d$ we denote the unique solution of the initial-value problem

$$Ld = 0, \quad d(0) = 0, \quad d'(0) = 1.$$  

Observe that

$$d(x) = \begin{cases} (e^{\mu x} - e^{\nu x})/(\mu - \nu), & \mu \neq \nu \\ xe^{\mu x}, & \mu = \nu \end{cases} \hfill (1.2)$$

where $\mu, \nu \in \mathbb{C}$ are such that $D^2 + \gamma D + \delta = (D - \mu)(D - \nu)$.

The spaces of generalized polynomials referred to above are defined as

$$\mathcal{P}_n := \text{span}\{d^n((\cdot - t)/n), t \in \mathbb{R}\}, \quad n \geq 0,$$ \hfill (1.3)

where $\text{span}\{S\}$ stands for the linear span of a set $S$. Elements of $\mathcal{P}_n$ will be called $\mathcal{P}$-polynomials of degree $n$. Note that $\mathcal{P}_0$ consists of constant functions and that $\mathcal{P}_1 = \mathcal{P}$. Unless $\gamma = \delta = 0$, the spaces $\mathcal{P}_n$ are not nested (i.e., $\mathcal{P}_n \not\subset \mathcal{P}_{n+1}$).
However, it is a simple matter to show, e.g., using (1.2), that for all positive integers 
k and n, it holds
\[ \mathcal{P}_n \subseteq \mathcal{P}_{nk}. \]  
(1.4)

In particular, \( \mathcal{P} \subseteq \mathcal{P}_n \), for all positive \( n \). Note that property (1.4) is a consequence of the scaling factor \( 1/n \) in (1.3). In the special case where \( \delta = 1, \gamma = 0 \), the space \( \mathcal{P}_n \) is the same as \( \mathcal{T}_n \), whereas for \( \delta = 0, \gamma = 0 \), the space \( \mathcal{P}_n \) is the space of ordinary polynomials of degree at most \( n \).

2. B-representation and Polar Forms

In this section we list a few useful facts about \( \mathcal{P} \)-polynomials. To avoid the uninteresting special case \( n = 0 \), from now on we will assume that \( n \geq 1 \). We already mentioned that the spaces \( \mathcal{P}_n \) are scaled versions of the spaces introduced in [10]. The proofs of the results below can be obtained along the same lines as in that paper.

Let \( a, b \in \mathbb{R} \) be such that \( a < b \) and \( d((b - a)/n) \neq 0 \) (and therefore also \( d((a - b)/n) \neq 0 \)). To construct a basis for \( \mathcal{P}_n \), we define
\[ b_{0,n}(x) := \frac{d((x-b)/n)}{d((a-b)/n)}, \quad b_{1,n}(x) := \frac{d((x-a)/n)}{d((b-a)/n)}, \quad x \in \mathbb{R}. \]  
(2.1)

For \( n = 1 \), these two functions form a basis for \( \mathcal{P} \). More generally, we have
\[ \mathcal{P}_n = \text{span} \{ B_i^n \}_{i=0}^n, \]
where
\[ B_i^n := \binom{n}{i} b_{0,n}^{-i} b_{1,n}^i, \quad i = 0, \ldots, n. \]

Thus every \( \mathcal{P} \)-polynomial of degree \( n \) can be written in the form
\[ P(x) = \sum_{i=0}^n c_i B_i^n(x), \quad x \in \mathbb{R}, \]  
(2.2)

where \( c_0, \ldots, c_n \in \mathbb{R} \). In accordance with standard terminology for ordinary polynomials, we will refer to (2.2) as the (generalized) B-representation of \( P \).

We also recall the definition of polar forms. Given a function \( P \in \mathcal{P}_n \), the polar form \( p(x_1, \ldots, x_n) \) of \( P \) is the unique function of \( n \) variables that is symmetric, diagonal i.e., \( p(x, \ldots, x) = P(x) \) for all \( x \in \mathbb{R} \), and that satisfies the recurrence relation
\[ p(\ldots, y, \ldots) = \frac{d((y - y_2)/n)}{d((y_1 - y_2)/n)} p(\ldots, y_1, \ldots) + \frac{d((y - y_1)/n)}{d((y_2 - y_1)/n)} p(\ldots, y_2, \ldots), \]  
(2.3)
for all \(y, y_1, y_2 \in \mathbb{R}\), such that \(d((y_2 - y_1)/n) \neq 0\). Note that a consequence of this recurrence is the identity

\[
P(x/n) = b_{0,n}(x)P(a/n) + b_{1,n}(x)P(b/n), \quad x \in \mathbb{R}, \ P \in \mathcal{P},
\]

whose interpretation is given in Section 6. The usefulness of polar forms stems from the fact that the coefficients \(c_0, \ldots, c_n\) in (2.2) can be expressed by evaluating the polar form \(P\) at \(t_i := (\underbrace{a, \ldots, a, b, \ldots, b}_{n-i, i})\), that is

\[
c_i = p(t_i), \quad i = 0, \ldots, n.
\]

To express \(p(x_1, \ldots, x_n)\) for an arbitrary \(n\)-tuple \((x_1, \ldots, x_n)\) in terms of the values \(c_0, \ldots, c_n\), we can utilize the recursion

\[
c_i^k := b_{0,n}(x_k)c_{i-1}^{k-1} + b_{1,n}(x_k)c_i^{k-1}, \quad i = k, \ldots, n, \quad k = 1, \ldots, n,
\]

where \(c_i^0 := c_i, i = 0, \ldots, n\). It follows from (2.3) that \(p(x_1, \ldots, x_n) = c_n^n\). Since in the special case \(x_1 = \ldots = x_n = x\) we obtain \(P(x) = p(x, \ldots, x) = c_n^n\), we see that equation (2.6) is an analog of the familiar de Casteljau algorithm.

We end this section with a technical lemma needed later.

**Lemma 2.1.** Let \(I = [a, b] \subset \mathbb{R}\) be a bounded interval. For \(P \in \mathcal{P}_n\), let \(p_n\) be the polar form of \(P\) and let \(p_{nk}\) be the polar form of \(P\) viewed as a function in \(\mathcal{P}_{nk}\). Then there exist constants \(C_0, C_1,\) and \(C_2,\) independent of \(k,\) such that

\[
\sup_{x_1, \ldots, x_{nk} \in I^{nk}} |p_{nk}(x_1, \ldots, x_{nk})| \leq C_0,
\]

\[
\sup_{x_1, \ldots, x_{nk} \in I^{nk}} \left| \frac{\partial}{\partial x_i} p_{nk}(x_1, \ldots, x_{nk}) \right| \leq \frac{C_1}{k}, \quad i = 1, \ldots, nk,
\]

\[
\sup_{x_1, \ldots, x_{nk} \in I^{nk}} \left| \frac{\partial^2}{\partial x_i \partial x_j} p_{nk}(x_1, \ldots, x_{nk}) \right| \leq \frac{C_2}{k^2}, \quad i, j = 1, \ldots, nk.
\]

**Proof:** Clearly,

\[
p_{nk}(x_1, \ldots, x_{nk}) = \frac{1}{(nk)!} \sum p_n \left( \frac{z_1}{k}, \ldots, \frac{z_n}{k} \right),
\]

where the sum is taken over all permutations \(y_1, \ldots, y_{nk}\) of \(x_1, \ldots, x_{nk}\) and where \(z_t := y_{(t-1)k+1} + \ldots + y_{tk}, t = 1, \ldots, n.\) This means that (2.7) holds with \(C_0 = \sup_{I^n} |p_n(\cdot)|\). Differentiating both sides of (2.10) partially with respect to \(x_i,\) we
arrive at (2.8), where \( C_1 = \sup_{I^n} \max_{\ell=1,\ldots,n} |p_n^\ell(\cdot)| \), and where \( p_n^\ell \) denotes the partial derivative of \( p_n \) with respect to the \( \ell \)-th variable. Here we used the fact that

\[
\left| \frac{\partial}{\partial x_i} p_n^m \left( \frac{z_1}{k}, \ldots, \frac{z_n}{k} \right) \right| = \left| \frac{1}{k} \sum_{\ell=1}^n p_n^\ell \left( \frac{z_1}{k}, \ldots, \frac{z_n}{k} \right) \frac{\partial z_\ell}{\partial x_i} \right| \leq \frac{C_1}{k} \sum_{\ell=1}^n \left| \frac{\partial z_\ell}{\partial x_i} \right| = \frac{C_1}{k}.
\]

Moreover, the constant \( C_1 \) is finite since \( p_n \) is continuously differentiable, in fact infinitely differentiable, and thus its partial derivatives are bounded on \( I^n \). Inequality (2.9) can be obtained in the same manner. In particular, in this case \( C_2 = \sup_{I^n} \max_{\ell,m=1,\ldots,n} |p_n^{\ell,m}(\cdot)| \), where \( p_n^{\ell,m} := \frac{\partial^2}{\partial x_\ell \partial x_m} p_n \).

3. Generalized Bernstein Operator

It is possible to define an analog of the classical Bernstein operator for the space \( \mathcal{P}_n \). This operator is defined for all bounded real-valued functions \( F \) on an interval \( I = [a, b] \) as

\[
B_n F := \sum_{i=0}^n F(\xi_i) B_i^n,
\]

where

\[
\xi_i := a + i \frac{b-a}{n}, \quad i = 0, \ldots, n. \tag{3.1}
\]

**Proposition 3.1.** The operator \( B_n \) reproduces functions in \( \mathcal{P} \) i.e.,

\[
B_n P \equiv P, \quad P \in \mathcal{P}.
\]

**Proof:** Using (2.5) it is sufficient to prove that for all \( P \in \mathcal{P} \),

\[
p(t_i) = P(\xi_i), \quad i = 0, \ldots, n.
\]

The polar form of \( P \) is easily obtained as (cf. [10])

\[
p(x_1, \ldots, x_n) = P \left( \frac{\sum_{j=1}^n x_j}{n} \right).
\]

Hence the assertion follows from the definition of \( \xi_i \). \( \blacksquare \)

A very similar result to Proposition 3.1 has been obtained in [10], where a slightly different space from \( \mathcal{P}_n \) has been used. The proof of the above proposition is virtually the same as the one in [10, Corollary 4.4], and we have included it here for convenience of the reader. We mention that for trigonometric polynomials this result seems to have been first established in [11] (see also [1,15]).
Unlike other generalizations of the Bernstein operator [1,11,15], the operator $B_n$ has the important property that every continuous function $F$ can be approximated arbitrarily well by the $\mathcal{P}$-polynomials $B_nF$. The discussion of this fact will fill the remainder of this section.

In order to arrive at a convergence result, it will be necessary to restrict the length of the interval $I$. From now on we will assume that

$$0 < b - a < \begin{cases} \infty, & \text{if } \gamma^2 - 4\delta \geq 0, \\ 2\pi/\sqrt{4\delta - \gamma^2}, & \text{otherwise}. \end{cases}$$

(3.2)

With this restriction, $d((b - a)/n) \neq 0$, $n \geq 1$, and hence the functions in (2.1) are well defined and so is the operator $B_n$. Moreover, by using Wronskian determinants (see e.g., [14]) it can be shown that the three functions $\{1, d, d'\}$, which span the null space of the differential operator $D(D^2 + \gamma D + \delta)$, form a Chebyshev system, in fact a complete Chebyshev system.

We first address the convergence of Bernstein polynomials $B_nF$, where $F$ is a constant function.

**Lemma 3.2.** There exists a constant $C \geq 0$, depending on $a, b, \delta, \gamma$, but not on $x$ and $n$, such that for every $x \in I$ and every $n \in \mathbb{N}$,

$$B_n1(x) - 1 = \delta \frac{(x - a)(b - x)}{2} n^{-1} + R(x, n)n^{-2},$$

(3.3)

where $|R(x, n)| \leq C$.

**Proof:** It will be sufficient to prove that $|R(x, n)| \leq C$ holds for all sufficiently large values of $n$. Consider the function

$$g(u, v) := \begin{cases} \frac{d(v(u-b))}{d(v(\alpha-b))} + \frac{d(v(u-a))}{d(v(b-a))}, & v \neq 0, \\ 1, & \text{otherwise}, \end{cases}$$

which can be viewed as a complex-valued function in two variables $u, v \in \mathbb{C}$. It follows from (1.2) that the function $d$ is entire and that it has a simple zero at the origin. This means $v = 0$ is a removable singularity of the rational function appearing in the definition of $g$. These facts, together with $\lim_{v \to 0} g(u, v) = 1, u \in \mathbb{C}$, which can be verified using (1.2), imply that for every fixed $u$ the function $g(u, v)$ is analytic in a neighborhood of the origin. Similarly, $g(u, v)$ is entire in variable $u$ for any fixed value of $v$. This means that $g$ is separately analytic and hence by a well-known theorem of Hartogs [13], the function $g$ is (jointly) analytic on $\mathbb{C} \times B_{\varepsilon}$, where $B_{\varepsilon} := \{v : |v| \leq \varepsilon\}$, and where $\varepsilon > 0$ is sufficiently small.

Fix $x \in I$ and consider the Taylor expansion of $g$ in the form

$$g(x, y) = g(x, 0) + gy(x, 0)y + g_{yy}(x, 0)y^2/2 + g_{yyy}(x, \eta(y))y^3/6, \quad g_y := \partial g/\partial y,$$

(3.4)
where \( y \in [-\varepsilon, \varepsilon] \) and \( \eta(y) \in [-\varepsilon, \varepsilon] \). Employing the explicit formulae (1.2) again, (3.4) can be rewritten (with a help of Mathematica) as

\[
g(x, y) = 1 + \delta(x - a)(b - x)y^2/2 + g_{yyy}(x, \eta(y))y^3/6. \tag{3.5}
\]

Since \( g \) is analytic, it follows that \( g_{yyy} \) is continuous and hence bounded on \( I \times [-\varepsilon, \varepsilon] \). The identity (3.3) now follows from (3.5), the fact that \( B_n 1(x) = g^n(x, n^{-1}) \), and the elementary identity

\[
(1 + \delta \frac{(x - a)(b - x)}{2} n^{-2} + r(x, n)n^{-3})^n = 1 + \delta \frac{(x - a)(b - x)}{2} n^{-1} + R(x, n)n^{-2},
\]

for some \( R(x, n) \), with \( r(x, n) := g_{yyy}(x, \eta(n^{-1}))/6 \). Here, for all \( n \geq \varepsilon^{-1} \), the function \( R(x, n) \) is bounded independently of \( x \) and \( n \), which follows from the boundedness of \( r(x, n) \). \( \Box \)

**Theorem 3.3.** Let \( I \) be an interval satisfying (3.2). Then for every continuous function \( F \), the sequence of functions \( \{B_n F\} \) converges uniformly to \( F \) on \( I \).

**Proof:** Since \( \{1, d, d'\} \) is a Chebyshev system, by the Korovkin Theorem [17, Theorem 8] we only need to verify the validity of the assertion for the three functions \( 1, d, \) and \( d' \). For \( F = d \) and \( F = d' \), this is an immediate consequence of Proposition 3.1, since both \( d \) and \( d' \) belong to \( \mathcal{P} \). The uniform convergence for \( F = 1 \) follows from Lemma 3.2. \( \Box \)

An interesting question is how fast is the convergence \( B_n F \to F \) in the case where \( F \) is sufficiently smooth. We mention in passing that it is possible to prove that

\[
\lim_{n \to \infty} n(B_n F(x) - F(x)) = \frac{(x - a)(b - x)}{2} LF(x), \quad x \in I,
\]

for every twice differentiable function defined on the interval \( I \), where \( L \) is defined in (1.1). This is an analog of the classical Voronovskaya Theorem [5,18].

4. Degree Elevation

In this section we restrict our analysis to the case \( I = [a, b] = [0, h], \) where \( h = b - a \) satisfies (3.2), which is without loss of generality since \( \mathcal{P}_n \) is translation invariant. Due to the nestedness of the spaces \( \mathcal{P}_n \) mentioned earlier, it is natural to consider the following degree elevation problem. Consider a function \( P \in \mathcal{P}_n \) given by

\[
P = \sum_{i=0}^{n} c_i B_i^n,
\]
and let $k$ be a positive integer. Since $\mathcal{P}_n \subset \mathcal{P}_{nk}$, the function $P$ can also be written as
\[
P = \sum_{i=0}^{nk} c_i^{nk} B_i^{nk},
\]
where the coefficients $c_i^{nk}$ can be given as
\[
c_i^{nk} = p_{nk}(0, \ldots, 0, h, \ldots, h)
\left(\begin{array}{c}
k \\
i
\end{array} \right)^{-1} \sum_{i_1 + \ldots + i_n = i} \left(\begin{array}{c}
k \\
i_1
\end{array} \right) \ldots \left(\begin{array}{c}
k \\
i_n
\end{array} \right) p_n \left(\begin{array}{c}
i_1 h \\
k
\end{array} \right) \ldots \left(\begin{array}{c}
i_n h \\
k
\end{array} \right),
\]
where $p_n$ and $p_{nk}$ are defined as in Lemma 2.1. This formula follows directly from (2.10). Moreover, the values of $p_n$ appearing on the right-hand side of (4.2) can be obtained as
\[
p_n \left(\begin{array}{c}
i_1 h \\
k
\end{array} \right) \ldots \left(\begin{array}{c}
i_n h \\
k
\end{array} \right) = \sum_{i=0}^{n} c_i B_i^n \left(\begin{array}{c}
i_1 h \\
k
\end{array} \right) \ldots \left(\begin{array}{c}
i_n h \\
k
\end{array} \right),
\]
where $b_i^n$ is the polar form of $B_i^n$, satisfying $b_i^n(t_j) = \delta_{ij}, i, j = 0, \ldots, n$. We note that (4.3) can be computed by a de Casteljau algorithm [10]. Moreover, an algorithm for computing the coefficients $c_i^{nk}$ is given in [3] for the scaled trigonometric case, along with various related results.

While (4.2) and (4.3) allow us to compute the coefficients $c_i^{nk}$, it is not clear from these formulae whether the newly obtained coefficients converge to $P$, as is the case for the classical degree elevation of polynomials (see e.g., [4,7,22]). We next show that the coefficients $c_i^{nk}$ converge to values of the function $P$ as we increase $k$.

**Theorem 4.1.** Let $P$ and $c_i^{nk}$ be defined as above. Then
\[
\lim_{k \to \infty} \max_{i=0, \ldots, nk} |P(ih/nk) - c_i^{nk}| = 0.
\]
To prove this theorem, we need a few auxiliary results.

**Proposition 4.2.** Let $n$ be a fixed positive integer and let $\{d_i^k\}, i = 0, \ldots, nk, k = 1, 2, \ldots$, be a bounded array of real numbers such that
(a) $\sum_{i=0}^{nk} d_i^k B_i^{nk}(x) \to 0$, as $k \to \infty$, for every $x \in I$,
(b) $|d_i^k - d_{i-1}^k| \leq C/k, i = 1, \ldots, nk$, for a constant $C$ independent of $k$.
Then $\max_{i=0, \ldots, nk} |d_i^k| \to 0$, as $k \to \infty$.

**Proof:** Consider the sequence of functions $\{F_k\}$, where $F_k$ is the piecewise linear interpolant of the values $d_i^k$ at the nodes $ih/nk, i = 0, \ldots, nk$. Using the assumption (b), it is not difficult to see that
\[
|F_k(x) - F_k(y)| \leq Cn|x - y|, \quad x, y \in I,
\]
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where $C$ is the same as in the proposition. This and the fact that the array of coefficients $d_i^k$ is bounded implies that the sequence $\{F_k\}$ is an equicontinuous family of functions. We show that this sequence converges uniformly to the zero function. If this is not the case, then there exists an $\varepsilon > 0$ and a subsequence $\{F_{k_l}\}$ of $\{F_k\}$ such that $\|F_{k_l}\| \geq \varepsilon$, for all $l$. By the Ascoli-Arzelà Theorem [24], this sequence contains a subsequence $\{F_{k_{l_m}}\}$ converging to a continuous function $F$, say. Clearly, $\|F\| \geq \varepsilon$ and hence $F \neq 0$. By Theorem 3.3, $\sum_{i=0}^{n_k} B_i^{n_k}$ converges uniformly to unity as $k \to \infty$, and hence it is uniformly bounded with respect to $k$. Moreover, the convergence $F_{k_{l_m}} \to F$ is uniform, and hence it is not difficult to show that

$$
\sum_{i=0}^{n_k} d_i^{k_{l_m}} B_i^{n_k} = \sum_{i=0}^{n_k} F_{k_{l_m}}(ih/n_k) B_i^{n_k} \to \sum_{i=0}^{n_k} F(ih/n_k) B_i^{n_k},
$$

as $m \to \infty$, where the convergence is uniform. However, by Theorem 3.3 the sum on the right converges uniformly to $F$, whereas the sum on the left converges pointwise to zero by assumption (a). Thus we see that necessarily $F \equiv 0$ which is a contradiction. We conclude that $F_k \to 0$ uniformly and hence $\max_{i=0,\ldots,n_k} |d_i^k| \to 0$, as $k \to \infty$.

**Proposition 4.3.** Let $P$ and $c_i^{nk}$ be defined as in (4.1). Then $|c_i^{nk} - c_i^{nk-1}| \leq C/k$, where $C$ is a constant independent of $k$.

**Proof:** Using the Taylor expansion of the polar form $p = p_{nk}$ of $P$, we have

$$p(x_1 + h_1, \ldots, x_n + h_n) - p(x_1, \ldots, x_n) = \sum_{j=1}^{n_k} \frac{\partial p(x_1, \ldots, x_n)}{\partial x_j} h_j + \frac{1}{2} \sum_{j=1}^{n_k} \frac{\partial^2 p(x_1, \ldots, x_n)}{\partial x_j \partial x_{n_k}} h_j h_{n_k},$$

where $\zeta_j = x_j + th_j, j = 1, \ldots, nk$, for some $t \in (0, 1)$. Specializing this identity to the case $x_j = 0, j = 1, \ldots, nk - i + 1, x_j = h, j = nk - i + 2, \ldots, nk$, and $h_j = \delta_{nk-i+1,j} h, j = 1, \ldots, nk$; and employing (2.5), we obtain

$$c_i^{nk} - c_i^{nk-1} = \frac{\partial p(0, \ldots, 0, h, \ldots, h)}{\partial x_{nk-i+1}} h + \frac{1}{2} \frac{\partial^2 p(\zeta_1, \ldots, \zeta_{nk})}{\partial x^2_{nk-i+1}} h^2.$$

The estimate $|c_i^{nk} - c_i^{nk-1}| \leq C/k$ now follows from Lemma 2.1.
Proof of Theorem 4.1: Observe first that it follows from Theorem 3.3 and (4.1) that

$$\sum_{i=0}^{nk} d_i^k B_i^{nk} := \sum_{i=0}^{nk} (P(ih/nk) - c_i^{nk}) B_i^{nk} \to 0,$$

uniformly as $k \to \infty$. By Proposition 4.3 and the fact that since $P$ is differentiable it is also Lipschitz continuous, we thus have $\max_{i=0, \ldots, nk} |d_i^k - d_{i-1}^k| \leq C/k$, for some $C$ independent of $k$. Now the assertion follows from Proposition 4.2.

5. Control Curves

Functions in the spaces $\mathcal{P}_n$ have a number of surprising geometric properties. To describe these properties we first introduce some terminology. Let $P = \sum_{i=0}^{n} c_i B_i^n$ be a $\mathcal{P}$-polynomial and let $\{r, s\}$ be a fixed basis of $\mathcal{P}$. The function $P$ gives rise to the curve $\mathbf{P} := \{P(x)(r(x), s(x)), x \in I\}$, hereafter called the $\mathcal{P}$-curve of $P$. The points $C_i := c_i(r(\xi_i), s(\xi_i)), i = 0, \ldots, n$, will be termed the control points of $\mathbf{P}$. Here the values $\xi_i$ are defined as in (3.1). The points $C_i$ are analogs of the classical control points [8] since by Proposition 3.1, if all $C_i$ lie on a $\mathcal{P}$-curve $\mathbf{P}$ associated with a polynomial $P \in \mathcal{P}$, then the $\mathcal{P}$-curve corresponding to these control points is the curve $\mathbf{P}$ itself. To define an analog of the familiar notion of a control polygon for $\mathcal{P}$-curves, let $q_i \in \mathcal{P}, i = 1, \ldots, n$, be the unique functions interpolating the values $c_{i-1}, c_i$ at $\xi_{i-1}, \xi_i$, respectively, that is

$$q_i(x) := \frac{d(x - \xi_i)}{d(\xi_{i-1} - \xi_i)} c_{i-1} + \frac{d(x - \xi_{i-1})}{d(\xi_i - \xi_{i-1})} c_i, \quad x \in [\xi_{i-1}, \xi_i]. \tag{5.1}$$

The curve $\mathbf{Q}$ consisting of the pieces $\{q_i(x)(r(x), s(x)), x \in [\xi_{i-1}, \xi_i]\}, i = 1, \ldots, n$, will be called the control curve of $\mathbf{P}$ (see Fig. 1).

Next, let $\mathbf{C} := \{(r(x), s(x)), x \in I\}$. Since under the restriction (3.2) $\mathcal{P}$ is a Chebyshev space, the curve $\mathbf{C}$ does not pass through the origin. Moreover, every line passing through the origin intersects $\mathbf{C}$ at most once. Hence every point $v \in \mathcal{V} := \{\alpha(r(a), s(a)) + \beta(r(b), s(b)), \alpha, \beta \geq 0\} \setminus \{(0, 0)\}$ can be expressed uniquely as $v = c(v)(r(x(v)), s(x(v)))$, for some $x(v) \in I$ and $c(v) > 0$. Let $T : \mathcal{V} \to \mathcal{V}$ be the mapping defined by

$$Tv := \frac{(r(x(v)), s(x(v)))}{c(v)} = \frac{v}{c^2(v)}.$$

Clearly, the points on $\mathbf{C}$ are invariant under $T$. In the special case where $r(x) = \cos(x), s(x) = \sin(x)$, the mapping $T$ is just the inversion with respect to the unit circle. It is well known that this inversion maps circles passing through the origin onto straight lines. The next lemma asserts that the mapping $T$ has a similar property, and hence it can be viewed as a generalized inversion.
Lemma 5.1. Let \( \{r, s\} \) be a basis for \( \mathcal{P} \) and suppose that \( t \in \mathcal{P} \) does not vanish on \( I = [a, b] \). Then the set of points \( \{P(x) := (r(x), s(x))/t(x), x \in I\} \) is a line segment connecting the points \( P(a) \) and \( P(b) \).

Proof: It will be sufficient to prove that \( P(x) \) lies on the line segment connecting \( P(a) \) and \( P(b) \). This is the case if and only if the area of the triangle with vertices \( P(a), P(b), \) and \( P(x) \) is zero, or if

\[
\det \begin{bmatrix} r(x)/t(x) & r(a)/t(a) & r(b)/t(b) \\ s(x)/t(x) & s(a)/t(a) & s(b)/t(b) \\ 1 & 1 & 1 \end{bmatrix} = 0.
\]

This determinant equals

\[
\frac{1}{t(x)t(a)t(b)} \det \begin{bmatrix} r(x) & r(a) & r(b) \\ s(x) & s(a) & s(b) \\ t(x) & t(a) & t(b) \end{bmatrix},
\]

which is zero because \( \mathcal{P} \) is two-dimensional. \( \blacksquare \)

It will be useful to consider the image of a \( \mathcal{P} \)-curve \( P \) under the mapping \( T \), called an inverse \( \mathcal{P} \)-curve. That is, assuming that \( P \in \mathcal{P}_n \) is positive on \( I \), the inverse \( \mathcal{P} \)-curve of \( P \) is the set \( TP = \{(r(x), s(x))/P(x), x \in I\} \). In general, the control curve \( Q \) of \( P \) is not a curve consisting of segments of straight lines. However, it turns out that the image of \( Q \) under \( T \) is polygonal. In order to make sure that \( TQ \) is well defined, the functions \( q_i \) must be positive in the corresponding intervals \([\xi_{i-1}, \xi_i] \). Hence from now on we shall assume that the coefficients \( c_0, \ldots, c_n \) are positive. Note that this also guarantees the positivity of \( P \) on \( I \).

Proposition 5.2. Let \( P \in \mathcal{P}_n \) and let \( Q \) be the control curve of the associated \( \mathcal{P} \)-curve \( P \). Then the curve \( TQ \) consists of segments of straight lines connecting the pairs of points \( TC_{i-1}, TC_i, i = 1, \ldots, n \).

Proof: Since \( Q \) is determined by the functions \( q_i \in \mathcal{P} \), by Lemma 5.1 the curve \( TQ \) is assembled from the line segments \( \{(r(x), s(x))/q_i(x), x \in [\xi_{i-1}, \xi_i]\} \), connecting the points \( TC_{i-1} = (r(\xi_{i-1}), s(\xi_{i-1}))/c_{i-1} \) and \( TC_i = (r(\xi_i), s(\xi_i))/c_i, i = 1, \ldots, n \). \( \blacksquare \)

Example. Fig. 1 shows an example of a \( \mathcal{P} \)-curve together with the associated control and inverse curves, corresponding to \( n = 3 \), \( d(x) = e^{2x} - e^x, I = [0, 1] \), and the coefficients \( c_0 = 0.9, c_1 = 0.6, c_2 = 0.5, c_3 = 0.7 \). The reference curve \( C \) is determined by \( r(x) = b_{0,1}(x) \) and \( s(x) = b_{1,1}(x) \), see (2.1). Fig. 2 illustrates convergence of degree elevation, addressed in Section 4. The figure displays a
sequence of three inverse control polygons (corresponding to the values $k = 2, 4, 8$), converging to the inverse $\mathcal{P}$-curve of a curve of degree $n = 2$, defined on $I = [0, 1]$, with coefficients $c_0 = 0.4, c_1 = 0.4, c_2 = 1$. Here the space $\mathcal{P}$ is the same as in Fig. 1, that is $\mathcal{P} = \text{span}\{e^x, e^{2x}\}$.

To compute the points of the inverse $\mathcal{P}$-curve $TP$, associated with a function $P \in \mathcal{P}_n$, we can proceed as follows. Given the coefficients $c_0, \ldots, c_n$ of $P$ and a value $x \in I$, we can employ the de Casteljau scheme (2.6) (with $x_1 = \ldots = x_n = x$) to obtain $P(x) = c_n^n$. The corresponding point of the inverse $\mathcal{P}$-curve is therefore given by $TP(x) = TC_n^n := T(c_n^n(r(x), s(x))) = (r(x), s(x))/c_n^n$. It will be instructive to discuss the geometric nature of this algorithm. Consider the functions $q_t^k \in \mathcal{P}$
defined as
\[ q_i^k(x) := \frac{d(x - \xi_i^{k-1})}{d(\xi_{i-1}^{k-1} - \xi_i^{k-1})} \xi_i^{k-1} + \frac{d(x - \xi_i^{k-1})}{d(\xi_{i-1}^{k-1} - \xi_i^{k-1})} \xi_i^{k-1}, \quad x \in [\xi_{i-1}^{k-1}, \xi_i^{k-1}], \]
where
\[ \xi_i^k := \frac{kx + (n - i)a + (i - k)b}{n}, \quad i = k, \ldots, n, \quad k = 0, \ldots, n, \quad (5.2) \]
and where the \( c_i^k \) are defined by (2.6). It is not difficult to see from (2.6) that \( q_i^k(\xi_i^k) = c_i^k \). Hence by Lemma 5.1, the point \( TC_i^k \) lies on the line segment between the points \( TC_{i-1}^{k-1} \) and \( TC_i^{k-1} \), where \( C_i^k := c_i^k(r(\xi_i^k), s(\xi_i^k)) \). Thus we proved the so-called convex hull property of the inverse \( \mathcal{P} \)-curves, a well-known property of the classical Bézier curves [8].

**Proposition 5.3.** The inverse of a \( \mathcal{P} \)-curve with control points \( C_0, \ldots, C_n \) satisfies the convex hull property i.e., it is contained in the convex hull of the points \( TC_0, \ldots, TC_n \).

The above property may be useful in the design of \( \mathcal{P} \)-curves since it suggests constructing the inverse \( \mathcal{P} \)-curves first, with a geometrically more intuitive control structure (see [2,19] for some examples).

6. Connection with Rational Curves

The results of the previous section are not surprising in view of the following remarkable property of inverse \( \mathcal{P} \)-curves. By Proposition 3.1, the inverse of a \( \mathcal{P} \)-curve corresponding to a polynomial \( P \in \mathcal{P}_n \) can be expressed in the form

\[ TP(x) = \frac{(r(x), s(x))}{P(x)} = \frac{\sum_{i=0}^{n} (r(\xi_i), s(\xi_i)) B_i^n(x)}{\sum_{i=0}^{n} c_i B_i^n(x)} = \sum_{i=0}^{n} c_i TC_i B_i^n(x), \quad x \in I. \]

(6.1)

As explained below, this means that inverse \( \mathcal{P} \)-curves are rational parametric curves.

First observe that for every basis polynomial \( B_i^n \in \mathcal{P}_n \) there exists a unique bivariate homogeneous polynomial \( H_i^n \) of degree \( n \) such that

\[ B_i^n(x) = H_i^n(r(x/n), s(x/n)), \quad x \in I. \]

(6.2)

This is a consequence of the fact that since \( r \) and \( s \) are linearly independent, every function in \( \mathcal{P}_n \) can be written in a unique way as a linear combination of the products \( r^{n-i}(x/n)s^i(x/n), i = 0, \ldots, n \). The functions \( H_i^n \) are commonly referred to as homogeneous Bernstein polynomials, see e.g., [20]. Defining the curve \( C_n := \{(r(x/n), s(x/n)), x \in I\}, \) (6.2) implies that every function \( P \in \mathcal{P}_n \) can be viewed as the restriction of a bivariate homogeneous polynomial to \( C_n \). This is an extension of the well-known fact that trigonometric functions are restrictions of bivariate
homogeneous polynomials to the circle. We also note that the curve $C_n$ allows the following interpretation of the functions $b_{0,n}$ and $b_{1,n}$ (see (2.1)). Consider the points $v_0 = (r(a/n), s(a/n))$, $v_1 = (r(b/n), s(b/n))$, and $v = (r(x/n), s(x/n)), x \in I$, which lie on $C_n$. It follows from (2.4) that

$$v = b_{0,n}(x)v_0 + b_{1,n}(x)v_1.$$  

This means that $b_{0,n}$ and $b_{1,n}$ can be viewed as generalized barycentric coordinates of $v$ with respect to $v_0$ and $v_1$.

On account of (6.2) and the homogeneity of $H_i^n$, the curve (6.1) can also be written in the form

$$\sum_{i=0}^n c_i TC_i H_i^n(\xi, \eta),$$  

(6.3)

where $(\xi, \eta)$ varies over all points from the wedge $U := \{\alpha v_0 + \beta v_1, \alpha, \beta \geq 0\}\setminus \{(0,0)\}$. Using homogeneity again, the same curve can be expressed in the form (6.3) but where now $(\xi, \eta) \in U$ such that $\xi + \eta = 1$. However, it is well known that the restriction of $H_i^n$ to the line $\xi + \eta = 1$ is an ordinary Bernstein polynomial of degree $n$, which means that the curve (6.1) is indeed rational. Another consequence of (6.3) is that the curve $TQ$ coincides with the usual control polygon of the rational curve $TP$ [8], a fact that is not obvious from the definitions of $P$ and $Q$. We summarize our observations in

**Proposition 6.1.** Let $P \in \mathcal{P}_n$ be a $\mathcal{P}$-polynomial with coefficients $c_i, i = 0, \ldots, n$. Then the inverse $\mathcal{P}$-curve $TP$, associated with $P$ and with a basis $\{r, s\}$ of $\mathcal{P}$, is a rational parametric curve with control points $TC_i$ and weights $c_i, i = 0, \ldots, n$.

The properties of $\mathcal{P}$-curves described above generalize similar results obtained in [20] for the trigonometric case. In this special case the resulting rational curves, called focal Bézier curves, have been studied in [6,20,23]. An explicit representation of focal curves in terms of ordinary Bernstein basis polynomials has been given in [2]. Moreover, it has been shown in [20] that the rational curves that are dual to focal Bézier curves have the striking property that their offsets are also rational. At present it is not clear whether a similar property carries over to the more general situation discussed in this paper.

Proposition 6.1 suggests an alternative to the method described in Section 5 for computing inverse $\mathcal{P}$-curves. Namely, since $TP$ is rational, we can apply the standard de Casteljau algorithm to evaluate the points of $TP$. A natural question is then whether the two methods lead to the same auxiliary control points $TC_i^k = T(c_i^k(r(\xi_i^k), s(\xi_i^k))) = (r(\xi_i^k), s(\xi_i^k))/c_i^k, i = k, \ldots, n, k = 1, \ldots, n$. This question is of importance if one wants to utilize de Casteljau algorithm to subdivide $TP$ (or the curve $P$) since the auxiliary points are known to give rise to the control points for the subdivided curves [8].
To settle the above question, consider the rational curve of the form

\[ \mathbf{R}(t) := \frac{\sum_{i=0}^{n} w_i D_i A_i^n(t)}{\sum_{i=0}^{n} w_i A_i^n(t)}, \quad (6.4) \]

with control points \( D_0, \ldots, D_n \in \mathbb{R}^2 \) and weights \( w_0, \ldots, w_n \in \mathbb{R} \), where \( A_i^n \) denote the ordinary Bernstein basis polynomials on interval \([0,1]\). For any \( t \in [0,1] \), the corresponding point \( \mathbf{R}(t) \) on the curve can be obtained by de Casteljau algorithm [8]:

\[ D_i^k := (1 - t) \frac{w_i^{k-1}}{w_i^k} D_i^{k-1} + t \frac{w_i^{k-1}}{w_i^k} D_i^{k-1}, \quad i = k, \ldots, n, \quad k = 1, \ldots, n, \]

where the weights are given as

\[ w_i^k := (1 - t) w_i^{k-1} + t w_i^{k-1}, \quad (6.5) \]

with \( w_i^0 := w_i, i = 0, \ldots, n \). To formulate our next result, we set \( D_i := TC_i \), \( w_i := c_i, i = 0, \ldots, n \), and \( t := b_{1,n}(x)/(b_{0,n}(x) + b_{1,n}(x)) \) in (6.4). Evidently, \( \mathbf{R}(t) = TP(x) \), hence (6.1) and (6.4) are different representations of the same parametric curve. We next show that the de Casteljau algorithm commutes with the generalized inversion, a fact that can also be derived, at least in the case of trigonometric \( \mathcal{P} \)-curves, from some observations in [23].

**Proposition 6.2.** The de Casteljau algorithm commutes with the inversion \( T \) in the sense that

\[ D_i^k = T C_i^k, \quad i = k, \ldots, n, \quad k = 1, \ldots, n. \]

**Proof:** Clearly, (2.6) combined with (6.5) implies \( w_i^k = (b_{0,n}(x) + b_{1,n}(x))^k c_i^k \), \( i = k, \ldots, n, k = 1, \ldots, n \). The proof of the proposition will be done by induction on \( k \). The assertion is trivial for \( k = 0 \). As for the induction step, we obtain

\[ D_i^k = (1 - t) \frac{w_i^{k-1}}{w_i^k} D_i^{k-1} + t \frac{w_i^{k-1}}{w_i^k} D_i^{k-1} \]

\[ = b_{0,n}(x) \frac{c_i^{k-1}}{c_i^k} T C_i^{k-1} + b_{1,n}(x) \frac{c_i^{k-1}}{c_i^k} T C_i^{k-1} \]

\[ = b_{0,n}(x)(r(\xi_i^{k-1}(x)), s(\xi_i^{k-1}(x))) + b_{1,n}(x)(r(\xi_i^{k-1}(x)), s(\xi_i^{k-1}(x))) \]

\[ = \frac{(r(\xi_i^k(x)), s(\xi_i^k(x)))}{c_i^k} = T C_i^k. \]

The second to last equality follows from

\[ b_{0,n}(x) p(\xi_i^{k-1}(x)) + b_{1,n}(x) p(\xi_i^{k-1}(x)) = p(\xi_i^k(x)), \quad (6.6) \]
which holds for every \( p \in \mathcal{P} \). To see this, assume first that \( \mu \neq \nu \) (cf. (1.2)). Since \( \mathcal{P} \) is two-dimensional, it will be sufficient to prove (6.6) for \( p(x) = e^{\kappa x} \), where \( \kappa \in \{\mu, \nu\} \). It follows from (5.2) and the definition of \( b_{0,n}(x) \), \( b_{1,n}(x) \) that both sides of (6.6) belong to the same two-dimensional space spanned by \( e^{(\mu + \kappa(k - 1))x/n} \) and \( e^{(\nu + \kappa(k - 1))x/n} \). Thus identity (6.6) will be established once we have shown that it holds for at least two values of \( x \), namely the values \( x \in \{a, b\} \). However, for these values, the validity of (6.6) is a consequence of

\[
b_{0,n}(a) = 1, \quad b_{0,n}(b) = 0, \quad b_{1,n}(a) = 0, \quad b_{1,n}(b) = 1,
\]

and

\[
\xi_{i-1}^{k-1}(a) = \frac{(n - i + k)a + (i - k)b}{n} = \xi_i^k(a), \quad \xi_{i-1}^k(b) = \frac{(n - i)a + ib}{n} = \xi_i^k(b).
\]

The proof of (6.6) for the case \( \mu = \nu \) is analogous.

Given the result of the last proposition, it comes as a surprise that inversion need not commute with degree elevation, as the next example shows.

**Example.** Let \([a, b] = [0, \pi/2]\) and \( \mathcal{P} = \text{span}\{\sin, \cos\} \). Consider the function \( P(x) = c_0 B_0^1(x) + c_1 B_1^1(x) = c_0 \cos x + c_1 \sin x \), \( c_0, c_1 \in \mathbb{R} \). Raising the degree of \( P \) gives \( P(x) = c_0 B_0^2(x) + \frac{\sqrt{2}}{2}(c_0 + c_1) B_1^2(x) + c_1 B_2^2(x) \), where \( B_0^2(x) = 2 \sin^2(\frac{x}{2}) \), \( B_1^2(x) = 4 \sin(\frac{x}{2}) \sin(\frac{x}{2}), B_2^2(x) = 2 \sin^2(\frac{x}{2}) \). Thus the corresponding inverse \( \mathcal{P} \)-curve \( TP \) is a rational curve with weights \( w_0 = c_0, w_1 = \frac{\sqrt{2}}{2}(c_0 + c_1) \), \( w_2 = c_1 \) and control points \( TC_0 = (1, 0)/w_0, TC_1 = (\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2})/w_1, TC_2 = (0, 1)/w_2 \). On the other hand, degree elevating the rational curve (see [8])

\[
\frac{c_0 \left( \frac{1}{c_0} A_0^1(t) + c_1 \frac{0.1}{c_1} A_1^1(t) \right)}{c_0 A_0^2(t) + c_1 A_1^2(t)}
\]

gives a rational curve with weights \( w_0' = c_0, w_1' = (c_0 + c_1)/2, w_2' = c_1 \) and control points \( D_0 = (1, 0)/w_0', D_1 = \frac{1.0 + (0.1)}{2}/w_1', D_2 = (0, 1)/w_2' \). Thus, degree elevation does not commute with inversion. Of course, the two quadratic rational curves, while having a different analytic form, represent the same curve in the plane. For a more detailed discussion of degree elevation in the trigonometric case, we refer the reader to [19].

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References


