On Degenerate Surface Patches

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Abstract
A local construction of a $GC^1$ interpolating surface to given scattered data in $R^3$ can give rise to degenerate Bernstein–Bézier patches. That means the parametrization at vertices is not regular in the sense that the length of the tangent vector to any curve passing through a vertex is zero at that vertex. This implies that the curvature of these curves tends to infinity whenever one approaches a vertex. This fact seems to have not a negative influence on the shape of the surface.

1 Interpolation problem

We have considered the problem of interpolating scattered data points in $R^3$ by a geometrically smooth ($GC^1$) surface. In order to be able to handle "complicated" surfaces with many data points, methods designed for that purpose are usually local in nature. For a classification of different approaches see [4]. Various attempts to attack the interpolation problem using degenerate polynomial patches are described in [3,5,1] and also in an article in these proceedings [2].

Now we will state the problem more precisely. Let us call the given set of scattered data points $V = \{V_i | V_i \in R^3\}_{i=1}^v$ and the given set of normal directions $N = \{(kN_i | k > 0) | N_i \in R^3/\{0\}\}_{i=1}^v$. We also assume given a triangulation $T$ of $V$, that is a set of triangles with vertices belonging to $V$: $T = \{\Delta_j | \Delta_j = (V_{j1}, V_{j2}, V_{j3}); \quad V_{j_k} \in V\}_{j=1}^t$. The data points $V_i$ are the $v$ vertices of the triangulation. The direction of the normal to the surface at the vertex $V_i$ is given by the vector $N_i$ of which the length is irrelevant. With the triangulation $T$ the topology of the surface interpolating the vertices of $V$ is to a large extent determined, in the sense that with $T$ a piecewise linear $C^0$ surface $s^0$ is given. Our aim is to modify this piecewise linear surface such that we obtain a $GC^1$ surface $s^1$.

Problem:
Find for given $V, N$ and $T$ a geometrically smooth surface $s^1$ interpolating $V$ and $N$.

We will represent the surfaces $s^0$ and $s^1$ by a collection of maps of the standard 2-simplex $S$ with vertices $(X_1, X_2, X_3)$ into patches with vertices $(V_{j_1}, V_{j_2}, V_{j_3})$ in $R^3$. Choose for in-
stance \( X_1 = (1, 0, 0), X_2 = (0, 1, 0), X_3 = (0, 0, 1) \). In order to be able to work with such a representation a triangulation \( T \) of \( V \) has to be given.

Representation of the interpolant \( s^0 \):

For each \( j \) (in each triangle \( \Delta_j \)) we consider a map \( L_j \) of the standard simplex in \( R^2 \) with vertices \( X_1, X_2, X_3 \) into \( R^3 \) given by:

\[
L_j(X_\lambda) = p_j^0(\lambda) = V_{j_1} \lambda_1 + V_{j_2} \lambda_2 + V_{j_3} \lambda_3.
\]

Here \( \lambda = (\lambda_1, \lambda_2, \lambda_3) \) are the barycentric coordinates of a point \( X_\lambda \) relative to the standard simplex \( S \):

\[
X_\lambda = \lambda_1 X_1 + \lambda_2 X_2 + \lambda_3 X_3, \quad \sum_{i=1}^3 \lambda_i = 1; \quad 0 \leq \lambda_i \leq 1.
\]

Thus we describe the surface \( s^0 \) by a collection of maps \( \{L_j\}_{j=1}^t \).

We will now similarly represent the surface \( s^1 \) by a collection of suitably defined maps \( \{P_j\} \).

These maps have to satisfy the following conditions:

1. \( P_j(X_{jk}) = V_{jk}, \quad j = 1, \ldots, t, \quad k = 1, 2, 3. \)

2. \( D_{\gamma_1} P_j(X_{jk}) \times D_{\gamma_2} P_j(X_{jk}) = d_{jk} N_{jk}, \quad j = 1, \ldots, t, \quad k = 1, 2, 3, \)

for some positive constants \( d_{jk} \).

3. Adjacent patches defined by \( \{P_j\}_{j=1}^t \) join in a geometrically smooth fashion.

With joining smoothly we mean that the resulting surface will be continuous and has a continuously varying unit normal vector. Note that this definition of geometric continuity is not based on a smooth reparametrization. The vectors \( \gamma_1 \) and \( \gamma_2 \) above and throughout this article satisfy the relation

\[
(\gamma_1 \times \gamma_2)((X_{j+1,k} - X_{jk}) \times (X_{j+2,k} - X_{j,k})) > 0. \tag{1}
\]

Here, \( uv \) is the scalar product of \( u \) and \( v \), the symbol "\( \times \)" denotes the usual vector product, and addition of indices is assumed to be mod 3.

2 Description of the algorithm

In order to determine the maps \( P_j (j = 1 \ldots t) \) satisfying the three conditions, it is useful to introduce a new map \( Q_j \) by

\[
Q_j(X) = q_j(\lambda) = \frac{\hat{q}_j(\lambda)}{\|\hat{q}_j(\lambda)\|} \quad \text{where} \quad \hat{q}_j(\lambda) = D_{\gamma_1} P_j(\lambda) \times D_{\gamma_2} P_j(\lambda).
\]

Note that \( Q_j \) may be defined by a whole class of functions \( q_j \):

\[
\{q_j(\lambda) | q_j(\lambda) = c(\lambda) \hat{q}_j(\lambda); c(\lambda) \in R \text{ for all } \lambda \text{ with } X_\lambda \in S \}.
\]

Functions \( q_j \) belonging to the same class will be considered equivalent and we will say that they produce equivalent patches or equivalent curves. The map \( P_j \) defines a triangular patch of the surface ("surface" patch) and the map \( Q_j \) defines a related patch ("normal" patch). Instead of constructing the collection of maps \( \{P_j\} \) we simultaneously construct the two collections of maps \( \{P_j\} \text{ and } \{Q_j\} \). Of course there is an interaction between the maps \( P_j \) and \( Q_j \) —
they have to be compatible in the sense that \( Q_j(X_\lambda) \) is orthogonal to the patch \( P_j \) at \( X_\lambda \). The conditions for the maps \( P_j \) and \( Q_j \) guaranteeing smoothness of the resulting composite interpolating surface can be expressed as follows:

1. Interpolation of \( V \):
   \[ P_j(X_k) = V_{jk}, \quad j = 1, \ldots, t, \quad k = 1, 2, 3. \]

2. Interpolation of \( N \):
   \[ Q_j(X_k) = d_{jk} N_{jk}, \quad d_{jk} > 0, \quad j = 1, \ldots, t, \quad k = 1, 2, 3. \]

3. Smoothness:
   Both \( \{ P_j \}_{j=1}^t \) and \( \{ Q_j \}_{j=1}^t \) define a \( C^0 \) surface.

4. Compatibility:
   \[ Q_j(X_\lambda) = d_j(\lambda) D_\gamma P_j(X_\lambda) \times D_\gamma P_j(X_\lambda), \quad d_j(\lambda) > 0, X_\lambda \in S, j = 1, \ldots, t. \]

In the algorithm we choose the functions \( p_j \) and \( q_j \) defining the maps \( P_j \) and \( Q_j \) to be polynomials of total degrees \( n \) and \( m \), respectively. Thus, in the Bernstein–Bézier form we have

\[ p_j(\lambda) = \sum_{|\alpha|=n} \frac{n!}{\alpha!} P_\alpha \lambda^\alpha \quad \text{and} \quad q_j(\lambda) = \sum_{|\alpha|=m} \frac{m!}{\alpha!} Q_\alpha \lambda^\alpha, \]

where as usual

\[ \alpha = (\alpha_1, \alpha_2, \alpha_3), \quad |\alpha| = \sum_{i=1}^n \alpha_i, \quad \frac{n!}{\alpha!} = \frac{n!}{\alpha_1! \alpha_2! \alpha_3!}, \lambda^\alpha = \lambda_1^{\alpha_1} \lambda_2^{\alpha_2} \lambda_3^{\alpha_3}. \]

The previously stated conditions can now be reformulated as:

1. \( P_{n,0,0}^j = V_{j_1}, \quad P_{0,n,0}^j = V_{j_2}, \quad P_{0,0,n}^j = V_{j_3} \)

2. \( Q_{n,0,0}^j = d_{j_1} N_{j_1}, \quad Q_{0,n,0}^j = d_{j_2} N_{j_2}, \quad Q_{0,0,n}^j = d_{j_3} N_{j_3} \)

3. On common edges \( \{ P_\alpha^j \} \) and \( \{ Q_\alpha^j \} \) associated with adjacent patches must coincide or in the case of \( \{ Q_\alpha^j \} \) define equivalent curves.

4. \( q_j(\lambda) D_\gamma p_j(\lambda) = 0 \), for an arbitrary vector \( \gamma \) and for all \( \lambda \) with \( X_\lambda \in S \).

Condition 4 imposes interrelations (compatibility conditions) between control points \( \{ P_\alpha^j \} \) and \( \{ Q_\alpha^j \} \).

The algorithm we used consists of two stages. In a first stage we choose the control points \( \{ P_\alpha^j \} \) and \( \{ Q_\alpha^j \} \) on the edges appropriately (satisfying conditions 1 through 4). At a second stage we choose the interior control points appropriately (satisfying condition 4).

The choice of the B-ordinates depends uniquely on “local” data. In the first stage that means the B-ordinates depend only on data of the corresponding edge and in the second stage the B-ordinates depend only on data of the corresponding triangle.
3 Degeneracy

Let us look at the compatibility around one vertex, say $V_1$, of a triangle $\{V_1, V_2, V_3\} \in T$. Since we deal only with the triangle $\{V_1, V_2, V_3\}$ we omit from now on the index $j$. It has been shown in [3] that condition 4 requires the following relation to hold

$$Q_{m-1,1,0}(P_{n-1,0,0} - P_{n,0,0}) = Q_{m-1,0,1}(P_{n-1,1,0} - P_{n,0,0}).$$

Notice that this relation is satisfied trivially if

$$P_{n-1,0,1} = P_{n,0,0} = P_{n-1,1,0}.$$

This suggests to consider degenerate patches i.e., patches with three coalescent control points at the vertices. First order geometric continuity of degenerate patches has been studied and sufficient and necessary conditions guaranteeing smoothness of patches at singularities have been described in [3].

As an example let us consider the degenerate patch of total degree $n = 4$, given by

$$p(\lambda) = \sum_{i=1}^{3} V_i(\lambda_i^4 + 4\lambda_i^3(1 - \lambda_i)) + 6P_{0,2,2}\lambda_2^2\lambda_3 + 6P_{2,0,2}\lambda_1^2\lambda_3 + 6P_{2,2,0}\lambda_1^2\lambda_2^2 + 12P_{2,1,1}\lambda_1^2\lambda_2\lambda_3 + 12P_{1,2,1}\lambda_1\lambda_2^2\lambda_3 + 12P_{1,1,2}\lambda_1\lambda_2\lambda_3^2.$$

Due to singularity of the parametrization at $V_1$ we have:

$$||D_{\gamma_1}p(X_1) \times D_{\gamma_2}p(X_1)|| = 0.$$  

However, for a right choice of control points $P_{2,2,0}, P_{2,1,1}, P_{2,0,2}$, the tangent plane and the direction of the normal vector at $V_1$ is well defined and the unit normal vector is equal to

$$\lim_{\lambda \to X_1} \frac{D_{\gamma_1}p(\lambda) \times D_{\gamma_2}p(\lambda)}{||D_{\gamma_1}p(\lambda) \times D_{\gamma_2}p(\lambda)||} = \frac{N_1}{||N_1||}.$$  

Thus, even at singular points the surface will satisfy the interpolation conditions and is geometrically smooth. The singular parametrization does not lead to kinks or cusps.

4 Curvature of curves on degenerate patches

In order to analyse curvature properties of degenerate patches at vertices let us consider curves passing through a degenerate vertex. Obviously, the parametrization of these curves is singular in the sense that the lengths of their tangent vectors are vanishing at the vertices. For example, for the boundary curve $b$ of a degenerate quartic patch given by

$$p(\lambda)|_{\lambda_3=0} = \tilde{p}(\lambda_1) = V_1(\lambda_1^4 + 4\lambda_1^3(1 - \lambda_1)) + 6P_{2,2,0}\lambda_2^2(1 - \lambda_1)^2 + V_2(4(1 - \lambda_1)^3\lambda_1 + (1 - \lambda_1)^4),$$

the length of the tangent vector of $b$ at $V_1$ is vanishing:

$$\lim_{\lambda_1 \to 1} ||\tilde{p}'(\lambda_1)|| = 0.$$
and so is the derivative of the arclength $s$ of $b$ with respect to $\lambda_1$:

$$\lim_{\lambda_1 \to 1} \frac{ds}{d\lambda_1}(\lambda_1) = 0.$$ 

For the curvature $\kappa$ of $p$ at $V_1$ we obtain

$$\lim_{\lambda_1 \to 1} \kappa(\lambda_1) = \lim_{\lambda_3 \to 1} \frac{\|\hat{p}'(\lambda_1) \times \hat{p}''(\lambda_1)\|}{\|\hat{p}'(\lambda_1)\|^3} = \infty.$$ 

Thus, the curves on degenerate patches have infinite curvature at the vertices. How important is the curvature in a point of the curve with a singular parametrization? All our tests performed in [3] using degenerate patches show no visual artefacts. Interpolating surfaces that we have obtained look smooth and do not seem to have an artificial behaviour in the vicinity of singularities in spite of the infiniteness of the curvatures. To illustrate that the curvature at a singular point may not be so important for the shape of a patch we make the following observations. These remarks support the fact that the phenomenon of infinite curvatures might not be a serious drawback from the practical point of view as is also suggested by the tests in [3].

**Observation 1.** Consider the curvature $\kappa(x)$ of the curve given by the function $f(x) = |x|^r$ on $[-1, 1]$ for different values of $r$:

$$\kappa(x) = \frac{r(r-1)x^{r-2}}{(1 + r^2x^{2r-2})^2}.$$ 

For $r = \frac{4m}{2m+1}$, the parametric form of this curve is given by

$$g(t) = \begin{pmatrix} t^{2m+1} \\ t^{4m} \\ 0 \end{pmatrix}, \quad t \in [-1, 1].$$

Some calculation shows

$$\kappa(t) = \mathcal{O}(t^{-1}),$$

which implies that the curvature tends to infinity for $t$ tending to zero for every positive integer $m$. On the other hand, for $r = 2$ we obtain $\kappa(0) = 2$. Obviously, if $m$ is sufficiently large the shapes of the two curves for $r = \frac{4m}{2m+1}$ and $r = 2$ become visually "identical" in spite of their essential difference with respect to their curvature properties.

**Observation 2.** Let $q$ be the nondegenerate quadratic curve given by

$$q(\lambda_1) = V_1 \lambda_1^2 + 2P_{2,2,0}\lambda_1(1 - \lambda_1) + V_2(1 - \lambda_1)^2.$$ 

Note that this quadratic parametric polynomial takes on the same positional and normal values at its end-points as the degenerate quartic curve $p$. Moreover, it is not difficult to see that both curves have equal unit tangent vectors for all $\lambda_1 \in [0, 1]$. Since the directions of tangent lines are essential for the shape of a surface, this may account for the fact that degenerate quartic curves look visually as smooth as the nondegenerate quadratic ones.

**Observation 3.** We have considered the curvature of the curve given by $\hat{p}$ in the case of $V_1 = (0, 0, 0), P_{2,2,0} = (1, 1, 0), V_2 = (2, 0, 0)$ in different points. As a parameter we used
<table>
<thead>
<tr>
<th>(s/s_p)</th>
<th>(s(s))</th>
</tr>
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<td>0.01</td>
<td>0.69</td>
</tr>
<tr>
<td>0.001</td>
<td>1.93</td>
</tr>
<tr>
<td>0.0005</td>
<td>2.69</td>
</tr>
</tbody>
</table>

Table 1: Values of curvatures of \(p\) at selected points.

the arclength \(s\) and \(s_p\) denotes the total length of the curve. In Table 1, is shown that the significant increase of curvature of a degenerate curve occurs only in a region very close to the singular point. For instance, it can be seen that at a distance of 0.0005 (measured relatively to the length \(s_p\) of \(b\)) the curvature of the curve is still relatively small. That means it would be necessary to sample this curve at more than 2000 points (provided the sampling is uniform with respect to the arclength) in order to be able to detect large curvatures near singular points. This also explains the fact that in our experiments we were not able to notice singularities visually, since we have used less than 100 sampling points per curve.

Finally, we remark that analogous results to those mentioned above for quartic degenerate curves could also be obtained for degenerate curves of arbitrary degrees.

Let us conclude with an open question. The directions of the normals are essential for the shape of a surface in shading algorithms (see [6]). This fact may explain, why we obtained nice pictures, even so we were using degenerate patches of total degree \(n = 4\) and \(n = 5\) (see [3]). However the question how the degeneracies are influencing the smoothness of the lines of reflection (assuming we have a linear light source) is still under investigation.

References


