Homogeneous Simplex Splines

by

Marian Neamtu 1)

Abstract. Homogeneous simplex splines, also known as cone splines or multivariate truncated power functions, are discussed from a perspective of homogeneous divided differences and polar forms. This makes it possible to derive the basic properties of these splines in a simple and economic way. In addition, a construction of spaces of homogeneous simplex splines is considered, which in the non-homogeneous setting is due to Dahmen, Micchelli, and Seidel. A proof for this construction is presented, based on knot insertion. Restricting the homogeneous splines to a sphere gives rise to spaces of spherical simplex splines.

1. Introduction

In this paper we focus on a class of multivariate splines, called homogeneous simplex splines. Perhaps better known as cone splines, or multivariate truncated powers, these functions were introduced in [7] and further studied e.g., in [6,14,5]. Our denomination for these splines is justified by the fact [8] that their restrictions to hyperplanes are essentially the well-known simplex splines, hereafter termed affine simplex splines.

The motivation for our study stems from a recent work [3,4], where a spherical analog of piecewise polynomials (defined on a triangulation of a planar region) has been introduced and analysed. These spherical splines are built from functions which are the restrictions of trivariate homogeneous Bernstein basis polynomials to the surface of a sphere. Here we go one step further and consider the homogeneous simplex splines, which are their direct generalization. In connection with multivariate polynomial interpolation, homogeneous simplex splines have been considered in [13]. Their univariate counterparts, the homogeneous B-splines, have been investigated in [11] and [24].

1) Department of Mathematics, Vanderbilt University, Nashville, TN 37240, U. S. A. neamtu@orpheus.cas.vanderbilt.edu, http://math.vanderbilt.edu/~neamtu. Partially supported by Vanderbilt University Research Council.
Our objective is to present a concise account of homogeneous simplex splines based on a systematic use of the functional of divided differences [16], in conjunction with polar forms [19]. Although many of the results are straightforward and expected generalizations of known facts about affine splines, in our opinion, the homogeneous (projective) setting allows for a more elegant treatment which enables a better insight into the structure of these functions. For example, truncated powers can be viewed as simplex splines with knots at infinity, whereas their affine description is complicated [16]. Moreover, we find the proofs presented here generally simpler than the ones obtained previously by other means. In fact, many of the basic properties of simplex splines are almost trivial consequences of the main definition (Definition 4.1 below) in which a simplex spline is defined in terms of divided differences. This resembles the traditional definition of univariate B-splines as a linear combination of truncated powers.

Finally, and perhaps most importantly, the framework of homogeneous splines makes it possible to define spaces of splines on an arbitrary differentiable manifold, not necessarily a hyperplane. For example, by restricting homogeneous splines to a sphere we obtain spherical simplex splines, introduced in [17], which we believe may prove useful in some important areas of application such as geophysics and other geosciences. Some experience with these splines has been reported in [17].

2. Polar Forms

Let \( \mathcal{H}_k \) be the \( \binom{k+s}{s} \)-dimensional space of homogeneous polynomials of degree \( k \) on \( \mathbb{R}^{s+1} \). For the sake of brevity, from now on we shall drop the adjective 'homogeneous', and will refer to these functions simply as polynomials. By \( F \) we denote the polar form of \( f \in \mathcal{H}_k \) [19]. Thus the arguments of \( F \) form a suite \( X \subset \mathbb{R}^{s+1} \) whose cardinality \( \#X \) equals the degree \( k \). A suite or a multiset is a collection of elements, some of which may occur in this collection with multiplicities (and therefore it is not a set, see [20]). The order of elements in \( X \) is irrelevant since \( F \) is symmetric. Moreover, \( F \) is multi-linear i.e., linear in each of its arguments. We follow the convention that linearity implies homogeneity. Finally, \( F \) is diagonal i.e., \( F(X) = f(x) \) whenever \( X \) consists of \( k \) copies of \( x \in \mathbb{R}^{s+1} \).

For later use, we recall the conditions on the order of contact of two polynomials at a point. Polynomials \( f \) and \( g \) of degree \( k \) have contact of order \( 0 \leq m \leq k \) at \( x \), if \( f - g \) and all its derivatives of (total) order up to \( m \) vanish at \( x \). Setting \( X \) to be the suite containing \( k - m \) copies of \( x \), this property is equivalent to the condition [18]

\[
F(X \cup Y) = G(X \cup Y),
\]

which should hold for all suites \( Y \subset \mathbb{R}^{s+1} \), with \( \#Y = m \).
3. Homogeneous Divided Differences

A useful tool in our investigation of homogeneous simplex splines, will be an analog of multivariate divided differences [16], defined below. We will use subscripts 1, . . . , s + 1 to denote Cartesian coordinates of vectors in $\mathbb{R}^{s+1}$, so that $x \in \mathbb{R}^{s+1}$ is given by $x = (x_1, \ldots, x_{s+1})^T$. Superscripts will count vectors in a given collection of vectors. Let $X$ be a set of vectors in $\mathbb{R}^{s+1}$ with cardinality $\#X = n + 1, n \geq s$. The elements of $X$ are called knots. We assume that the knots are in generic position, i.e., such that every subset of $X$ of cardinality $s + 1$ forms a collection of linearly independent vectors. In particular, the zero vector $0$ is not contained in $X$.

For an ordered set (or sequence) $Y = \{y_1, \ldots, y_s\} \subset \mathbb{R}^{s+1}$ and a vector $y \in \mathbb{R}^{s+1}$, we define

$$d(Y, y) := \det \begin{pmatrix} y_1 & \cdots & y^i & y_1 \\ \vdots & \ddots & \vdots & \vdots \\ y^1_{s+1} & \cdots & y^i_{s+1} & y_{s+1} \end{pmatrix}.$$  

If $V$ is the sequence $\{y^1, \ldots, y^s, y\}$, we abbreviate $d(Y, y)$ as $d(V)$, and for a finite suite $Z$, we set $d(Y, Z) := \prod_{z \in Z} d(Y, z)$ and $Y \setminus Z := \{y \in Y, y \notin Z\}$. In order not to be overburdened by parentheses, we write $Y \setminus Z$, $Y \cup Z$, and $Y \setminus z \cup x$, instead of $Y \setminus \{z\}$, $Y \cup \{x\}$, and $(Y \setminus z) \cup x$, respectively.

If $Y$ is fixed, $d(Y, y)$ is a linear function in $y$, whereas fixing $y$ gives a multilinear function in $Y$. This suggests to define a 'dual' space of $\mathcal{H}_k$ as

$$\mathcal{H}_k^* := \text{span}\{d^k(\cdot, y), y \in \mathbb{R}^{s+1}\},$$  

where span($A$) denotes the linear span of a set $A$. Thus $\mathcal{H}_k^*$ contains functions which are homogeneous polynomials of degree $k$ in each of the variables contained in $Y$. In case $s = 1$, the two spaces $\mathcal{H}_k$ and $\mathcal{H}_k^*$ coincide. For $s > 1$, $\mathcal{H}_k$ and $\mathcal{H}_k^*$ are isomorphic and hence

$$\dim \mathcal{H}_k^* = \binom{k + s}{s}.  \hspace{2cm} (3.2)$$

This is obvious from the fact that for every $Y$ there is a unique vector $y \in \mathbb{R}^{s+1}$ such that $d(Y, x) = y \cdot x$, for all $x \in \mathbb{R}^{s+1}$, and the fact that $\mathcal{H}_k = \text{span}\{d^k(Y, \cdot), Y \subset \mathbb{R}^{s+1}, \#Y = s\}$.

Let $f(Y)$ be a real-valued function of $Y \subset \mathbb{R}^{s+1}, \#Y = s$, such that for $s > 1$, it holds

$$f(\ldots, y^i, \ldots, y^i, \ldots) = (-1)^{k+1} f(\ldots, y^i, \ldots, y^i, \ldots), \quad i \neq j,  \hspace{2cm} (3.3)$$

where for the remainder of the paper, the numbers $s, k$, and $n$ are related by the identity $n = k + s$. Note that every $f \in \mathcal{H}_k^*$ satisfies (3.3).
Definition 3.1. The divided difference of a function $f$ satisfying (3.3) with knots $X$ is defined as

$$[X]f := \sum_{Y \subseteq X \setminus z} \frac{f(Y)}{d(Y, X \setminus Y)}. \quad (3.4)$$

Note that (3.3) ensures that (3.4) is well defined since each term on the right-hand side of (3.4) is independent of the ordering of $Y$ (the products $d(Y, X \setminus Y)$ appearing in (3.4) contain $k + 1$ terms, each of which is antisymmetric, and hence $f(Y)/d(Y, X \setminus Y)$ is symmetric).

Clearly, $[X]$ is a linear functional which does not depend on the ordering of $X$. Moreover, $[RX](f \circ R^{-1}) = [X]f$, whenever $R$ is a linear isometric transformation on $\mathbb{R}^{s+1}$. We have the following useful identity.

Lemma 3.2. Let $n > s$ and $z \in X$. Then

$$[X]d(\cdot, z)f(\cdot) = [X \setminus z]f. \quad (3.5)$$

Proof: By definition,

$$[X]d(\cdot, z)f(\cdot) = \sum_{Y \subseteq X \setminus z} \frac{d(Y, z)f(Y)}{d(Y, X \setminus Y)} = \sum_{Y \subseteq X \setminus z} \frac{f(Y)}{d(Y, X \setminus (Y \cup z))} = [X \setminus z]f,$$

where we used the fact that $d(Y, z) = 0$ if $z \in \text{span}(Y)$. □

Corollary 3.3. Let the numbers $\lambda_y, y \in X$, be such that

$$\sum_{y \in X} \lambda_y y = x. \quad (3.6)$$

Then

$$[X]d(\cdot, x)f(\cdot) = \sum_{y \in X} \lambda_y [X \setminus y]f. \quad (3.7)$$

Next, we present an interesting identity for 'polar interpolation', expressing a polar form in terms of divided differences.

Proposition 3.4. Let $X$ be a collection of $n + 1 = k + s + 1$ knots. Then there exists a unique polynomial $p \in \mathcal{H}_{k+1}$ whose polar form $P$ attains prescribed values at all subsets of $X$ of cardinality $k + 1$. This polar form is given by

$$P(Z) = [X]d(\cdot, Z)P(X \setminus \cdot), \quad Z \subset \mathbb{R}^{s+1}, \#Z = k + 1. \quad (3.8)$$
**Proof:** To show the existence of $P$, observe that the right-hand side of (3.8) is symmetric and multi-linear in $Z$. Therefore it suffices to show that (3.8) also satisfies the interpolation conditions i.e., that for all $Z \subset X, \#Z = k + 1$,

$$P(Z) = \sum_{Y \subset X \atop \#Y = s} \frac{d(Y, Z)}{d(Y, X \setminus Y)} P(X \setminus Y). \quad (3.9)$$

However, this is obvious from

$$\frac{d(Y, X \setminus W)}{d(Y, X \setminus Y)} = \delta_{YW}, \quad Y, W \subset X, \#Y = \#W = s. \quad (3.10)$$

The uniqueness part of the assertion follows once we have shown that if $P(Z) = 0$, for all $Z \subset X, \#Z = k + 1$, then $P$ must be identically zero. Therefore, consider a set $V \subset X, \#V = k$ and let $x \in \mathbb{R}^{s+1}$. Moreover, let $\lambda_v, v \in X \setminus V$ be numbers such that

$$\sum_{v \in X \setminus V} \lambda_v v = x,$$

which clearly exist since the points in $X \setminus V$ are in generic position. The polar form $P$ is multi-linear and hence

$$P(V \cup x) = \sum_{v \in X \setminus V} \lambda_v P(V \cup v) = 0.$$

In a similar way we can prove (e.g., by induction) that $P(V \cup U) = 0$ for all $V \subset X, \#V = k - \ell, \ell = 1, \ldots, k$, where $U$ is the suite consisting of $\ell + 1$ copies of $x$. In particular, this means $P(x, \ldots, x) = p(x) = 0$, and thus $P$ is identically zero.

Next we construct a basis for the space $\mathcal{H}_{k+1}$ which is more general than the one in (3.1). The result below is a direct consequence of (3.10).

**Proposition 3.5.** Let $\mathcal{X} := \{d(\cdot, X \setminus Y), Y \subset X, \#Y = s\}$. Then

$$\dim \text{span}(\mathcal{X}) = \dim \mathcal{H}_{k+1}.$$

**Proof:** By (3.10), the functions in the collection $\mathcal{X}$ are linearly independent and hence $\dim \text{span}(\mathcal{X}) = \#\mathcal{X} = \binom{k+s+1}{s}$. ■

A ‘dual’ form of the polar interpolation stated in Proposition 3.4 is the following Lagrange interpolation for functions from the space $\mathcal{H}_{k+1}$.
Proposition 3.6. Let $X$ be a collection of $n + 1 = k + s + 1$ knots. Then there exists a unique polynomial $p \in \mathcal{H}_{k+1}^s$ attaining prescribed values $p(Y)$ for all $Y \subset X, \#Y = s$, given by

$$p(W) = [X]d(W, X\setminus \cdot)p(\cdot), \quad W \subset \mathbb{R}^{s+1}, \#W = s. \quad (3.11)$$

Proof: Note first that the value $p(Y)$ may depend on the ordering of $Y$. However, the expression (3.11) accommodates this fact since expanding the divided difference $[X]$ leads to an expression of the form

$$\sum_{Y \subseteq X \atop \#Y = s} \frac{d(W, X\setminus Y)}{d(Y, X\setminus Y)}p(Y), \quad (3.12)$$

which is well defined since the knots are in generic position. Obviously, each of the terms in this expression is independent of the ordering of $Y$ since $p \in \mathcal{H}_{k+1}^s$. Moreover, the coefficients in (3.12) satisfy

$$\frac{d(W, X\setminus Y)}{d(Y, X\setminus Y)} = \delta_{YW}, \quad Y, W \subset X, \#Y = \#W = s.$$ 

This shows that the right-hand side of (3.11) is a polynomial in $\mathcal{H}_{k+1}^s$ satisfying the interpolation conditions. Moreover, by Proposition 3.5 the functions $d(\cdot, X\setminus Y)$, $Y \subset X, \#Y = s$, are linearly independent and form a basis for $\mathcal{H}_{k+1}^s$. Thus the polynomial satisfying the interpolation conditions is unique. \[\square\]

Corollary 3.7. Suppose the polynomial $p \in \mathcal{H}_{k+1}^s$ is such that $p(Y) = 0$ for all $Y \subset X, \#Y = s$. Then $p$ is identically zero.

4. Homogeneous Simplex Splines

We now employ divided differences to define homogeneous simplex splines, henceforth referred to as simplex splines. Suppose $X$ is a set of $n + 1$ knots in generic position such that $\text{cone}(X) := \{x = \sum_{y \in X} \lambda_y y, \lambda_y \in \mathbb{R}_+\}$, the conical hull of $X$, is a proper cone i.e., a cone in $\mathbb{R}^{s+1}$ not containing a nontrivial linear subspace. Equivalently, the convex hull $\text{conv}(X)$ of $X$ does not contain the zero vector. Also, let $z \in \mathbb{R}^{s+1} \setminus 0$ (see Remark 1).

Definition 4.1. The (homogeneous) simplex spline $M(\cdot|X)$ of degree $k = n - s$ with knots $X$ is a function, defined for $k = 0$ and $x \in \mathbb{R}^{s+1}$, by

$$M(x|X) := \frac{\chi_x(x)}{|d(X)|^1}, \quad (4.1)$$
where \( \chi_x \) is the characteristic function of the simplicial cone \( \text{cone}(X) \). For \( k > 0 \),
\[
M(x|X) := [X]d^k(\cdot, x)d(\cdot, z)M(x| \cup z). \tag{4.2}
\]
If \( d(Y, z) = 0 \), for some \( Y \subset X \), \#\( Y = s \), the corresponding term in (4.2) (obtained by expanding the divided difference as in (3.4)) is discarded.

To justify this definition, we need to show that the simplex spline \( M \) does not depend on the choice of \( z \). Let us first observe that (4.1) and (4.2) are consistent in the sense that (4.2) remains valid for \( k = 0 \). We omit the technical details of the proof of this intuitively clear fact (however, see Remark 1).

**Lemma 4.2.** Let \( n = s \) and let \( z \in \mathbb{R}^{s+1} \setminus 0 \). Then
\[
M(x|X) = [X]d(\cdot, z)M(x| \cup z), \quad x \in \mathbb{R}^{s+1}.
\]

Using (3.5), (3.7), and Lemma 4.2, the following two results are immediate consequences of Definition 4.1.

**Lemma 4.3.** If \( n > s \), then
\[
M(x|X \cup x) = M(x|X). \tag{4.3}
\]

**Proposition 4.4.** Let \( u \in \mathbb{R}^{s+1} \setminus 0 \) and let \( \lambda_y, y \in X \) be such that
\[
\sum_{y \in X} \lambda_y y = u. \tag{4.4}
\]

Then
\[
M(x|X) = \sum_{y \in X} \lambda_y M(x|X \setminus y \cup u). \tag{4.5}
\]

Setting \( u = x \) in Proposition 4.4 and invoking identity (4.3), readily proves

**Corollary 4.5.** Let the numbers \( \lambda_y, y \in X \), be given as in (3.6). Then
\[
M(x|X) = \sum_{y \in X} \lambda_y M(x|X \setminus y). \tag{4.6}
\]

The independence of \( M \) on the choice of \( z \) is now clear from Corollary 4.5 by an induction argument on \( n \). In fact, it follows from the recurrence (4.6) that the functions \( M \) defined here are identical with the well-known cone splines or multivariate truncated power functions since these also satisfy (4.1) and (4.6) (see Remark 2).
A special case of the knot insertion formula (4.5) can be obtained as follows. Suppose $V \subset \mathbb{R}^{s+1}$, $\#V = s + 1$, is a set of vectors in generic position. Then for every $u \in \mathbb{R}^{s+1}$, it holds

$$u = \sum_{v \in V} b_v(u|V)v, \quad (4.7)$$

where

$$b_v(u|V) := \frac{d(V\setminus v, u)}{d(V\setminus v, v)}, \quad v \in V. \quad (4.8)$$

The functions $b_v$ can be viewed as homogeneous barycentric coordinates of $u$ with respect to the set $V$ [3]. Note that these are the unique functions from $\mathcal{H}_1$ with the property

$$b_v(u|V) = \delta_{u,v}, \quad u, v \in V,$$

and thus they are linearly independent. For $V \subset X$, equation (4.7) is a special case of (4.4) and hence (4.5) reduces to

$$M(x|X) = \sum_{v \in V} b_v(u|V)M(x|X \cup v \cup u). \quad (4.9)$$

5. Properties of Homogeneous Simplex Splines

Simplex splines are by now well understood and it is not our intention here to rederive or redevelop their entire theory. Rather, we want to show how a number of properties of these functions easily (in fact, almost trivially) follows from Definition 4.1.

It is clear from (4.2) that $M$ is a piecewise polynomial of degree $k$, which is also homogeneous of the same degree. To elaborate on the smoothness of $M$, we first give an explicit formula for polar forms of the polynomial pieces corresponding to $M$. Suppose $U$ is a $p$-region of $M$ i.e., a region in $\mathbb{R}^{s+1}$ which is not intersected by any face cone$(Y), Y \subset X, \#Y = s$. Moreover, let $p_U$ be the polynomial $M|_U$ obtained by restricting $M$ to $U$, and let $u$ be an arbitrary point in int$(U)$, the interior of $U$.

**Lemma 5.1.** The polar form of $p_U$ is

$$P_U(Z) = [X]d(\cdot, Z)d(\cdot, z)M(u \cup z), \quad \#Z = k. \quad (5.1)$$

**Proof:** The symmetry of $P_U$ and its multi-linearity are obvious. We show the remaining diagonal property by induction on $k$. For $k = 0$, the assertion follows by Lemma 4.2 since in that case, $U$ is just the interior of cone$(X)$ and hence $M(u|X) = M(x|X)$ for all $x$ in this interior. Using one step of the recurrence (4.6), the induction step is obvious.

A simple consequence of the explicit formula (5.1) is the familiar fact that $M$ is a function of optimal smoothness.
Corollary 5.2. Let $X$ be such that $k > 0$ i.e., $\#X = k + s + 1 > s + 1$. Then $M(\cdot | X)$ is a $C^{k-1}$ function.

Proof: We prove that $M$ is $C^{k-1}$ smooth across every face cone($W$), where $W \subset X, \#W = s$. Let $U$ and $U'$ be two neighbouring p-regions of $M$ such that $U \cap U' \subset$ cone($W$), and let $u \in \text{int}(U), u' \in \text{int}(U')$. Also, let $x \in U \cap U'$. By Lemma 5.1 and (2.1), we must show

$$[X]d(\cdot, Z)d(\cdot, z)(M(u | \cdot \cup z) - M(u' | \cdot \cup z)) = 0,$$

for all $Z$ such that $x \in Z$. For this, it will be convenient to choose $z$ such that $z \in X \setminus W$, and hence by (3.5), (5.2) becomes

$$[X \setminus z]d(\cdot, Z)(M(u | \cdot \cup z) - M(u' | \cdot \cup z)) = 0,$$

or

$$\sum_{Y \subset X \setminus z \atop \#Y = s} \frac{d(Y, Z)(M(u | Y \cup z) - M(u' | Y \cup z))}{d(Y, X \setminus (Y \cup z))} = 0. \quad (5.3)$$

We show that each term in (5.3) vanishes. We consider two cases. If $Y \neq W$ then, by the definition of $U$ and $U'$, we have $M(u | Y \cup z) - M(u' | Y \cup z) = 0$. This is because by the choice of $z$, either both $u$ and $u'$ are contained in cone($Y \cup z$) or they both fall outside this region. On the other hand, if $Y = W$, then $d(Y, Z) = 0$ (since $x \in Z$ and $x \in \text{cone}(W)$), and so the corresponding term in (5.3) also vanishes. \hfill \blacksquare

Next, we consider collections of simplex splines and discuss their linear independence. Suppose $K$ is a set of $m + 1$ knots in generic position, where $m \geq n \geq s$. Let $S := \{M(\cdot | L), L \subset K, \#L = n + 1\}$. This is the set of the so-called complete configurations of simplex splines associated with $K$. In the affine setting, this collection has been considered in [8]. The first result expresses linear dependence relations for simplex splines in terms of similar relations for certain multivariate polynomials in the dual space $\mathcal{H}^{*}_{m-k}$.

Proposition 5.3. A collection of numbers $c_L \in \mathbb{R}, L \subset K, \#L = n + 1, \#L = n + 1$, satisfies

$$\sum c_L M(x | L) = 0, \quad x \in \mathbb{R}^{s+1}, \quad (5.4)$$

if and only if

$$\sum c_L d(Y, K \setminus L) = 0, \quad Y \subset \mathbb{R}^{s+1}, \#Y = s, \quad (5.5)$$

where it is understood that the sums are taken over all $L \subset K, \#L = n + 1$.

Proof: Using (3.5) and (4.2), it follows that (5.4) is equivalent to

$$[K] \left( \sum c_L d(\cdot, K \setminus L) \right) d^{\#}(\cdot, x)d(\cdot, z)M(x | \cdot \cup z) = 0. \quad (5.6)$$
Expanding the divided difference in this expression, we obtain a linear combination of functions of the form $d^k(Y, x)M(x|Y \cup z), Y \subset K, \#Y = s$. Assuming that $z$ is in generic position with respect to the set $K$, these functions (viewed as functions of $x$) are linearly independent since each but one of them is a $C^\infty$ function across a fixed cone($Y$). Note that $d^k(Y, x)M(x|Y \cup z)$ is only $C^{k-1}$ across cone($Y$). Hence, (5.6) can be zero only if
\[
\sum c_Ld(Y, K\setminus L) = 0, \quad (5.7)
\]
for all $Y \subset K, \#Y = s$. However, by Corollary 3.7 this means that (5.7) is true for all $Y \subset \mathbb{R}^{s+1}, \#Y = s$. ■

**Corollary 5.4.** Let $S$ be as above. Then
\[
\dim \text{span}(S) = S := \left(\frac{m - k}{s}\right). \quad (5.8)
\]

**Proof:** By Proposition 5.3, the statement is equivalent to
\[
\dim \text{span}(K) = S,
\]
where $K := \{d(\cdot, K\setminus L), L \subset K, \#L = n + 1\}$. Since $K \subset \mathcal{H}_{m-k}^n$, it is clear by (3.2) that $\dim \text{span}(K) \leq S$. To prove that the dimension cannot be strictly less than $S$, we proceed as follows. Let $X$ and $Z$ be two disjoint sets such that $K = X \cup Z$, where $\#X = m - k$ and $\#Z = k + 1$. Consider the collection of simplex splines $\mathcal{C} := \{M(\cdot|Y \cup Z), Y \subset X, \#Y = s\}$, which is a subset of $S$ with $\#\mathcal{C} = S$. Thus the assertion will be proved once we have shown that $\dim \text{span}(\mathcal{C}) = S$. Equivalently, we can show that $\dim \text{span}\{d(\cdot, X\setminus Y), Y \subset X, \#Y = s\} = S$. However, this follows from Proposition 3.5. ■

**Corollary 5.5.** Let $S$ be as above. Suppose
\[
\sum c_LM(x|L) = 0, \quad x \in \mathbb{R}^{s+1}. \quad (5.9)
\]
Then also
\[
\sum c_LM(x|L \cup W) = 0, \quad x \in \mathbb{R}^{s+1}, \quad (5.10)
\]
for every finite $W \subset \mathbb{R}^{s+1}$.

**Proof:** Using (4.2) and Proposition 5.3 it is not difficult to see that both (5.9) and (5.10) are equivalent to (5.5). ■

We conclude the section with a generalization of (4.2).
Theorem 5.6. Let $Z$ be in generic position with respect to $X$ and such that $\#Z = k + 1$. Then

$$M(x|X) = [X]d(\cdot, Z)M(x) \cdot U Z, \quad x \in \mathbb{R}^{s+1}. \quad (5.11)$$

Proof: While the proof can be done directly by inserting the knots $Z$ into the spline $M$ using (4.5) and Corollary 5.5, we give a proof based on Corollary 5.4. First observe that the right-hand side of (5.11) is a linear combination of $S := \binom{n+1}{s}$ simplex splines from the collection $C := \{M(\cdot|Y \cup Z), Y \subset X, \#Y = s\}$. By a similar argument as in the proof of Corollary 5.4, these splines are linearly independent.

Next, consider the space which is the linear span of the collection of complete configurations of simplex splines of degree $k$ associated with knots $X \cup Z$. By Corollary 5.4, the dimension of this space equals $S$. This means the spline $M(\cdot|X)$, which belongs to this space, can be represented as a linear combination of splines from $C$. By the linear independence, this representation is unique. Thus it remains to prove that the coefficients in this representation correspond to (5.11). However, these coefficients are exactly as in polar interpolation (3.9) and hence the result follows from (3.10). □

6. Spline Spaces

In this section we generalize a construction of simplex spline spaces given in [9] to the homogeneous setting.

Let $P$ be a finite set of points in $\mathbb{R}^{s+1} \setminus 0$ and let $\mathcal{P} := \text{cone}(P)$. Suppose $\mathcal{V}$ is a partition of $\mathcal{P}$ into simplicial cones. As usual, we require that $\mathcal{V}$ be proper i.e., such that it contains no degenerate simplicial cones and such that two simplicial cones in $\mathcal{V}$ are either disjoint or share a common face. To simplify the presentation, we assume $\mathcal{P} = \mathbb{R}^{s+1}$. Thus $\mathcal{V}$ is a collection of simplicial cones $V \in \mathcal{V}$ whose union is all of $\mathbb{R}^{s+1}$. This assumption and the finiteness of $P$ are not essential and they can be removed with some additional technical assumptions. We will abuse the notation slightly and denote by $V$ both a simplicial cone in $\mathcal{V}$ and also the set of direction vectors (or knots) corresponding to this cone.

To describe the construction of spaces of homogeneous simplex splines of degree $k$, each knot $p$ in $P$ is associated with a cloud of knots of cardinality $k + 1$, which includes the knot $p$. In this way each simplicial cone $V \in \mathcal{V}$ corresponds to a collection of knots $x^{v, \beta_v}, v \in V, \beta_v = 0, \ldots, k$, where $x^{v,0} := v$. By $\beta$, we denote the ordered set $\{\beta_v\}_{v \in V} \subset \mathbb{Z}_+$ and we use the standard notation $|\beta| := \sum_{v \in V} \beta_v$. For $|\beta| = k$, we define the following sets: $X^V_{\leq \beta} := \{x^{v, \alpha_v}, \alpha_v = 0, \ldots, \beta_v, v \in V\}$, $X^V_{\beta} := \{x^{v, \beta_v}, v \in V\}$, $X^V_{\leq \beta} \setminus X^V_{\beta}$. When considering an individual simplicial cone $V$ from $\mathcal{V}$, we shall drop the superscript $V$ from the notation. We introduce an ordering of $X^V_{\beta}$ as follows. We assume the ordering of knots in each of the cones
Fig. 1. Inserting knots $z^0, \ldots, z^k$ into the spline associated with simplicial cone $V$ ($s = 2$).

$V$ to be such that $d(V) > 0$. This induces an ordering on $X_\beta$ i.e., if $V$ is ordered as $v^0, \ldots, v^s$, then $X_\beta$ is $x^{v_0, \beta v_0}, \ldots, x^{v_s, \beta v_s}$. Clearly, each collection $X_{\leq \beta}^V$ contains $n + 1$ knots and hence gives rise to a homogeneous simplex spline $M(\cdot|X_{\leq \beta}^V)$. It will be convenient to normalize this simplex spline as

$$N_{\beta}^V(\cdot) := d(X_{\beta}^V)M(\cdot|X_{\leq \beta}^V).$$

With this notation, the partition $\mathcal{V}$ together with the corresponding clouds defines a spline space

$$\mathcal{S}_k(\mathcal{V}) := \text{span}\{N_{\beta}^V, |\beta| = k, V \in \mathcal{V}\}. \quad (6.1)$$

As shown in the affine case [9], this space has many desirable properties and it is not surprising that the same is true in the homogeneous setting. One main reason for the usefulness of $\mathcal{S}_k(\mathcal{V})$ stems from the reproductive property of this space. In particular, following [9] one can show that this space contains all polynomials in $\mathcal{H}_k$. In the sequel we shall give a slightly different proof of this fact which uses knot insertion. To that end, we need some auxiliary results. Let $Z = \{z^0, \ldots, z^k\}$. For $V \in \mathcal{V}$, we consider the following linear combination

$$\sum_{|\beta| = k} c_{\beta} N_{\beta}. \quad (6.2)$$

By inserting the knots $Z$ into this spline (see Fig. 1), it is possible to express (6.2) as a linear combination of simplex splines associated with the $s + 1$ simplices
with knots $V \setminus v \cup z^0, v \in V$. Let the numbers $c^{[\ell]}_{\beta}, |\beta| = k - \ell$, be obtained by applying the following recursive scheme [21] to the sequence $c_{\beta}, |\beta| = k$. Namely,

$$
c^{[\ell]}_{\beta} := \sum_{v \in V} b_v(z^{\ell-1}|X_{\beta})c^{[\ell-1]}_{\beta+e^v}, \quad |\beta| = k - \ell, \quad \ell = 1, \ldots, k,
$$

(6.3)

where $c^{[0]}_{\beta} := c_{\beta}$, and where $e^v$ denotes the ordered set $(\delta_{u,v})_{u \in V}$. Just as in the case of polynomials in the B-form, restricting the $c^{[\ell]}_{\beta}$ to the indices $\beta$ of which at least one coordinate is zero, gives rise to the following knot insertion (or subdivision) formula, obtained for the affine setting (for $s = 2$) in [23]. First, we need some notation. By $X_{\leq \beta, v}$ we denote the set obtained from $X_{\leq \beta}$ by replacing $x^{v,0}, \ldots, x^{v,\beta_v}$ with $z^{k-|\beta|}, \ldots, z^{k-|\beta|+\beta_v}$ (in order). For example, if $|\beta| = k$, we obtain $X_{\leq \beta, v}$ by replacing $x^{v,0}, \ldots, x^{v,\beta_v}$ in $X_{\leq \beta}$ with $z^0, \ldots, z^{\beta_v}$. The definitions of $X_{\leq \beta, v}$ and $X_{\beta, v}$ are similar. Note that in this notation, $X_{\leq \beta+e^v, v} = X_{\leq \beta, v} \cup z^0$ and $X_{\beta+e^v, v} = X_{\beta, v}$, for $|\beta| = k - 1$. Furthermore, let $N_{\beta, v}$ be the normalized simplex spline with knots $X_{\leq \beta, v}$ (the normalizing factor being $d(X_{\beta, v})$).

**Proposition 6.1.** Let $c_{\beta, v} := c^{[\gamma_v]}_{\gamma}, |\beta| = k, v \in V$, where $\gamma$ is such that $\gamma_u = \beta_u$, for $u \neq v$, and $\gamma_v = 0$. Then

$$
\sum_{|\beta|=k} c_{\beta} N_{\beta} = \sum_{v \in V} \sum_{|\beta|=k} c_{\beta, v} N_{\beta, v}.
$$

(6.4)

**Proof:** We prove the assertion by induction on $k$. For $k = 0$, it is a restatement of Lemma 4.2 with $z = z^0$. Next, let $k > 0$. We will insert knot $z^0$ into the left-hand side of (6.4). The knot insertion formula (4.9) and a simple manipulation yield

$$
\sum_{|\beta|=k} c_{\beta} N_{\beta} = \sum_{|\beta|=k} c_{\beta} d(X_{\beta}) \sum_{v \in V} b_v(z^0|X_{\beta})M(\cdot|X_{\leq \beta \setminus x^{v,\beta_v} \cup z^0})
$$

$$
= \sum_{v \in V} \sum_{|\beta|=k} c_{\beta, v} N_{\beta, v}
$$

$$
+ \sum_{|\beta|=k-1} \sum_{v \in V} c_{\beta+e^v} d(X_{\beta+e^v})b_v(z^0|X_{\beta+e^v})M(\cdot|X_{\leq \beta \cup z^0})
$$

$$
= \sum_{v \in V} \sum_{|\beta|=k} c_{\beta, v} N_{\beta, v} + \sum_{|\beta|=k-1} c^{[1]}_{\beta} d(X_{\beta})M(\cdot|X_{\leq \beta \cup z^0}).
$$

The right-hand side of (6.4) can be written as

$$
\sum_{v \in V} \sum_{|\beta|=k} c_{\beta, v} N_{\beta, v} = \sum_{v \in V} \sum_{|\beta|=k} c_{\beta, v} N_{\beta, v}
$$

$$
+ \sum_{v \in V} \sum_{|\beta|=k-1} c_{\beta+e^v, v} d(X_{\beta+e^v, v})M(\cdot|X_{\leq \beta+e^v, v})
$$

$$
= \sum_{v \in V} \sum_{|\beta|=k} c_{\beta, v} N_{\beta, v} + \sum_{v \in V} \sum_{|\beta|=k-1} c^{[1]}_{\beta, v} d(X_{\beta, v})M(\cdot|X_{\leq \beta, v} \cup z^0),
$$

13
where \( c^{[1]}_{\beta,v} := c^{[1+\beta,v]}_\gamma \), for \( \gamma_u = \beta_u, u \neq v \), and \( \gamma_v = 0 \). Thus by Corollary 5.5, (6.4) holds if

\[
\sum_{|\beta|=k-1} c^{[1]}_{\beta} d(X_\beta) M(\cdot|X_{\leq \beta}) = \sum_{v \in V} \sum_{|\beta|=k-1} c^{[1]}_{\beta,v} d(X_{\beta,v}) M(\cdot|X_{\leq \beta,v}),
\]

or

\[
\sum_{|\beta|=k-1} c^{[1]}_{\beta} N_\beta = \sum_{v \in V} \sum_{|\beta|=k-1} c^{[1]}_{\beta,v} N_{\beta,v},
\]

which is true by the induction hypothesis (for the sequence \( \{z^1, \ldots, z^k\} \)).

We also need the following

**Lemma 6.2.** Let \( p \in \mathcal{H}_k \) and suppose \( c_\beta = P(X_{< \beta}) \). Then

\[
c_{\beta,v} = P(X_{< \beta,v}), \quad |\beta| = k, \quad v \in V.
\]

**Proof:** We show the following more general identity

\[
c^{[\ell]}_\beta = P(X_{< \beta} \cup \{z^0, \ldots, z^{\ell-1}\}), \quad |\beta| = k - \ell, \quad \ell = 1, \ldots, k.
\]

Proceeding by induction on \( \ell \), the assertion is true for \( \ell = 0 \) by the definition of \( c^{[0]}_\beta \). For \( \ell > 0 \), we have, by multi-linearity of \( P \),

\[
c^{[\ell]}_\beta = \sum_{v \in V} b_v(z^{\ell-1}|X_\beta) P(X_{< \beta} \cup x^{v,\beta,v} \cup \{z^0, \ldots, z^{\ell-2}\})
\]

\[
= P(X_{< \beta} \cup \{z^0, \ldots, z^{\ell-1}\}).
\]

Combining Proposition 6.1 and Lemma 6.2 leads to [9]

**Theorem 6.3.** Let \( p \in \mathcal{H}_k \). Then

\[
p = \sum_{V \in V} \sum_{|\beta|=k} p^{V}(X_{< \beta}) N_\beta^{V}.
\]

**Proof:** Let \( x \in \mathbb{R}^{k+1} \) and let \( Z \) be the sequence containing \( k + 1 \) copies of \( x \). By Proposition 6.1 and Lemma 6.2, we obtain

\[
\sum_{V \in V} \sum_{|\beta|=k} p(X_{< \beta}) N_\beta^{V}(x) = \sum_{V \in V} \sum_{v \in V} \sum_{|\beta|=k} P(X_{< \beta,v}) N_{\beta,v}^{V}(x)
\]

\[
= \sum_{V \in V} \sum_{v \in V} \sum_{|\beta|=k} P(X_{< \beta,v}) N_{\beta,v}^{V}(x) + \sum_{V \in V} \sum_{v \in V} \sum_{\beta_u = k} P(X_{< \beta,v}) N_{\beta,v}^{V}(x).
\]
We prove that the first expression in (6.6) is equal to \( p(x) \), while the second vanishes. Note that for \( \beta_v = k \), we have \( P(X_{\leq \beta_v}) = P(z^0, \ldots, z^{k-1}) = P(x, \ldots, x) = p(x) \). Moreover, \( N_{\beta_v}^V = d(V \setminus v \cup x) M(V \setminus v \cup x) \), and

\[
\sum_{v \in V} d(V \setminus v \cup x) M(x|V \setminus v \cup x) = d(V) M(x|V) = \chi_v(x).
\]

In the above expressions we have assumed that the set \( V \setminus v \cup x \) is ordered in the same way as \( V \), where \( v \) is replaced by \( x \). This shows that the first expression in (6.6) is indeed equal to \( p(x) \). As for the second part, note that for two adjacent simplices in \( V \), the normalized simplex splines corresponding to their common face are the same, except that the orientation of the two simplices forces these simplex splines to be of opposite signs and hence they cancel each other. 

7. Simplex Splines on Star-Like Differentiable Manifolds

Suppose \( \mathcal{M} \) is an \( s \)-dimensional star-like differentiable manifold in \( \mathbb{R}^{s+1} \) i.e., such that every ray \( \{ \lambda v, \lambda \in \mathbb{R}_+ \}, v \in \mathbb{R}^{s+1} \setminus 0 \), intersects \( \mathcal{M} \) at most once. Provided the vertices of \( V \) lie on \( \mathcal{M} \), restricting the simplicial cones \( V \in V \) to \( \mathcal{M} \) gives rise to a partition \( V_M \) on \( \mathcal{M} \) consisting of 'surface simplices'. For example, if \( \mathcal{M} \) is a hyperplane in \( \mathbb{R}^3 \) not passing through the origin, we obtain a triangulation on \( \mathcal{M} \), whereas if \( \mathcal{M} \) is the unit sphere in \( \mathbb{R}^3 \) centered at the origin, the partition is a spherical triangulation consisting of spherical geodesic triangles. A natural way to define a spline space on \( \mathcal{M} \) is to take the restriction of \( S_k(V) \) to \( \mathcal{M} \). Below we discuss the above two special cases in more detail.

7.1. Affine Simplex Splines with Knots at Infinity

Suppose \( \mathcal{M} \) is a hyperplane in \( \mathbb{R}^{s+1} \) not passing through the origin. Without loss of generality we can assume that \( \mathcal{M} \) is given by

\[
\mathcal{M} := \{ x \in \mathbb{R}^{s+1}, x_{s+1} = 1 \}.
\]

It was proved in [8] that homogeneous simplex splines (with knots in \( \mathcal{M} \)) restricted to \( \mathcal{M} \) are the affine simplex splines. This is obvious from the following reasoning. Let \( X \) be a set of knots in \( \mathcal{M} \). For \( x \in \mathcal{M} \), the equality (3.6) forces the \( \lambda_y \) to be such that

\[
\sum_{y \in X} \lambda_y = 1. \tag{7.1}
\]

However, then the recurrence (4.6) together with (3.6) and (7.1) becomes (up to a normalizing factor) the familiar recurrence relation for affine simplex splines, discovered in [15]. The advantage of the more general homogeneous setting is that
it is possible to extend the class of affine simplex splines by including knots which are points at infinity. As a consequence, truncated power functions as defined in [16] and the familiar univariate truncated powers, can be viewed as instances of simplex splines. We illustrate this fact with the following example. A thorough discussion of univariate B-splines with knots at infinity appeared in [24].

Example 7.1. Let $s = 1$. By $\infty$ we denote the point at infinity of $\mathcal{M}$ whose homogeneous coordinates are $(1, 0)$. Consider the homogeneous simplex spline (B-spline) $M(x|x^0, \ldots, x^{k+1})$ with knots $x^0 := (0,1), x^1 = \ldots = x^{k+1} := (1,0)$. Suppose $B(t|t_0, \ldots, t_{k+1}), t_0 = 0, t_1 = \ldots = t_{k+1} = \infty$, is the restriction of $M$ to $\mathcal{M}$. Employing definition (4.1) and recurrence (4.6) it follows that $B$ is identical with the univariate truncated power function of degree $k$, that is

$$ B(t|t_0, \ldots, t_{k+1}) = t^k_+ := t^k \max\{0, \text{sgn}(t)\}, \quad t \in \mathbb{R}. $$

### 7.2. Spherical Splines

Let $\mathcal{M}$ be the unit sphere in $\mathbb{R}^{s+1}$ centered at the origin. By restricting the space $S_k(\mathcal{V})$ to $\mathcal{M}$, we obtain a space of spherical splines. It is known [1] that for $s = 1$, these splines, called circular splines, are essentially trigonometric splines (see also [10,24]). This is because the restriction of $\mathcal{H}_k$ to a circle (centered at the origin) gives the space of trigonometric polynomials. For $s > 1$, the restriction of $\mathcal{H}_k$ leads to spherical polynomials considered in [2]. These are functions which are linear combinations of spherical harmonics of various degrees. Note that the restricted space $\mathcal{P}_k := \mathcal{H}_k|\mathcal{M}$ is nested in the sense that $\mathcal{P}_k \subset \mathcal{P}_{k+2}$. This means $\mathcal{P}_k$ contains constant functions whenever $k$ is even. In [2], a representation of functions in $\mathcal{P}_k$ was given in terms of spherical Bernstein basis polynomials. These are functions defined for a spherical simplex $V \subset \mathcal{M}$, by

$$ B_\beta(x) := \frac{k!}{\beta!} b^\beta(x|V), \quad |\beta| = k, \quad x \in \mathcal{M}, \quad (7.2) $$

where $\beta! := \prod_{v \in V} \beta_v!$ and $b^\beta := \prod_{v \in V} b^\beta_v$. The functions $b_v$ are the homogeneous barycentric coordinates defined in (4.8) (in fact, their restrictions to the sphere $\mathcal{M}$). They were discussed extensively in [2]. Piecewise spherical polynomials defined on a triangulation of $\mathcal{M}$ were considered in detail (for $s = 2$) in [3,4].

It is not surprising that spherical Bernstein polynomials are instances of normalized spherical simplex splines, which are the restrictions of homogeneous simplex splines to $\mathcal{M}$. To see this, let the clouds of knots corresponding to a simplex $V$ be such that $x^{v,\alpha_v} = v$, for all $\alpha_v = 1, \ldots, \beta_v, v \in V$. Then

$$ B_\beta = N_\beta|\mathcal{M}. $$

16
This follows from a special case of the recursion (4.6), given by

\[ N_\beta(x) = \sum_{v \in V} b_v(x|X_{\beta})N_{\beta - e^v}(x), \quad x \in \mathbb{R}^{s+1}, \]

and the fact that for the knot collection above, \( b_v(x|X_{\beta}) = b_v(x|V) \). Note that the same recurrence applies to (7.2).

8. Remarks

**Remark 1.** In Definition 4.1 and the subsequent considerations we have tacitly assumed that the point \( z \) is such that \( \text{cone}(X \cup z) \) is a proper cone. Note that in the affine case \( i.e., \) the case \( X \subset \{ x \in \mathbb{R}^{s+1}, x_{s+1} = 1 \} \), this condition is automatically fulfilled. In the homogeneous setting it is needed, however, since otherwise Lemma 4.2 might not be true. For example, suppose \( s = 1, \) \( X := \{(1,1),(-1,1)\} \), and \( z = (0,-1) \). In this case Lemma 4.2 is violated since the supports of \( M(\cdot|\{(1,1),(0,-1)\}) \) and \( M(\cdot|\{-1,1\},(0,-1)) \) do not match the support of \( M(\cdot|\{-1,1\},(1,1)) \). For our purposes this assumption is not a serious restriction since in all our considerations the choice of \( z \) is free and thus can be made ad hoc so as to meet this requirement. One could remove this assumption by redefining the lowest order simplex splines (4.1) so as to take into account the ordering of the set \( X \). In that case, depending on the ordering of \( X \), the support of \( M \) would be either \( \text{cone}(X) \) or the complement of this cone \( \mathbb{R}^{s+1} \setminus \text{cone}(X) \). We have not pursued this idea in this paper.

**Remark 2.** For the convenience of the reader, we recall the definition of a cone spline, introduced in [7] (see also [6, 14]). The cone spline \( C(\cdot|X) \) is a distribution defined by

\[ \langle C, f \rangle = k! \int_{\mathbb{R}^{s+1}} f(Xt)dt, \quad (8.1) \]

which must hold for every continuous compactly supported function \( f \). Here, \( \langle , \rangle \) denotes the usual inner product on \( \mathbb{R}^{s+1} \), and \( Xt := \sum_{y \in X} t_y y \). In (8.1) we used the normalization \( k! \), since then for knots in generic position, \( C \) coincides with the function \( M \) defined in Definition 4.1. It is known that the support of \( C(\cdot|X) \) is \( \text{cone}(X) \). If \( \text{cone}(X) \) is nondegenerate \( i.e., \) if \( \text{span}(X) = \mathbb{R}^{s+1} \), \( C(\cdot|X) \) can be identified with a real-valued function continuous inside its support. The recurrence relation (4.6) was first proved in [7] and later in [14]. A simple proof is also given in [5]. The knot insertion formula (4.5) for affine simplex splines has been discovered in [14].

**Remark 3.** A special case of (4.2) was considered in [16]. There the description of the result is more complicated since it is formulated in the less convenient affine setting.
Remark 4. In the affine case, Theorem 6.3 gives rise to a Schoenberg operator for simplex splines [9]. In the general homogeneous setting it is not clear how to define an analog of this operator. In fact, such an analog may not exist, as already pointed out in [2] for the special case of Bernstein polynomials. This is partly because the space \( \mathcal{H}_1 \) is not reproduced by homogeneous splines of degree \( k > 1 \). On the other hand, (for \( s > 1 \)) there is no other \( (s + 1) \)-dimensional space which is invariant under isometries on \( \mathbb{R}^{s+1} \) [2]. This also means that there may be no analog of control points for homogeneous simplex splines which would possess most of the favorable properties of the affine control points.

Remark 5. Theorem 6.3 implies that polynomials in \( \mathcal{H}_k \) are contained in the space \( S_k(\mathcal{V}) \). In the affine case, it has been proved [22] that in fact this space contains a class of piecewise polynomials of degree \( k \) corresponding to the partition \( \mathcal{V} \). Using the methods presented here this result can also be easily established for homogeneous splines.

Remark 6. In this paper we have worked exclusively with knots in generic position. If the knots are not in generic position, a special treatment is needed since then homogeneous divided differences are not defined. This may be the case if some of the knots coincide. In order to extend (if possible) our results to the general case, a natural approach is to view every simplex spline with a general set of knots as a (possibly distributional) limit of a sequence of simplex splines with generic knots. For example, using this limit argument one can show that recurrence (4.6) is true for every homogeneous simplex spline.

Remark 7. Let \( Q \) be a homogeneous quadratic form on \( \mathbb{R}^{s+1} \) and consider the quadric manifold \( \mathcal{M} \) defined by

\[
\mathcal{M} := \{ x \in \mathbb{R}^{s+1}, Q(x) = 1 \}.
\]

Restricting the homogeneous simplex splines to \( \mathcal{M} \), we obtain a class of splines containing the two examples considered in Section 7 as a special case. By choosing \( Q(x) = x_2^2 - x_1^2 \), \( s = 1 \), \( \mathcal{M} \) becomes a hyperbola and the spline space corresponding to \( \mathcal{M} \) is essentially the space of hyperbolic splines (see e.g., [24]).

Remark 8. Spherical simplex splines are closely related to certain multivariate trigonometric simplex splines defined in [12].

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