What is the Natural Generalization of
Univariate Splines to Higher Dimensions?

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Abstract. In the first part of the paper, the problem of defining multivariate splines in a natural way is formulated and discussed. Then, several existing constructions of multivariate splines are surveyed, namely those based on simplex splines. Various difficulties and practical limitations associated with such constructions are pointed out. The second part of the paper is concerned with the description of a new generalization of univariate splines. This generalization utilizes the novel concept of the so-called Delaunay configurations, used to select collections of knot-sets for simplex splines. The linear span of the simplex splines forms a spline space with several interesting properties. The space depends uniquely and in a local way on the prescribed knots and does not require the use of auxiliary or perturbed knots, as is the case with some earlier constructions. Moreover, the spline space has a useful structure that makes it possible to represent polynomials explicitly in terms of simplex splines. This representation closely resembles a familiar univariate result in which polar forms are used to express polynomials as linear combinations of the classical B-splines.

§1. Introduction

This paper describes the material I presented at the Fifth International Conference on Mathematical Methods for Curves and Surfaces, held in Oslo in June 2000. In my talk, I addressed the topic of a meaningful generalization of the classical univariate splines to higher dimensions. The quest for finding such generalizations is certainly not new. Many researchers in the theory of multivariate splines have addressed this or similar questions. Indeed, there is a great variety of multivariate generalizations of splines available today. However, there does not seem to exist a generalization that is commonly agreed to be the “right” one. This fact perhaps prompted Carl de Boor to conclude his survey “What is a multivariate spline?” [8] with a touch of irony: “If this leaves you a bit wondering what multivariate splines might be, I am pleased. For I don’t know myself.”

Given the large number of possible approaches to multivariate splines, it is clear that the term “natural generalization” in the title is necessarily vague
unless we agree on its precise meaning. Therefore, let us first discuss which properties a multivariate spline should ideally possess. My own understanding of “spline” in this paper is that it should be a piecewise polynomial of a given fixed degree. Second, the individual polynomial pieces of the spline should be associated with polygonal (polyhedral) regions. This means that the domain of definition of the spline should be partitioned by segments of straight lines (hyperplanes), i.e., it should be a rectilinear partition. In particular, we will assume that the partition is a union of polytopes, each of which is the convex hull of a finite set of points, called knots. The third important property of splines is that they are “local”. Thus, not only should the spline space have a finite local dimension, but also the space should contain splines of compact support. Ideally, a good spline space should be such that every spline in the space can be written as a combination of compactly supported splines.

Mathematicians usually endeavor to obtain results in their greatest possible generality. Therefore, it is desirable to have a generalization of splines that applies to all degrees, (almost) arbitrary knot locations, and all spatial dimensions. Moreover, in the one-dimensional case the spline construction should reduce to the familiar univariate splines. We must also postulate that the obtained spline space is large enough so that it can be used to effectively approximate other functions. Namely, the spline space should contain constants and, more generally, all polynomials up to a given degree, ideally equal to the spline degree. The requirement of polynomial reproduction is now standard, since it is well known that the degree of approximation is closely related to the degree of local polynomial reproduction, see e.g. [6,131].

Before we formulate the above requirements more rigorously, we need some notation and a few technical assumptions. We will be concerned with s-variate splines i.e., functions whose domain is \( \mathbb{R}^s \), \( s = 1, 2, \ldots \). The degree of a polynomial or a spline will be denoted by \( n \), \( n = 0, 1, \ldots \). The symbol \( K \) will stand for the set of knots, i.e., points in \( \mathbb{R}^s \). To avoid the consideration of “boundary conditions”, we assume throughout that the convex hull \([K]\) of \( K \) equals \( \mathbb{R}^s \). Moreover, we only allow knot-sets without accumulation points, or, equivalently, \( K \) must be such that \( K \cap \Omega \) is finite for every compact \( \Omega \subseteq \mathbb{R}^s \).

Lastly, \( K \) will be “generic” in the sense (to be made more precise later) that certain properties of the corresponding spline space will be required to hold only for “almost all” knot-sets, not necessarily all knot-sets. This will help us avoid various special cases (such as knot-multiplicities), which, while deserving attention in practical considerations of splines, do not have direct bearing on the central issue of existence of a natural generalization of splines. The set \( K \), together with a set of rules of how the knots in \( K \) are connected, will give rise to a partition of \( \mathbb{R}^s \) and hence also to a spline space, denoted by \( S_n \), of all real-valued piecewise polynomials of a given degree \( n \) and smoothness \( C^{n-1} \).

This smoothness is also sometimes called optimal since a higher than optimal smoothness would necessarily imply \( S_n \subseteq \Pi_n \), where, as usual, \( \Pi_n \) stands for the space of polynomials of total degree at most \( n \). This, however, is not a spline space in our terminology since clearly \( \Pi_n \) does not contain functions of compact support, and hence \( S_n \) cannot be spanned by such functions. The
above assumption that all splines in $S_n$ are optimally smooth is motivated by the fact that univariate splines with generic knots (i.e., knots without multiplicities) also have optimal smoothness.

We are now ready to define more precisely the problem of finding a natural generalization of splines, which we will refer to as the fundamental problem.

**Fundamental Problem.** For any integers $s \geq 1$ and $n \geq 0$, and any generic set of knots $K$, construct a spline space $S_n$ on $\mathbb{R}^s$, such that

(A) Each spline in $S_n$ is a piecewise polynomial of degree $n$, associated with a rectilinear partition determined by $K$;
(B) Each spline in $S_n$ has optimal smoothness i.e., $S_n \subset C^{n-1}(\mathbb{R}^s)$;
(C) Splines in $S_n$ reproduce polynomials i.e., $\Pi_n \subset S_n$;
(D) $S_n$ is locally finite dimensional, i.e., $\dim S_n|_{\Omega} < \infty$, for all compact $\Omega \subset \mathbb{R}^s$;
(E) There exists a countable collection $B_n$ of compactly-supported functions which forms a basis for $S_n$, in the sense that for every $x \in \mathbb{R}^s$, all but a finite number of functions in $B_n$ vanish at $x$, and every spline in $S_n$ can be uniquely represented as a linear combination of the form $\sum_{B \in B_n} c_B B$, where $c_B \in \mathbb{R}$, $B \in B_n$;
(F) For $s = 1$, the space $S_n$ reduces to the classical space of univariate splines and the functions in $B_n$ are the ordinary univariate B-splines.

Several remarks on the above formulation are in order. Condition (E) implies, among other things, that the functions in $B_n$ are linearly independent. Moreover, a consequence of (D) and (E) is that for every compact $\Omega \subset \mathbb{R}^s$, there is a finite number of functions in $B_n$ with support in $\Omega$. The elements of $B_n$ will be called (multivariate) B-splines. Requirement (F) expresses the desire, discussed earlier, that the construction of $S_n$ must reduce to the familiar splines in one dimension. Strictly speaking, this condition is somewhat vague since in principle one could have different constructions for $s \geq 2$ and $s = 1$. However, our implicit assumption is that the sought-for construction of splines operates essentially in the same way for all dimensions $s$. Thus, in this sense the well-known multivariate box splines do not satisfy (F) since for $s = 1$, such splines are different from the usual univariate splines (except for special knot-sets). In Section 3, we will give examples of other multivariate spline spaces that satisfy (A)–(E), but are not “natural” since they fail (F).

The reader will notice that there are other important requirements that one might want to impose on the space $S_n$. For example, next to the linear independence of the functions in $B_n$, it is often imperative to have a stronger property of stability of $B_n$. Also, it is advantageous to have efficient algorithms for numerical manipulation with the splines, such as recurrence relations for their evaluation. No doubt, the wish list of additional properties of splines could be much longer. However, there is no point in contemplating such a list when in fact it is not clear whether the fundamental problem has a solution. Indeed all types of multivariate splines that come to my mind are established by relaxing some of the requirements (A)–(F). For example, it is known that the classical piecewise polynomials on triangulations do not in general satisfy
(B) and (E) unless the smoothness degree is strictly less than the optimal degree. The already mentioned box splines also do not solve the fundamental problem, since the space \( S_n \) is not even defined for non-uniform knots \( K \). These and several other examples of splines (see e.g. Section 3) show that it is conceivable that requirements (A)–(F) may be too restrictive. In fact, one reason why I became interested in the subject of this paper was a realization that none of the available spaces of multivariate splines seemed to provide a solution to the fundamental problem.

In this paper I will address the fundamental problem by first explaining that the most obvious candidates for spaces \( S_n \) satisfying (A)–(F) are those spanned by simplex splines. In the next section, I briefly discuss the history of these functions and recall their definition, along with some of their properties. In Section 3, I review several constructions of simplex spline spaces that are currently available and explain why they do not satisfy all the above six properties. Then, in Sections 4 and 5, a new type of simplex spline spaces is described, consistent with the requirements of the fundamental problem. The main idea of the construction is based on a new concept of a Delaunay configuration, introduced in this paper, which is a generalization of the classical Delaunay triangulation.

\section{Simplex Splines}

There are two basic approaches to multivariate splines. We can first define the space \( S_n \) and then check whether it has the desired properties, e.g. whether it possesses a compactly-supported basis. Examples of spaces defined by this principle are the classical piecewise polynomials on planar triangulations. The alternative approach is to start with appropriate compactly-supported piecewise polynomials, and then form \( S_n \) as the linear span of these functions.

The first approach is not appropriate in our situation. This is because by requirement (B), the underlying rectilinear partition for the spline space will likely be very complicated. Note that we cannot expect the sought-for space \( S_n \) to coincide with the usual space of piecewise polynomials on triangulations or other relatively simple partitions of \( \mathbb{R}^s \), since it is known that there may be no nontrivial compactly-supported splines of optimal smoothness on such partitions, let alone a basis consisting of such splines. Thus, it may be difficult or impossible to “guess” what a suitable partition should be. Note that this is trivial in the univariate case since partitions of \( \mathbb{R} \) are unions of intervals.

The second approach seems much more tractable since it suggests that we first design appropriate individual “B-splines” and only then proceed with defining the corresponding spline space. The reason why this may be an easier path to follow is that there is already a well-understood class of multivariate compactly-supported splines, the so-called polyhedral splines. In particular, let \( n \geq 1, \sigma \) be a bounded convex polyhedron in \( \mathbb{R}^{n+s} \) with positive volume, and let \( X \subset \mathbb{R}^s \) be the canonical projection of the vertices of \( \sigma \) onto \( \mathbb{R}^s \). Recall that the canonical projection of a point \( v = (v_1, \ldots, v_s, \ldots, v_{s+n}) \in \mathbb{R}^{n+s} \) onto \( \mathbb{R}^s \) is defined as \( v_{|\mathbb{R}^s} \) := \( (v_1, \ldots, v_s) \). The polyhedral spline \( M_\sigma \) is defined
as the volumetric projection of $\sigma$ onto $\mathbb{R}^s$, that is

$$M_\sigma(x) := \frac{\text{vol}_n \{ v \in \sigma : v|_{\mathbb{R}^s} = x \}}{\text{vol}_{n+s} \sigma},$$

(1)

where $\text{vol}_k(A)$ stands for the $k$-dimensional volume of a set $A$. In the special case $n = 0$, $\sigma$ is a polyhedron in $\mathbb{R}^s$, and one defines

$$M_\sigma := \frac{\chi_\sigma}{\text{vol}_s \sigma},$$

(2)

where $\chi_\sigma$ is the characteristic function of $\sigma$. Polyhedral splines were introduced in [11]. They are known to be piecewise polynomials of degree $n$ and of optimal smoothness if $X$ is in generic position. By definition, polyhedral splines are obviously compactly supported with support $[X]$, non-negative, and the normalization in the above two identities is such that the integral of $M_\sigma$ is one. Special cases of polyhedral splines are the already mentioned box splines and simplex splines, obtained when $\sigma$ is a parallelepiped and a simplex, respectively.

Since every polyhedron can be decomposed as a union of simplices, every polyhedral spline is a linear combination of simplex splines. Therefore, from now on we will restrict ourselves, without loss of generality, to the special case where $\sigma$ is a simplex. It turns out that in this situation $M_\sigma(x)$ only depends on the projection $X$, which justifies a new and more appropriate notation $M(x|X) := M_\sigma(x)$, used henceforth. The geometric idea of the definition of $M(\cdot|X)$ via volumetric projection is depicted in Figure 1. In this figure, a bivariate simplex spline of degree one is obtained by volumetrically projecting a three-dimensional simplex $\sigma$ onto the plane. This gives a pyramid-like function whose support is the convex hull of the four knots $X = \{x_0, x_1, x_2, x_3\}$, being the projections of the vertices of $\sigma$ onto $\mathbb{R}^2$. A typical bivariate quadratic simplex spline is displayed in Figure 2, along with its support, the convex hull.
Fig. 2. A bivariate quadratic simplex spline and its support \((s = 2, n = 2)\).

of five knots (since in the quadratic case, \(\sigma\) is a simplex in \(\mathbb{R}^4\)). In general, a collection \(X\) of \(n + s + 1\) knots in \(\mathbb{R}^n\) gives rise to an \(s\)-variate simplex spline of degree \(n\).

For a number of reasons, simplex splines are an excellent point of departure for attempting to solve the fundamental problem. First, they are defined for all \(s\) and \(n\), and also they are optimally smooth \((C^{n-1})\) for generic knot sets \(X\). In fact, “generic” for simplex splines means that the knots in \(X\) must be in general position, \(i.e.,\) no \(s + 1\) of them are allowed to lie in a common hyperplane in \(\mathbb{R}^n\). For example, the quadratic spline in Figure 2 has its knots in general position (since no three of them are collinear), hence the spline is tangent-plane continuous \((C^1)\). Thus, simplex splines satisfy (A) and (B) since any linear combination of simplex splines is a piecewise polynomial of degree \(n\) and optimal smoothness. Moreover, the definition of \(M(\cdot|X)\) is consistent with (F) in the sense that \(M(\cdot|X)\) is just the univariate B-spline for \(s = 1\). Simplex splines and univariate B-splines share many useful properties, see \(e.g.,\) [30,87]. For example, simplex splines can be evaluated by the Michelli recurrence [85], expressing simplex splines in terms of their lower-degree versions. In particular,

\[
M(x|X) = \frac{n + s}{n} \sum_{y \in X} \lambda_y M(x|X \setminus \{y\}), \quad x \in \mathbb{R}^n,
\]

where the numbers \(\lambda_y \in \mathbb{R}, y \in X\), are chosen so that \(\sum_{y \in X} \lambda_y y = x\) and \(\sum_{y \in X} \lambda_y = 1\). Note that this is a generalization of the familiar B-spline recurrence. Beside the original proof of Michelli, alternative proofs of the recurrence were discovered in [11,15,18,65,68,84,91,105] (and possibly elsewhere).

There are also other reasons, described in more detail in the next section, why simplex splines are an ideal starting point for our investigation. In particular, one can show that simplex splines have the striking property that if their linear combinations reproduce constants, then this will automatically imply that they can represent all polynomials up to the degree of the simplex splines. Another fact worth mentioning is that every piecewise polynomial on a rectilinear partition of \(\mathbb{R}^n\) can be expressed as a linear combination of
appropriately chosen simplex splines. Thus, assuming that piecewise polynomials in the sought-for space $S_n$ are combinations of simplex splines is not a real restriction.

We finish this section with a few historical remarks and comments on references. Simplex splines were formally introduced in 1976 by de Boor [5], who followed a suggestion of Schoenberg [112]. However, simplex splines were already known to statisticians several years earlier (see e.g. [126]). We refer the reader to the surveys [6,7,10,30,47,70,113,115,123], for many results on simplex splines. The paper [31] dwells on the origins of simplex splines in statistics. There are several recent books containing chapters on simplex splines, including [1,4,87]. Since the introduction of simplex splines, a steady stream of papers has been written each year on this subject, perhaps with the exception of the last few years, during which there seemed to be less visible activity in the area. We include here a list of references that contains an up-to-date set of 130 research articles, surveys, dissertations, and unpublished papers. To my best knowledge, the list is complete. However, in case it is not, the reader is encouraged to send me additions.

§3. Available Constructions of Simplex Spline Spaces

While individual simplex splines are very appealing functions mathematically, it is their linear combinations that are of main interest in applications. To solve the fundamental problem, we need to choose appropriate collections $C_n$ of knot-sets $X \subset K$ of size $n + s + 1$, so that the associated simplex spline spaces

$$S_n := \text{span}\{M(\cdot|X) : X \in C_n\} := \left\{ \sum_{X \in C_n} c_X M(\cdot|X) : c_X \in \mathbb{R} \right\}$$

will satisfy conditions (A)-(F). Before we address this issue in more detail, it is tempting to review the existing constructions of simplex spline spaces, to see whether they do not already provide a solution to the fundamental problem. We shall see that this is not the case.

3.1. Complete Configurations of Simplex Splines

For the purpose of this subsection only, suppose that $K$ is finite. A possible way to construct a space $S_n$ is to consider the so-called complete configurations of knot-sets $X \subset K$ of size $n + s + 1$, that is

$$C_n := \{ X \subset K : \#X = n + s + 1 \},$$

where $\#X$ denotes the size of the set $X$. Complete configurations were investigated in [28] and also in [62,66]. From the practical point of view, the space $S_n$ corresponding to such collections $C_n$ is not very useful. For example, the simplex splines generated by the knot-sets in $C_n$ are linearly dependent and also can have “large” supports i.e., can be non-local. Nevertheless, the space $S_n$ is
helpful in gaining theoretical insight into simplex splines. In particular, it is known [66] that \( S_n \) is precisely the space of optimally smooth piecewise polynomials of degree \( n \) on the partition of \([K]\), generated by all \((s-1)\)-dimensional faces (or simplices) of the form \([Y]\), where \( Y \subset K, \#Y = s \). Figure 3 shows a typical example of such a partition in \( \mathbb{R}^2 \). The above implies that every piecewise polynomial of optimal smoothness and compact support in \([K]\), which reduces to a polynomial of degree at most \( n \) in each region bounded but not intersected by the faces \([Y]\), is a linear combination of simplex splines.

Another well-known fact about \( S_n \), proved in [28] and independently in [66], is that

\[
\dim S_n = \binom{\#K - n - 1}{s}.
\]

A simple consequence of this identity, obtained by setting \( \#K = n + s + 1 \), is that a simplex spline with \( n + s + 1 \) knots in general position is the unique (up to normalization) piecewise polynomial of degree \( n \) and (global) smoothness \( C^{n-1} \), supported on the partition of \([K]\), generated by all \((s-1)\)-dimensional faces \([Y]\), where \( Y \subset K, \#Y = s \). This property, which is sometimes referred to as the minimal support property, is reminiscent of a familiar property of univariate B-splines, and can be used as an alternative definition of simplex splines. The term “minimal” refers to the fact that taking \( \#K < n + s + 1 \) implies that \( S_n \) contains the zero function only. Thus, for example, the quadratic simplex spline in Figure 2 is the unique \( C^1 \) piecewise quadratic function supported on the convex hull of the knots \( \{x_0, \ldots, x_4\} \), which is a single polynomial in each of the displayed regions \( \Omega_j \). There is no such non-trivial spline if we only take four knots, instead of five.

The above dimension formula also shows why choosing \( C_n \) as the complete configuration associated with \( K \) is not a good idea. Namely, clearly, \( \#C_n = \binom{\#K}{s} \), which is larger than the dimension of \( S_n \). Consequently, the simplex splines corresponding to \( C_n \) are linearly dependent. This explains why researchers looked for better ways to define suitable sets \( C_n \).
3.2. A Geometric Construction

The geometric definition (1) of simplex splines generated considerable interest among researchers since it immediately gives a recipe for selecting suitable collections \( c_n \). Specifically, de Boor [5] proposes to consider the cylinder or “slab” \( \mathbb{R}^s \times \Omega \), where \( \Omega \subset \mathbb{R}^n \) is a suitable convex polytope of unit volume, and subdivide this slab into nontrivial simplices. The corresponding simplex splines will then automatically form a partition of unity. This is because the volumetric projection of the above slab, which is the union of the mentioned simplices, is obviously the constant function with value \( \text{vol}_n(\Omega) = 1 \). To obtain a \textit{linearly independent} collection of simplex splines, we must choose \( \Omega \) to be a simplex [21,27]. For example, to define linear univariate splines, we can set \( \Omega = [0,1] \) and then triangulate the slab \( \mathbb{R} \times [0,1] \), as shown in Figure 4.

Similarly, to obtain bivariate quadratic splines, one can take the four-dimensional slab \( \mathbb{R}^2 \times \Omega \subset \mathbb{R}^4 \), where \( \Omega \) is a 2-simplex, and triangulate it, \textit{i.e.}, decompose it into four-dimensional simplices, each of which gives rise to a quadratic simplex spline (such as the one in Figure 2).

One can show that by the above geometric construction, the associated spline space \( S_n \) will contain not only constants, but also \textit{all} polynomials of degree \( n \). This is yet another surprising property of simplex splines, which is in general not satisfied by other polyhedral spline spaces, \textit{e.g.} box splines. The first proof of the fact that polynomial reproduction is obtained “for free” from the partition of unity was given in [16] and is also explained in many other papers, including [6,27,30,47,50,69,70]. The proof is based on the already mentioned fact that if \( \sigma \subset \mathbb{R}^{n+s} \) is a simplex, then \( M_\sigma \) is determined by the projection \( X \) of the vertices of \( \sigma \) onto \( \mathbb{R}^s \) \textit{i.e.}, the actual shape of \( \sigma \) does not matter as long as \( X \) stays the same. Note that this property is unique to simplex splines and does not hold for other polyhedra \( \sigma \).

The above geometric idea was elaborated on in detail in [50] (see also [47]). There, explicit formulae were given for representing bivariate polynomials as linear combinations of simplex splines, closely resembling similar identities for univariate splines. However, despite such results and the apparent elegance of the geometric construction, this idea did not gain popularity, perhaps as a consequence of the need to triangulate four or higher-dimensional polyhedra. Therefore, other means of obtaining appropriate collections \( c_n \) were proposed.
3.3. Triangulations of Simploids

We have seen that a serious disadvantage of the geometric construction is that the user has to construct triangulations in $\mathbb{R}^{n+s}$, a task that is practically not feasible for $n + s > 3$. A computationally more attractive method of decomposing the slab $\mathbb{R}^s \times \Omega$ into simplices was proposed in [27] and independently in [69].

This method proceeds in two stages. In the first stage, we start with a triangulation $\Delta$ of the knots $K$ i.e., a decomposition of $\mathbb{R}^n$ into simplices $T \in \Delta$, whose vertices are the given knots. This gives rise to a decomposition of the slab into polyhedra of the form $T \times \Omega$, called simploids in [27]. Hence, to find a triangulation of $\mathbb{R}^n \times \Omega$, it is sufficient to triangulate the individual simploids $T \times \Omega$, provided that this triangulation is done in such a way that it results in a global triangulation of $\mathbb{R}^n \times \Omega$, see [27, 69]. This turns out to be a comparatively simpler task since simploids can be triangulated in a canonical way, see [6, 27, 29, 47, 69, 123]. In fact, the knot-sets corresponding to the simplices of this canonical triangulation can be obtained using simple combinatorial rules, and hence computing an actual triangulation of the simploids is not necessary. One can show that in this way one obtains $(n+s)$ simplex splines per simplex $T$, spanning $\Pi_n$ on each $T$. This is because the simplex splines obtained in this way have multiple knots and in fact they are equal to the Bernstein polynomials associated with the simplices $T \in \Delta$. Thus, this construction simply gives rise to a space $S_n$ that is precisely the space of all piecewise polynomials of degree $n$, associated with the triangulation $\Delta$. In particular, the space $S_n$ satisfies (C). However, $S_n$ also contains functions that are discontinuous along the boundaries of the simplices $T \in \Delta$, hence $S_n$ fails to satisfy condition (B). This is a consequence of the fact that the individual simplex splines that span $S_n$ are themselves discontinuous, due to their multiple knots.

The second stage of the method consists in modifying or “smoothing” the space $S_n$ so that all of its elements are optimally smooth. Roughly, this can be achieved by removing the multiplicities of the knots of the above simplex splines, by perturbing them or “pulling them apart”. This is why sometimes this construction is referred to as the pulling-apart method. Thus, the smoothed-out space $S_n$ is spanned by modified simplex splines obtained as perturbed versions of Bernstein polynomials. We refer the reader to the original papers [27, 69], for a detailed description of this method.

Examples of the pulling-apart process and the resulting knot-sets are displayed in Figures 5-8. In particular, Figure 5 shows the situation in the univariate case, where the simplices $T$ are just intervals. In this case the simploids are rectangles partitioning the slab $\mathbb{R} \times [0,1]$, each giving rise to two discontinuous linear Bernstein polynomials, i.e., simplex splines with double knots. By perturbing the triangles of the depicted triangulation of $\mathbb{R} \times [0,1]$, and then projecting volumetrically those triangles onto $\mathbb{R}$, we obtain continuous simplex splines, shown in the figure, which are simply hat functions. Thus, the pulling-apart step removed the multiplicity of the double knots (shown as filled circles in the figure) and replaced them with pairs of distinct
Fig. 5. $C^0$ linear splines obtained by pulling apart the knots ($s = 1, n = 1$).

Fig. 6. Two discontinuous simplex splines and their perturbations ($s = 1, n = 1$).

Fig. 7. The pulling-apart method for bivariate quadratic splines ($s = 2, n = 2$).

knots, consisting of the original knots (filled circle) and their perturbations (empty circles). The smoothing effect of the pulling-apart process on the linear Bernstein polynomials corresponding to a single interval can be better seen in Figure 6.

Figure 7 shows an analogous situation for bivariate quadratic simplex splines. Here, a planar region is first triangulated and then each knot, which is thought of as a triple knot, is separated into three distinct knots, the original one (filled circle), and two “auxiliary” knots (empty circles). Figure 8 displays
the collection of six knot-sets of size five, associated with the shaded triangle $T$ from Figure 7. In Figure 8, the filled circles represent the “active” knots forming the given knot-set of size five, whereas the empty circles are the remaining knots associated with the shaded triangle, not belonging to the knot-set.

It should be noted that by the described construction, the resulting space $S_n$ will still contain $\Pi_n$, i.e., the perturbation of the knots will not compromise property (C). It is also worth stressing that, as we have seen, to obtain $S_n$ it is not necessary to triangulate the mentioned $(n + s)$-dimensional slab, we only have to deal with triangulations in $\mathbb{R}^s$. The collections of simplex splines constructed by the described method can be shown to have several additional properties. In particular, the simplex splines in such collections are linearly independent and also, under some mild conditions, stable. Nevertheless, as will be discussed in Section 3.5, the spline space $S_n$ does not satisfy all properties (A)–(F). Moreover, there are several other shortcomings of the presented construction that necessitated a search for a better solution.

### 3.4. DMS-Splines

Except for the method proposed later in this paper, the most recent construction of simplex spline spaces, often called DMS-splines or triangular B-splines, was given in [32].

Unlike the previous two methods, this construction is not based on the geometric principle. However, one can still think of the simplex splines that one obtains as being generated by a perturbation of Bernstein polynomials or, more precisely, perturbation of the de Casteljau algorithm for these polynomials. In particular, this construction also employs auxiliary or pulled-apart knots. The difference with the method of Section 3.3 is that the knots are used to construct collections of simplex splines in a “more symmetric” way. For details on this construction, the interested reader is referred to [32] and the survey [113].

Figures 9–11 give examples of collections of knot-sets obtained by this method. The univariate case is explained in Figure 9. Here, the filled circles along the real axis represent the original knots and the empty circles are the auxiliary knots (there is only one per original knot in the linear case $n = 1$). Two simplex splines corresponding to a single interval are depicted in
Fig. 9. Construction of DMS-splines \((s = 1, n = 1)\).

Fig. 10. Perturbations of two discontinuous simplex splines by the DMS-method.

Fig. 11. Construction of knot-sets for DMS-splines \((s = 2, n = 2)\).

Figure 10, for the original (not pulled-apart) knots and also for their perturbations. Clearly, compared to the analogous figure for the method of Section 3.3 (cf. Figure 6), this construction is more symmetric. This can also be seen in Figure 11, which is a counterpart of Figure 8. The figure displays the collection of six knot-sets of size five, associated with the shaded triangle \(T\) from Figure 7, giving rise to six quadratic simplex splines. In contrast with Figure 8, the collections in Figure 11 are more symmetric. For example, renumbering the three vertices (the original knots) of the shaded triangle will result in the same configuration of knot-sets, as opposed to the construction of the previous subsection.

Superficially, the difference between DMS-splines and the splines of Section 3.3 may seem marginal and unimportant. However, a closer inspection reveals a significant difference. The present construction, due to its inherent symmetry, has more interesting and useful structure. Among other things, it is possible to derive elegant explicit formulae for the representation of polynomials as combinations of simplex splines. It will be instructive to dwell on this aspect of DMS-splines in more detail. The space \(S_n\) of DMS-splines can
be written as

$$S_n = \text{span}\{M_{j,T} : j = 1, \ldots, J(n,s), T \in \Delta\},$$

where $J(n,s) := \binom{n+s}{s}$, $\Delta$ is a triangulation of $\mathbb{R}^*$, and $\{M_{j,T}\}^{J(n,s)}_{j=1}$ are the simplex splines corresponding to a given $T \in \Delta$. Let $\Pi_n(\Delta)$ denote the space of piecewise polynomials of degree $n$ and optimal smoothness $C^{n-1}$, associated with the triangulation $\Delta$. It was shown in [114], that every $f \in \Pi_n(\Delta)$ can be expressed as a linear combination of the simplex splines, namely

$$f = \sum_{T \in \Delta} \sum_{j=1}^{J(n,s)} F_T(K_{j,T}) N_{j,T},$$

(6)

where $N_{j,T}$ are appropriate normalizations of the simplex splines $M_{j,T}$. We also used the following notation. Since $f$ is a piecewise polynomial on the triangulation $\Delta$, its restriction $f_T$ to a simplex $T \in \Delta$ is a single polynomial. In the above formula, $F_T$ stands for the polar form of $f_T$ (see Section 4, for a precise definition of polar forms) and $K_{j,T}$ are appropriately chosen collections of knots of size $n$. Since the details of formula (6) will not be needed here, we refer the reader to the original article [114] and the surveys [113,115], for a discussion. However, it is important to note that (6) closely resembles a well-known univariate formula, discussed in Section 4.6. Moreover, (6) and $\Pi_n \subset \Pi_n(\Delta)$ imply $\Pi_n \subset S_n$, which was already proved in [32]. This means that the construction of DMS-splines is consistent with requirement (C). Finally, identity (6) also proves the aforementioned fact, that every optimally smooth piecewise polynomial on a rectilinear partition can be written as a linear combination of simplex splines. This is because every rectilinear partition can be triangulated in such a way that any piecewise polynomial $f$ of optimal smoothness on this partition is also a piecewise polynomial on the triangulation. Hence by (6), $f$ is a combination of simplex splines. We note in passing that this assertion is true, even without the assumption of optimal smoothness, if we allow simplex splines with multiple knots.

3.5. Existing Constructions do not Solve the Fundamental Problem

Unfortunately, all of the above constructions suffer from various shortcomings that are not encountered in the univariate situation. We have already explained that the construction via complete configurations is not acceptable. As for the other three methods, they generate, strictly speaking, a family of spline spaces, not a single space for a given collection of knots $K$. This family either depends on how we triangulate the slab in Section 3.2, or on how we choose the set of auxiliary (or pulled-apart) knots. The least elegant ingredient of these constructions is probably that it is not clear how one can choose the auxiliary knots in a natural way. This complication may explain why so far simplex splines have not gained popularity in applications.

Of course, in principle one could make a generic (e.g. random) choice of the auxiliary knots. However, not surprisingly, this may result in spline spaces
Simplex Splines

with "less structure". For example, for the method of Section 3.3, a generic choice of the auxiliary knots will in the univariate case not lead to the classical splines. On the other hand, a clever selection of these knots does give rise to the ordinary univariate splines, see [27,29,69]. Unfortunately, such selection does not seem to have an analog in higher dimensions. This led de Boor [7] to the grim conclusion that it is "not likely that simplex splines will be used as a basis for a good subspace of a given smooth [piecewise polynomial] space of functions". This should be contrasted with his more optimistic view on DMS-splines. In his survey [10], de Boor writes: "Unfortunately, the first scheme [of Section 3.3] did not lead to a spline space with easily constructed quasi-interpolant schemes. However, very recently, a scheme has become available, in [32], that, in hindsight, appears to be the 'right' one.

Despite the above praise and a considerable appeal of DMS-splines, e.g. exhibited by formula (6), these splines are not a solution to the fundamental problem either. In contrast to the geometric construction of Section 3.3, where there is not enough symmetry, in the case of DMS-splines there is "too much" symmetry in some sense. This symmetry makes it impossible to choose the auxiliary knots so as to obtain the classical univariate splines in the case $s = 1$. Thus, DMS-splines are not consistent with requirement (F), except in the special case when the original knots $K$ are not pulled apart, i.e., when the splines remain discontinuous and hence not optimally smooth.

To conclude, we see that in the context of our criteria (A)–(F), none of the four described constructions of multivariate splines can be viewed as "natural". We need to go back to the drawing board and explore alternative ways of defining simplex spline spaces.

§4. A New Construction

Below, we describe a new construction of spline spaces that does not require the introduction of auxiliary knots, and which nevertheless still exhibits many desirable properties. We begin with a glimpse at univariate splines.

4.1. Univariate Splines without Total Ordering of the Real Line

In the univariate case, as we know, the issue of selecting knots for B-splines is resolved trivially by choosing knot-sets formed by consecutive knots. Thus, in the one-dimensional case, suitable collections of knots arise naturally as a byproduct of the total ordering of the real line. Since there is no such useful ordering in higher-dimensional Euclidean spaces, another principle of selecting appropriate collections of knot-sets $C_n$ for simplex splines must be found. However, then, in view of property (F), a similar principle should also apply in the univariate case. Hence, there should be a way to dispense with the total ordering and still be able to group the knots into appropriate collections. Moreover, this new selection principle should give rise to the usual consecutive knots for $s = 1$.

Figure 12 displays a set of knots in the plane, along with two collections of knot-sets of size five. There is little doubt which of the two should be picked
by any reasonable selection principle. It is clear that the knot-set with smaller convex hull is the obvious choice. As for the other knot-set, its knots seem “too far” from each other, which is unacceptable if we want to avoid simplex splines with “large” supports. This discussion suggests the following heuristic selection principle for constructing the collections $\mathcal{C}_n$.

**Heuristic Principle.** Given a knot-set $X \subset K$ of size $n + s + 1$, $X$ should be an element of $\mathcal{C}_n$ if and only if $X$ contains “nearby” knots.

While obviously very vague, intuitively this principle expresses a natural requirement that $\mathcal{C}_n$ should be defined by choosing knot-sets corresponding to simplex splines with “small” supports. The following two examples show that this heuristic principle can be formulated rigorously.

**Example 1.** Define $\mathcal{C}_n$ as the set of all knot-sets $X \subset K$ of size $n + s + 1$ such that $[X] \cap K = X$ i.e., the convex hull of $X$ does not contain any knots from $K$ not belonging to $X$.

**Example 2.** Define $\mathcal{C}_n$ as the set of all knot-sets $X \subset K$ of size $n + s + 1$ such that $\text{diam}(X \cup \{x\}) > \text{diam}(X)$, for every $x \in K \setminus X$, where $\text{diam}(A)$ denotes the diameter of a set $A$.

It is easy to see that in both examples, the construction of $\mathcal{C}_n$ is well defined in any dimension $s$, since to determine $\mathcal{C}_n$ we need only to compute the convex hull of a finite number of points and its diameter, which clearly can be done for all $s \geq 1$. Also note that for $s = 1$, both principles give rise to knot-sets of consecutive knots, which confirms that the use of total ordering of the real line can be avoided.

It is another question whether the collections $\mathcal{C}_n$ from Examples 1 and 2 are actually suitable for constructing meaningful simplex spline spaces $\mathcal{S}_n$ for $s > 1$. The answer is negative in both cases and for different reasons, even though these collections are identical and the “correct” ones for $s = 1$. The collection $\mathcal{C}_n$ from the first example is not appropriate in higher dimensions since there are too many knot-sets in this collection. Namely, observe that
the condition \([X] \cap K = X\) does not prevent \(C_n\) from containing knot-sets whose convex hulls are “long” and “thin”. However, a more detailed analysis shows that the space \(S_n\), corresponding to \(C_n\) via relation (4), is rich enough so that \(\Pi_n \subseteq S_n\). The collection \(C_n\) obtained in Example 2 has the opposite drawback. This time, \(C_n\) is too small since \(S_n\) does not contain all polynomials of degree \(n\).

There are many other similar selection criteria that lead to knot-sets of consecutive knots in the univariate case, but fail to give rise to good collections in the multivariate case (see Example 4 below). Nevertheless, the above examples illustrate that the idea of selecting knots according to whether the knots are nearby or not, makes sense, at least in principle. We only need to define accurately what “nearby” means. A natural place to look for a good definition is in the theory of Voronoi diagrams, which are well-known concepts from computational geometry. Voronoi diagrams record information about proximity relations between point-sets and can help us determine whether points are close or far from each other.

### 4.2. Voronoi Diagrams and Higher Order Voronoi Diagrams

The Voronoi diagram of a set of knots \(K\) is defined as the partition of \(\mathbb{R}^s\) into subsets that are closest to one of the points in \(K\). Thus, for each \(x \in K\) there is a closed set \(c(x) \subseteq \mathbb{R}^s\), called a Voronoi cell, such that the distance of every \(y \in c(x)\) from \(x\) is less than or equal the distance of \(y\) from any other point in \(K\). Voronoi cells are known to be polyhedral, i.e., intersections of finitely many closed half-spaces. Figure 13 shows the Voronoi diagram of a set of points in the plane. Voronoi diagrams belong to standard tools in many areas of mathematics and computational geometry. For more on Voronoi diagrams, we refer the reader to [133,134].

Voronoi diagrams can be generalized to the so-called higher order Voronoi diagrams, introduced in 1975 by Shamos and Hoey [137] (see also [133,134]).
The usefulness of these generalizations of Voronoi diagrams stems from the fact that they allow us to find, for any given point $x \in \mathbb{R}^s$ and any number $k$, a set of $k$ points from $K$ closest to $x$. The precise definition is as follows.

**Definition 3.** The Voronoi diagram of order $k$ of the set $K$ is the subdivision of $\mathbb{R}^s$ into regions, called Voronoi cells (of degree $k$), such that each cell contains the same $k$ nearest points from $K$. In particular, let $X \subset K$ be such that $\#X = k$. The Voronoi cell $c(X)$ is defined as

$$c(X) := \{ x \in \mathbb{R}^s : d(x,X) \leq d(x,Y), Y \subset K, \#Y = k \},$$

where $d(x,Y) := \max_{y \in Y} \| x - y \|$, and $\| \cdot \|$ is the Euclidean norm.

Note that some Voronoi cells could be empty sets. Voronoi diagrams of order one are the usual Voronoi diagrams. Just as in the case of ordinary
Fig. 16. Sabin’s suggestion works well for univariate knot-sets.

diagrams, the cells \( c(X) \) are convex polyhedral sets. Figures 14 and 15 show the Voronoi diagrams of orders two and three, respectively, for the point-set from Figure 13.

Let us now return to the problem of constructing appropriate collections of knot-sets \( C_n \), using the heuristic principle mentioned earlier. Definition 3 motivates the following alternative to the constructions in Examples 1 and 2.

**Example 4.** Define \( C_n \) as the set of all knot-sets \( X \subset K \) of size \( n + s + 1 \), which correspond to nonempty Voronoi cells of order \( n + s + 1 \).

This suggestion makes sense since if \( c(X) \) is nonempty, then all knots in \( X \) are “nearby” in that there is a region in \( \mathbb{R}^s \), namely \( c(X) \), such that the points in this region are closer to the knots \( X \) than to any other knots from \( K \). Clearly, this means that points in \( X \) cannot be too far apart.

The above interesting choice for \( C_n \) was proposed in 1988 by Malcolm Sabin at the *First Oslo Conference on Curves and Surfaces*. In his paper for the proceedings of that conference [107], Sabin discussed the problem of using bivariate quadratic simplex splines to define useful spline spaces. In that paper, he first considers splines with knots on a uniform triangular grid, after which he suggests using higher order Voronoi diagrams to choose knot-sets for quadratic simplex splines. He writes: “The fifth order Voronoi diagram ... identifies quintuples of vertices on the basis of locality”. However, he continues on a pessimistic note: “I thought this was a lovely idea until I discovered that it would not produce the regular grid example. Indeed it is difficult to imagine any plausible locality-based selection of five ... which would omit the vertex in the middle of the regular grid ...”

It is not difficult to see that in the univariate case, Sabin’s idea works well in that it leads to collections of consecutive knots. This is illustrated in Figure 16, displaying Voronoi diagrams of orders one, two, and three for a set of eight knots on the real line. For example, the interval marked as 3–5 in this figure is the Voronoi cell of order three, corresponding to the knots numbered 3,4,5.

Unfortunately, Sabin’s suggestion does not give rise to appropriate collections of knot-sets in higher dimensions. This is explained in Figure 17, representing the Voronoi diagram of order three for a set of four knots (numbered 1 through 4 in the figure). It can be seen that in this specific example,
the Voronoi diagram consists of four nonempty Voronoi cells of order three. For example, the Voronoi cell marked 1-2-4 is the region of points in \( \mathbb{R}^2 \) that are closest to knots 1, 2, and 4 (or, equivalently, farthest to knot 3). Thus, according to Sabin, the collection \( \mathcal{C}_0 \) should consist of the four triples \( \mathcal{C}_0 = \{\{1, 2, 3\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 4\}\} \). The space \( \mathcal{S}_0 \) in this case is therefore spanned by the characteristic functions of the triangles whose vertices are these four triples of knots. It is easy to see that these four functions are linearly dependent. Hence, Sabin’s recipe leads to a collection of “too many” simplex splines.

In the next section we will see that Sabin’s suggestion can be modified to give rise to more appropriate collections \( \mathcal{C}_n \).

### 4.3. Delaunay Triangulations

Returning to the example in Figure 17, it is clear that instead of considering the collection \( \mathcal{C}_0 \) of all four triples of knots, it would be more natural to consider just two of those knot-sets, for example \( \mathcal{C}_0 = \{\{1, 2, 3\}, \{2, 3, 4\}\} \). These knot-sets give rise to two simplex splines that are clearly linearly independent (since their supports are two non-overlapping triangles) and whose linear combinations reproduce constants on the convex hull of the four knots. More generally, if one wants to construct collections \( \mathcal{C}_0 \) corresponding to an arbitrary set of knots \( K \), then instead of using the approach of Example 4, it is preferable to start with a triangulation \( \Delta \) of the knots \( K \). This leads to a space \( \mathcal{S}_0 \), spanned by the lowest-degree simplex splines, namely the characteristic functions of the triangles (simplices) of the triangulation \( \Delta \).

This brings us to the concept of the dual graph of the Voronoi diagram, obtained by connecting two points from \( K \) by an edge if their associated Voronoi cells are neighbors, i.e., share a common \((s - 1)\)-dimensional face. From now on, we will assume that the knots \( K \) are such that no \( s + 2 \) of them are co-spherical, i.e., lying on a common \((s - 1)\)-dimensional sphere. For
$s = 2$, this assumption means that no four knots from $K$ are co-circular. A consequence of this slight restriction is that any given point in $\mathbb{R}^s$ can belong to at most $s + 1$ Voronoi cells. In his 1934 paper [132], B. N. Delaunay proved that the dual graph of the Voronoi diagram is a triangulation of the convex hull $[K]$, now called the Delaunay triangulation. Note that in our case $[K] = \mathbb{R}^s$, by an earlier assumption on $K$. The Delaunay triangulation corresponding to the point-set from Figure 13 is shown in Figure 18.

For our purposes, it will be convenient to interpret Delaunay’s result in function-theoretic language. Namely, denoting the Delaunay triangulation by $\Delta$, this result can be restated as

$$\sum_{T \in \Delta} \text{vol}_s(T)M(x|V(T)) = 1, \quad x \in \mathbb{R}^s,$$

(7)

where $V(T)$ are the vertices of the simplex $T \in \Delta$. This follows from formula $M(x|V(T)) = \chi_T/\text{vol}_s(T)$, which is a consequence of (2). Hence, identity (7) asserts that appropriate linear combinations of the lowest-degree simplex splines, corresponding to the collection of knot-sets

$$\mathcal{C}_0 = \{V(T), T \in \Delta\},$$

(8)

reproduce constants. Strictly speaking, equality (7) is true only for all $x$ in the interior of the simplices of $\Delta$, but one can redefine the lowest-degree simplex splines on the boundary of the simplices so as to achieve pointwise equality everywhere on $\mathbb{R}^s$.

To summarize, in the lowest-degree case $n = 0$, it is a better idea to utilize Delaunay triangulations to construct collections $\mathcal{C}_0$, instead of Voronoi diagrams of order $s + 1$, as suggested by Sabin. This is because the collection (8) clearly gives rise to independent simplex splines and yet their number is large enough so that $\prod_0 \subseteq \mathcal{S}_0$. 
Fig. 19. Voronoi diagram of order two and Delaunay configuration of degree one.

4.4. Delaunay Configurations

Motivated by (7) and (8), the main idea of our new construction can be explained as follows. Since the lowest-degree splines are obtained from Delaunay triangulations, it is conceivable that splines of degrees $n \geq 1$ can be generated using appropriate higher-degree analogs of such triangulations, called here Delaunay configurations. Just as Delaunay triangulations can be derived from Voronoi diagrams, Delaunay configurations will be intimately related to the already mentioned higher order Voronoi diagrams.

To motivate the definition of such configurations, let us consider the particular planar Voronoi diagram of order two, displayed in Figure 19. The large-font pairs of numbers in some of the Voronoi cells represent the pairs of knots associated with a given cell. Thus, for example, the cell marked 7–16 consists of all points in the plane that are closer to knots 7 and 16 than to any other pair from the given set of knots. To explain the idea behind Delaunay configurations, let us recall that a Voronoi vertex is a point in the plane (or in $\mathbb{R}^s$) which is common to exactly three (or $s+1$) Voronoi cells (recall that by our assumption, there are no points in $\mathbb{R}^2$ ($\mathbb{R}^s$) belonging to four ($s+2$) or more Voronoi cells). It is not difficult to see that every Voronoi vertex is the incenter of a circle circumscribed to a triple of knots. In fact, there are precisely two types of such Voronoi vertices, depicted in the figure. The first type corresponds to the Voronoi vertex common to cells 7–9, 7–11, 9–11, hence it is the center of the circle passing through knots 7,9,11. The second depicted Voronoi vertex is common to cells 7–9, 7–11, 7–16 and is the center of the circle passing through knots 9,11,16. The essential difference between these two types of vertices is that the first mentioned circle contains no knots in its interior, whereas the second circle contains one knot, namely the knot 7. Therefore, by a well-known fact about Delaunay triangulations, in the first case the triangle with vertices 7,9,11 is a Delaunay triangle, while in
the second case, the triangle with vertices 9,11,16 is not a Delaunay triangle. The Delaunay triangle gives rise, as we mentioned earlier, to a simplex spline of degree zero (a constant simplex spline). On the other hand, the quadruple (7,9,11,16) corresponds to a (linear) simplex spline of degree one. This quadruple will be called a Delaunay configuration of degree one ("one" refers to the number of knots inside the circle circumscribed to the triangle with vertices 9,11,16) or simply a Delaunay quadruple. We shall see that simplex splines corresponding to such collections will give rise to a spline space with several striking properties.

The above discussion motivates the following definition.

**Definition 5.** The family of pairs

$$\Delta_n := \{ X = (X_B, X_I) \},$$

such that

$$X_B, X_I \subset K, \ \#X_B = s + 1, \ \#X_I = n,$$

and such that the closed ball $$\Omega \subset \mathbb{R}^s$$, circumscribed to $$X_B$$, contains $$X_I$$ in its interior and no other knots from $$K$$, i.e., $$X_B \cup X_I = \Omega \cap K$$ and $$X_I = \text{int}(\Omega) \cap K$$, is called the (oriented) Delaunay configuration of degree $$n$$ associated with the knots $$K$$.

Figure 20 shows examples of Delaunay configurations of varying degrees. In particular, the triple of points $\{12, 21, 24\}$ is a Delaunay triangle, $\{1, 2, 19, 28\}$ is a Delaunay quadruple, and both $\{3, 4, 13, 14, 31\}$ and $\{13, 17, 31, 4, 27\}$ are Delaunay quintuples (i.e., Delaunay configurations of degree two). Figure 21 shows a Delaunay configuration of degree three.

Before we embark on discussing the relevance of the above notion for constructing multivariate splines, a few remarks on Definition 5 are in order. The concept of a Delaunay configuration seems to be new. Although it is implicit in several papers on higher order Voronoi diagrams, see *e.g.* [138], it
Fig. 21. Boundary and interior knots of a Delaunay configuration of degree three.

does not seem to have been studied as a separate entity. It is immediately clear from the well-known properties of Delaunay triangulations that these are Delaunay configurations of degree \( n = 0 \), i.e., \( \Delta_0 = \Delta \). It should be noted that Delaunay configurations for \( n \geq 1 \) do not form a partition of \( \mathbb{R}^s \) i.e., a Delaunay configuration is not a tessellation. Clearly, for every \( X \in \Delta_n \), the sets \( X_B \) and \( X_I \) are disjoint. In our notation, the subscript “\( B \)” stands for “boundary” and “\( I \)” for “interior” knots.

4.5. Partition of Unity and Reproduction of Polynomials

We now return to the central issue of this section and present a construction of appropriate collections \( C_n \) of knot-sets, along with the accompanying simplex spline spaces \( S_n \).

First recall that a Delaunay configuration of degree \( n \) corresponds to a collection of knot-sets of size \( n + s + 1 \). This suggests that we define \( S_n \) as the space spanned by the simplex splines whose knot-sets are precisely the elements of the Delaunay configuration \( \Delta_n \). As we have pointed out earlier, this certainly makes sense in the lowest-degree case \( n = 0 \), since then \( S_0 \) contains constant functions, as a consequence of (7). In fact, this property carries over to all degrees \( n \).

To formulate the above more precisely, let us first recall the definition of polar forms of polynomials. The concept of a polar form or “blossom”, while known for quite some time in an algebraic context, has been introduced into the spline theory by de Casteljau and independently by Ramshaw (see [135], for an introduction). Polar forms have proven to be a convenient mathematical tool for describing (piecewise) polynomial functions and for analyzing various spline algorithms. Given an \( s \)-variate polynomial \( p \in \Pi_n \), the polar form \( P \) of \( p \) is defined as the unique function of \( n \) vector variables \( x_1, \ldots, x_n \in \mathbb{R}^s \), which is symmetric, affine in each of these variables, and such that \( p \) is equal to \( P \) on the diagonal, i.e., \( P(x, \ldots, x) = p(x), x \in \mathbb{R}^s \). For \( n = 0 \), we define \( P := p \).

We are now ready to state the announced result, that every polynomial \( p \in \Pi_n \) can be expressed as a linear combination of the simplex splines whose knot-sets are Delaunay configurations of degree \( n \).
**Theorem 6.** Let \( p \in \Pi_n \) and let \( P \) be the polar form of \( p \). Also, let \( \Delta_n \) be the Delaunay configuration of degree \( n \) of the set of knots \( K \subset \mathbb{R}^s \). Then

\[
p = \sum_{X \in \Delta_n} P(X_I)N(\cdot | X),
\]

where \( N(\cdot | X) \) are normalized simplex splines, defined by

\[
N(\cdot | X) := \left( \frac{n+s}{s} \right)^{-1} \text{vol}_s [X_B] M(\cdot | X),
\]

and \( M(\cdot | X) \) is the simplex spline with knots \( X_B \cup X_I \). In particular,

\[
\Pi_n \subset \mathcal{S}_n,
\]

where \( \mathcal{S}_n \) is the spline space corresponding to \( \Delta_n \), i.e.,

\[
\mathcal{S}_n := \text{span}\{M(\cdot | X), X \in \Delta_n\}.
\]

**Proof:** The theorem is proved in [95]. However, let us remark that for \( n = 0 \), assertion (9) follows from the fact that \( N(\cdot | X) \) is the characteristic function of the Delaunay simplex \([X_B]\) and that such simplices form a partition of \( \mathbb{R}^s \). Hence, (9) is just a restatement of (7).

For \( n \geq 1 \), the proof employs the identity

\[
\sum_{X \in \Delta_n} P(X_I)N(\cdot | X) = \sum_{X' \in \Delta_{n-1}} P(X'_I, x)N(\cdot | X').
\]

This identity can be proved using recurrence (3) along with some combinatorial properties of Delaunay configurations. Namely, there is a natural way to use (3) in our setting to evaluate \( N(\cdot | X) \), which is a scalar multiple of \( M(\cdot | X) \). In particular, we can choose the coefficients \( \lambda_y, y \in X \), in (3) to be such that all but \( s + 1 \) of them are zero, where the nonzero values are associated with the \( s + 1 \) knots \( y \in X_B \). Since these knots are in general position, it is well known that the coefficients \( \lambda_y \) are uniquely determined. The recurrence relation allows us to rewrite the left-hand side of (11) as a linear combination of simplex splines of degree \( n - 1 \). The proof of (11) is completed by a judicious manipulation of this linear combination. Next, iterating identity (11) \( n \) times, one obtains

\[
\sum_{X \in \Delta_n} P(X_I)N(x | X) = \sum_{X'' \in \Delta_{n-2}} P(X''_I, x, x)N(\cdot | X'') = \ldots
\]

\[
= \sum_{X''' \in \Delta_{n-3}} P(x, \ldots, x)N(x | X''')
\]

\[
= p(x) \sum_{X'''' \in \Delta_n} \chi[X''''\setminus x]N(x | X''') = p(x),
\]
for all $x \in \mathbb{R}^s$, where we again used that $\Delta_0$ is a tessellation of $\mathbb{R}^s$. □

An immediate consequence of (10) is that the normalized simplex splines add up to one. This follows from (9) by setting $p \equiv 1$, or $P \equiv 1$. The geometric nature of the normalization is illustrated for a Delaunay configuration of degree three in Figure 22.

**Corollary 7.** The normalized simplex splines form a partition of unity, i.e.,

$$ \sum_{X \in \Delta_n} N(\cdot|X) = 1. $$

### 4.6. The Univariate Case

It will be interesting to compare identities (9) and (10) with their univariate counterparts. Consider an increasing sequence of knots $K = \{x_i\}_{i \in \mathbb{Z}} \subset \mathbb{R}$. The univariate normalized B-spline (see [136]) with knots $x_i, \ldots, x_{i+n+1}$ is given by

$$ N(\cdot|x_i, \ldots, x_{i+n+1}) = \frac{x_{i+n+1} - x_i}{n+1} M(\cdot|x_i, \ldots, x_{i+n+1}), $$

where $M$ is the univariate B-spline of degree $n$, normalized to have unit integral. Observe that (12) is the exact analog of (10). This is because the knot-set $\{x_i, \ldots, x_{i+n+1}\}$ can be thought of as an element of the Delaunay configuration of $K$ of degree $n$, with $X = (X_B, X_I)$, where the “boundary” knots are $X_B = \{x_i, x_{i+n+1}\}$ and the “interior” knots are given as $X_I = \{x_{i+1}, \ldots, x_{i+n}\}$. Note that the Delaunay configuration of degree $n$ of the knot-set $K$ is precisely the set of all $(n+2)$-tuples of consecutive knots from $K$. Thus, setting $s = 1$ and noticing that $\text{vol}_1[X_B] = x_{i+n+1} - x_i$, shows that (10) reduces to (12).

The univariate counterpart of (9) is

$$ p = \sum_{i \in \mathbb{Z}} P(x_{i+1}, \ldots, x_{i+n}) N(\cdot|x_i, \ldots, x_{i+n+1}), $$

where $p$ is now a univariate polynomial of degree at most $n$, and $P$ is the polar form of $p$. This is a well-known polynomial reproduction formula for univariate splines, see [135].
Fig. 23. Two different Delaunay configurations of degree one with identical knots.

§5. Are Simplex Splines B-Splines?

Does the spline space $S_n$ introduced in the previous section provide a solution to the fundamental problem? We have seen in Section 4.6 that $S_n$ reduces to the univariate splines for $s = 1$ and hence requirement (F) is met. Conditions (A) and (B) are consequences of the property of simplex splines that, for generic knots, they are piecewise polynomials of optimal smoothness. Requirement (C) is the assertion of Theorem 6. As for (D), this follows from the fact that for any given compact set $\Omega \subset \mathbb{R}^d$ there is a finite number of simplex splines overlapping $\Omega$, which in turn is a consequence of the local nature of Delaunay configurations [95].

Unfortunately, the collection of simplex splines $B_n := \{M(\cdot|X), X \in \Delta_n\}$, that spans $S_n$, may not satisfy property (E). This is because the simplex splines in $B_n$ can be dependent, i.e., these splines are not necessarily B-splines. The reason for this is that some Delaunay configurations might be different as pairs and yet the same as sets. In particular, there could be two different configurations $(X^1_B, X^1_I)$ and $(X^2_B, X^2_I)$, corresponding to the same set of points, i.e., $X^1_B \cup X^1_I = X^2_B \cup X^2_I$, but $X^1_B \neq X^2_B$, $X^1_I \neq X^2_I$. An example of this situation is displayed in Figure 23, depicting four knots from a set $K$ (the remaining knots from $K$ are not shown in the figure, but are assumed to lie outside both circles). In this figure, two Delaunay configurations are formed from the four knots, namely, $X^1 = (X^1_B, X^1_I) = \{\{1, 2, 3\}, \{4\}\}$ and $X^2 = (X^2_B, X^2_I) = \{\{1, 3, 4\}, \{2\}\}$. Hence, $M(\cdot|X^1)$ and $M(\cdot|X^2)$ are trivially dependent since they are identical.

One way to deal with the above difficulty is to identify all such dependent splines in $B_n$. Equivalently, instead of using the Delaunay configurations introduced in Definition 5, one could consider “un-oriented” Delaunay configurations, consisting of knot-sets of the form $X_B \cup X_I$, instead of the pairs $(X_B, X_I)$. Indeed, it turns out that by removing multiple entries in the collection $B_n$, the resulting simplex splines are B-splines in the sense of property (E) and, in particular, these splines are independent. Thus, with this straightforward modification of $B_n$, the space $S_n$ is a solution to the fundamental problem. Note, however, that for certain purposes it may still be important to treat the configurations of knots that are the same as sets, as being different, for example when employing identity (9).

Before we address another possibility of how to get rid of the described dependencies in the collection $\{M(\cdot|X), X \in \Delta_n\}$, let us recall an interesting
identity for univariate splines, which is a generalization of the reproduction formula (13), see [135]. Let \( f \) be a univariate spline of degree \( n \) with knots \( K = \{ x_i \}_{i \in \mathbb{Z}} \). Then

\[
f = \sum_{i \in \mathbb{Z}} F_i(x_{i+1}, \ldots, x_{i+n}) N(|x_i, \ldots, x_{i+n+1}|). \quad (14)
\]

Here, the symbol \( F_i \) stands for the polar form of the polynomial \( f|_{(x_j, x_{j+1})} \), the restriction of \( f \) to an interval \( (x_j, x_{j+1}) \), where \( j \) is any integer such that \( i \leq j \leq i + n \). Equivalently, \( F_i \) can be taken as the polar form of any polynomial piece of \( f \) that “lives” inside the interval \([x_i, x_{i+n+1}]\), the support of the B-spline \( M(|x_i, \ldots, x_{i+n+1}|) \). In spite of the freedom that we have to choose \( F_i \), formula (14) is well defined since one can show that the value of \( F_i \) at the knots \( x_{i+1}, \ldots, x_{i+n} \) is the same, no matter which interval \( (x_j, x_{j+1}) \) is picked for the definition of \( F_i \).

Identity (14) allows us to express any univariate piecewise polynomial explicitly as a linear combination of B-splines. We have already encountered a multivariate analog of this identity in this paper. Namely, we have seen in Section 3.4 that formula (6) makes it possible to express every \( f \in \Pi_n(\Delta) \), i.e., every optimally smooth piecewise polynomial on a triangulation \( \Delta \), as a linear combination of simplex splines. However, formula (6) is not a full analog of (14) since it holds only for single polynomials or piecewise polynomials on triangulations, but not for all DMS-splines.

Thus, a natural question is whether identity (14) has a generalization for our new type of splines. It turns out (see Theorem 9 below) that, just as with the space of DMS-splines, the space \( \mathcal{S}_n \) introduced in Section 4 also contains \( \Pi_n(\Delta) \), where \( \Delta = \Delta_0 \). As a consequence, (9) can be extended to all piecewise polynomials from \( \Pi_n(\Delta) \). However, again, there does not seem to be an extension of (9) that would express every spline in \( \mathcal{S}_n \) as a combination of simplex splines from \( \mathcal{B}_n \). An additional aesthetic flaw of (9) is that the arguments \( X_I \) in (9) are equal for all simplex splines with the same interior knots, and hence those simplex splines have identical coefficients in this formula. Therefore, the simplex splines in \( \mathcal{B}_n \) are “redundant”, in some sense.

This brings us to a second possibility of how one can handle the linear dependence of the simplex splines \( \{ M(|X), X \in \Delta_n \} \). We could define a new set of compactly-supported splines by adding all simplex splines in \( \{ N(|X), X \in \Delta_n \} \) sharing the same set of interior knots. The linear span of these new functions, denoted by \( \mathcal{S}'_n \), will be a smaller space of splines, i.e., \( \mathcal{S}'_n \) will be a proper subspace of \( \mathcal{S}_n \). Nevertheless, \( \mathcal{S}'_n \) will still be large enough since it will contain all polynomials of degree \( n \).

Definition 8. Let \( n \geq 1 \), \( I_n := \{ X_I : X = (X_B, X_I) \in \Delta_n \} \), and \( I \in I_n \). A multivariate B-spline \( B_I \) is the function defined as

\[
B_I := \sum_{X \in \Delta_n, X_I = I} N(|X|).
\]
It is an immediate consequence of (9) that the B-splines \( \mathcal{B}_n^I := \{B_I, I \in \mathcal{I}_n\} \) form a partition of unity and the reproduction formula (9) can be written as
\[
p = \sum_{I \in \mathcal{I}_n} P(I)B_I, \quad p \in \Pi_n,\]
which confirms that the new space \( \mathcal{S}_n^I := \text{span}\{B_I, I \in \mathcal{I}_n\} \subset \mathcal{S}_n \) still contains \( \Pi_n \). In fact, this space, together with the collection \( \mathcal{B}_n^I \), also solves the fundamental problem. Note that in the univariate case, there is only one simplex spline per set \( I \) of interior knots. Hence, the new multivariate B-splines reduce to the univariate B-splines for \( s = 1 \), which establishes property (F).

We finish this section with a striking generalization of formula (9), which, unlike (6), is a complete analog of (14). Let \( f \in \mathcal{S}_n^I \) and \( I \in \mathcal{I}_n \). Just as in the univariate case, it can be shown that if \( f_I \) is a single polynomial piece of \( f \) on a region in the support of \( B_I \), bounded but not intersected by any hyperplane containing \( s \) knots, then the polar form \( F_I \) of \( f_I \), evaluated at \( X_I \), does not depend on which polynomial piece \( f_I \) we pick. That is, the number \( F_I(X_I) \) is well defined as long as the polynomial \( f_I \) corresponds to a region inside the support of \( B_I \). With this notation, we obtain

**Theorem 9** [95]. Every spline in the space \( \mathcal{S}_n^I \) can be written uniquely as a linear combination of B-splines. Namely,
\[
f = \sum_{I \in \mathcal{I}_n} F_I(I)B_I,\]
for every \( f \in \mathcal{S}_n^I \). Moreover, the space \( \mathcal{S}_n^I \) contains all optimally smooth piecewise polynomials of degree \( n \), associated with the Delaunay triangulation \( \Delta \), i.e., \( \Pi_n(\Delta) \subset \mathcal{S}_n^I \).

§6. Conclusions

The results of the previous two sections show that simplex splines can give rise to interesting spaces of multivariate splines. In particular, our construction yields spline spaces for which polynomials can be represented explicitly in an elegant and simple way. This representation is virtually identical to the univariate case, which is in contrast to earlier constructions, such as the construction of DMS-splines in [32].

On the other hand, we leave many obvious questions unanswered, such as: How can one compute Delaunay configurations?, Can the splines be efficiently and stably evaluated by means of a “de Boor-like” algorithm?, Do the splines have good approximation properties?

While the new splines seem very natural in many respects, it is also not clear at this time how useful they might become for applications. In this paper, I viewed the described fundamental problem as a purely mathematical problem, whose consideration may or may not have practical significance. In my opinion a natural construction of splines should be mathematically elegant.
and conceptually simple. However, this does not necessarily mean simplicity in the computational sense. Therefore, at this stage of the investigation it is too early to tell whether the splines constructed here can compete successfully with other types of multivariate splines currently in use. Nevertheless, I hope that the described results will spark sufficient interest to continue the research along the new directions outlined in this paper.

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