Bernstein-Bézier Polynomials on Spheres and Sphere-Like Surfaces

by

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Abstract. In this paper we discuss a natural way to define barycentric coordinates on general sphere-like surfaces. This leads to a theory of Bernstein-Bézier polynomials which parallels the familiar planar case. Our constructions are based on a study of homogeneous polynomials on trihedra in $\mathbb{R}^3$. The special case of Bernstein-Bézier polynomials on a sphere is considered in detail.

1. Introduction

Bernstein-Bézier (BB-) polynomials defined on triangles are useful tools for constructing piecewise functional and parametric surfaces defined over triangulated planar domains. They play an extremely important role in CAGD (computer-aided geometric design), data fitting and interpolation, computer vision, and elsewhere (see *e.g.* the books [Farin '88, Hoschek & Lasser '93]).

In many applications we need to work on the sphere, or on sphere-like surfaces. Researchers have been searching for a number of years for an appropriate analog of the BB-polynomials in the spherical setting, but have been hampered by the perceived lack of a reasonable way to define barycentric coordinates on spherical triangles. In fact, recently, [Brown & Worsey '92] showed that such coordinates (satisfying a reasonable looking list of conditions) do not exist.

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The purpose of this paper is to show that despite the results in [Brown & Worsey '92], there is in fact a simple and natural way to define barycentric coordinates on spherical triangles, and that they can be used to define associated spaces of BB-polynomials which exhibit most of the important properties of the classical BB-polynomials on planar triangulations.

The key to our construction of barycentric coordinates for general sphere-like surfaces is to omit the usual requirement that they form a partition of unity. It turns out that the associated BB-polynomials can be interpreted as particular trivariate BB-polynomials which are homogeneous.

The methods discussed in this paper have immediate applications to a variety of important practical problems involving interpolation and data fitting of a function defined on a sphere or sphere-like surface. Indeed, because of the close analogy with standard Bernstein-Bézier techniques, virtually all of the classical methods based on BB-polynomials on planar triangulations can be carried over to the spherical setting. We give a detailed treatment of several of these methods in a separate paper [Alfeld et al '95c]. Our spherical Bernstein-Bézier methods could also be of interest in the design of surfaces (especially closed surfaces), although some of the geometric properties of planar Bernstein-Bézier methods do not carry over.

The spherical BB-polynomials constructed here can also be used to define spaces of *splines* (see Remark 1 in Sect. 7) whose domains are triangulations of the sphere. We discuss these spline spaces in detail in [Alfeld et al '95b].

The paper is organized as follows. In Sect. 2 we introduce trihedral coordinates in $\mathbb{R}^3$ which will later be restricted to spheres or sphere-like surfaces. In Sect. 3 we study associated homogeneous BB-polynomials, and show that they have the same properties as in the standard Bernstein-Bézier theory. Among other things, we discuss the de Casteljau algorithm, subdivision, and necessary and sufficient conditions for smoothly joining two such polynomials across a plane through the origin. The reason for developing this homogeneous theory is that it provides a useful framework for developing a Bernstein-Bézier theory on sphere-like surfaces. This is done by restricting the trivariate polynomials to a surface; the details can be found in Sect. 4.

The next three sections deal with the sphere as a special case of a sphere-like surface. In Sect. 5 we discuss spherical barycentric coordinates, and develop their properties, including several (such as rotational invariance) which are particular to the sphere. In Sect. 6 we show that our choice of spherical barycentric coordinates is the only one which satisfies a short list of natural properties. Spherical BB-polynomials are treated in Sect. 7, where we also investigate associated surface patches in $\mathbb{R}^3$ and discuss the problem of defining suitable control nets.

The restrictions of spherical BB-polynomials to great circles turn out to be certain trigonometric polynomials. We give a detailed treatment of BB-polynomials on circular arcs in [Alfeld et al '95a].

The reader who is primarily interested in the sphere may want to start reading the paper in Sect. 5, referring back to earlier sections when needed for proofs and more general results.

Several months after submitting this article, one of us presented the results in Vienna, and H. Pottmann recognized that our spherical barycentric coordinates had already been studied in 1846 by A. F. Möbius [Möbius 1846]. He derived several interesting properties of these coordinates, including the properties which we rediscovered in Theorems 5.2 and 5.3.

2. Trihedral Coordinates

In this section we introduce a special set of coordinates in $\mathbb{R}^3$ which will be used later to construct barycentric coordinates on the sphere and sphere-like surfaces.

Definition 2.1. *Let $V = \{v_1, v_2, v_3\}$ be a basis for $\mathbb{R}^3$. We call*

$$T := \{v \in \mathbb{R}^3 \; : \; v = b_1 v_1 + b_2 v_2 + b_3 v_3 \quad \textit{with} \quad b_i \geq 0\}$$

the trihedron generated by V. Each $v \in \mathbb{R}^3$ can be written in the form

$$v = b_1 v_1 + b_2 v_2 + b_3 v_3. \tag{2.1}$$

We call b_1, b_2, b_3 the trihedral coordinates of v with respect to V.

If we choose $V = \{e_1, e_2, e_3\}$, where the e_i are the usual unit coordinate vectors in $\mathbb{R}^3$, then the corresponding trihedron is just the first octant, and the trihedral coordinates of a point $v \in \mathbb{R}^3$ are the usual Cartesian coordinates.

Equation (2.1) defining the trihedral coordinates can be written as a system of three equations for the b_i's:

$$\begin{pmatrix} v_1^x & v_2^x & v_3^x \\ v_1^y & v_2^y & v_3^y \\ v_1^z & v_2^z & v_3^z \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} v^x \\ v^y \\ v^z \end{pmatrix}, \tag{2.2}$$

where v^x denotes the x-coordinate of v, etc. The matrix in (2.2) is nonsingular since v_1, v_2, v_3 form a basis for $\mathbb{R}^3$. Using Cramer's rule, we immediately have

$$b_1 = \frac{\det\,(v, v_2, v_3)}{\det\,(v_1, v_2, v_3)}, \quad b_2 = \frac{\det\,(v_1, v, v_3)}{\det\,(v_1, v_2, v_3)}, \quad b_3 = \frac{\det\,(v_1, v_2, v)}{\det\,(v_1, v_2, v_3)}, \tag{2.3}$$

where

$$\det\,(v_1, v_2, v_3) := \det \begin{pmatrix} v_1^x & v_2^x & v_3^x \\ v_1^y & v_2^y & v_3^y \\ v_1^z & v_2^z & v_3^z \end{pmatrix},$$

and so forth. Equations (2.3) show that the b_i's are ratios of volumes of tetrahedra.

Clearly, for all $\alpha \in \mathbb{R}$, $b_i(\alpha v) = \alpha b_i(v)$, $i = 1, 2, 3$, which implies that the b_i are homogeneous linear functions of v. Since they are also trivially linearly independent,

$$\operatorname{span}\{b_1, b_2, b_3\} = \mathcal{L},$$

where $\mathcal{L}$ is the space of trivariate linear homogeneous polynomials.

It follows immediately from Definition 2.1 that

$$b_i(v_j) = \delta_{ij}, \quad i, j = 1, 2, 3, \tag{2.4}$$

and

$$b_i(v) > 0 \quad \text{for all } v \text{ in the interior of } T.$$

The trihedral coordinates of a point v are invariant under rotation. In fact, we can prove even more:

Theorem 2.2. *Let R be any nonsingular matrix. Then*

$$b_i^R(Rv) = b_i(v), \qquad i = 1, 2, 3,$$

where b_i^R are the trihedral coordinates with respect to $\{Rv_1, Rv_2, Rv_3\}$.

Proof: Multiplying (2.1) by R, we have $Rv = b_1 Rv_1 + b_2 Rv_2 + b_3 Rv_3$. ■

The next result also follows immediately from the definition of trihedral coordinates.

Theorem 2.3. *Let T be a trihedron generated by $\{v_1, v_2, v_3\}$. Then the three planes spanned by pairs of the v_i divide $\mathbb{R}^3$ into eight trihedra. The functions b_1, b_2, b_3 have constant signs on each of the eight trihedra. In particular, $v \in T$ if and only if $b_i \geq 0$, $i = 1, 2, 3$.*

When the v_i's are the unit coordinate vectors, the eight regions of Theorem 2.3 become the eight octants in the ordinary Cartesian coordinate system.

3. Homogeneous Bernstein-Bézier Polynomials

In this section we will be interested in a certain subspace of the space $\mathcal{P}_d$ of trivariate polynomials of total degree d. The dimension of $\mathcal{P}_d$ is $\binom{d+3}{3}$. One way to construct a basis for it is to start with four non-coplanar points v_i, $i = 1, \ldots, 4$, and use them to define the standard barycentric coordinates of a point $v \in \mathbb{R}^3$:

$$v = \sum_{i=1}^{4} b_i v_i, \qquad \text{where} \qquad \sum_{i=1}^{4} b_i = 1.$$

Then a basis for $\mathcal{P}_d$ is given by the classical *Bernstein polynomials*

$$B_{ijkl}^d(v) := \frac{d!}{i!j!k!l!} b_1^i b_2^j b_3^k b_4^l, \qquad i + j + k + l = d.$$

We recall the definition of homogeneous functions.

Definition 3.1. *A function f defined on $\mathbb{R}^3$ is homogeneous of degree d provided $f(\alpha v) = \alpha^d f(v)$ for all real numbers α and all $v \in \mathbb{R}^3$.*

We are interested in the space $\mathcal{H}_d$ of polynomials of degree d which are homogeneous of degree d. The proof of the following lemma is elementary.

Lemma 3.2. *The space $\mathcal{H}_d$ is an $\binom{d+2}{2}$ dimensional subspace of $\mathcal{P}_d$. Moreover, if we choose v_4 to be the origin in the above construction of the Bernstein polynomials, then the set*

$$\{B^d_{ijk0} \ : \ i+j+k=d\} \tag{3.1}$$

forms a basis for $\mathcal{H}_d$.

The polynomials in (3.1) play a key role in our paper. For ease of notation, it is convenient to drop the last subscript, leading us to the following definition:

Definition 3.3. *Let T be a trihedron generated by $\{v_1, v_2, v_3\}$, and let $b_1(v)$, $b_2(v)$, $b_3(v)$ denote the trihedral coordinates as functions of $v \in \mathbb{R}^3$. Given an integer $d \geq 0$, we define the homogeneous Bernstein basis polynomials of degree d on T to be the set of polynomials*

$$B^d_{ijk}(v) := \frac{d!}{i!j!k!} b_1(v)^i b_2(v)^j b_3(v)^k, \qquad i+j+k=d. \tag{3.2}$$

We call

$$p(v) := \sum_{i+j+k=d} c_{ijk} B^d_{ijk}(v) \tag{3.3}$$

a homogeneous Bernstein-Bézier (HBB-) *polynomial of degree d.*

In view of (2.4),

$$p(v_1) = c_{d00}, \quad p(v_2) = c_{0d0}, \quad p(v_3) = c_{00d}. \tag{3.4}$$

To evaluate p at other points in $\mathbb{R}^3$, we may use the classical de Casteljau algorithm:

Theorem 3.4. *Suppose we want to evaluate the HBB-polynomial (3.3) at a point w with trihedral coordinates b_1, b_2, b_3.*

Set $c^0_{ijk} := c_{ijk}$, $i+j+k=d$
For $l = 1$ to d
For $i+j+k = d-l$
$\quad c^l_{ijk} := b_1 c^{l-1}_{i+1,j,k} + b_2 c^{l-1}_{i,j+1,k} + b_3 c^{l-1}_{i,j,k+1}$.

Then $p(w) = c^d_{000}$.

Proof: Let $B^0_{000}(w) \equiv 1$. Then it can be shown by induction that

$$c^l_{ijk} = \sum_{r+s+t=l} c_{i+r,j+s,k+t} B^l_{rst}(w), \qquad i+j+k=d-l, \tag{3.5}$$

and we have

$$c_{000}^d = \sum_{r+s+t=d} c_{rst} B_{rst}^d(w) = p(w). \quad \blacksquare$$

The following is the analog of the classical subdivision algorithm for bivariate BB-polynomials. Its proof is exactly the same as in the planar case.

Theorem 3.5. *Let $\{c_{ijk}^l\}$ be the coefficients produced by the de Casteljau algorithm using trihedral coordinates b_1, b_2, b_3 corresponding to a point w lying in T. Then*

$$p(v) = \begin{cases} \sum_{i+j+k=d} c_{0jk}^i B_{ijk;1}^d(v), & v \in T_1 = \{w, v_2, v_3\}, \\ \sum_{i+j+k=d} c_{i0k}^j B_{ijk;2}^d(v), & v \in T_2 = \{v_1, w, v_3\}, \\ \sum_{i+j+k=d} c_{ij0}^k B_{ijk;3}^d(v), & v \in T_3 = \{v_1, v_2, w\}, \end{cases}$$

where the $B_{ijk;\nu}^d$ are the Bernstein-Bézier basis functions associated with the triangles T_ν, $\nu = 1, 2, 3$.

We now establish necessary and sufficient conditions for two HBB-polynomials to join together smoothly across a plane through the origin in the sense that the polynomials and their usual directional derivatives as trivariate functions are continuous as we cross the plane.

Theorem 3.6. *Let T and $\widetilde{T}$ be trihedra with vertices $V = \{v_1, v_2, v_3\}$ and $\tilde{V} = \{v_4, v_2, v_3\}$, where $v_4 = \sum_{i=1}^3 a_i v_i$. Let*

$$p(v) := \sum_{i+j+k=d} c_{ijk} B_{ijk}^d(v) \tag{3.6}$$

and

$$\tilde{p}(v) := \sum_{i+j+k=d} \tilde{c}_{ijk} \widetilde{B}_{ijk}^d(v), \tag{3.7}$$

where $\{B_{ijk}^d\}$ and $\{\widetilde{B}_{ijk}^d\}$ are the Bernstein-Bézier basis functions associated with T and $\widetilde{T}$. Then p and $\tilde{p}$ and all of their derivatives up to order m agree on the face shared by T and $\widetilde{T}$ if and only if

$$\tilde{c}_{ijk} = \sum_{r+s+t=i} c_{r,j+s,k+t} B_{rst}^i(v_4) \tag{3.8}$$

for all $i = 0, \ldots, m$ and all j, k such that $i + j + k = d$.

Proof: Suppose

$$P(v) := \sum_{i+j+k+l=d} C_{ijkl} B_{ijkl}^d(v)$$

and

$$\tilde{P}(v) := \sum_{i+j+k+l=d} \tilde{C}_{ijkl} \widetilde{B}^d_{ijkl}(v),$$

where

$$C_{ijkl} := \begin{cases} c_{ijk}, & \text{if } l = 0 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad \tilde{C}_{ijkl} := \begin{cases} \tilde{c}_{ijk}, & \text{if } l = 0 \\ 0, & \text{otherwise,} \end{cases} \tag{3.9}$$

and $B^d_{ijkl}(v)$ are the usual BB-polynomials of degree d associated with the tetrahedron with vertices $\{v_1, v_2, v_3, 0\}$ and $\widetilde{B}^d_{ijkl}(v)$ are those associated with the tetrahedron with vertices $\{v_4, v_2, v_3, 0\}$. It is well known [de Boor '87] that these polynomials join with C^m continuity if and only if

$$\tilde{C}_{ijkl} = \sum_{r+s+t+u=i} C_{r,j+s,k+t,l+u} B^i_{rstu}(v_4), \qquad i = 0, \ldots, m. \tag{3.10}$$

In view of (3.9), we can choose $l = u = 0$. In this case, (3.10) holds if and only if (3.8) holds. But $P = p$ and $\tilde{P} = \tilde{p}$, and the proof is complete. ■

4. Sphere-Like Surfaces

Throughout the remainder of the paper we denote the unit sphere in $\mathbb{R}^3$ and centered at the origin by S.

Definition 4.1. *Given an infinitely differentiable positive function ρ defined on the unit sphere S, we define a sphere-like surface in $\mathbb{R}^3$ to be the set*

$$\mathcal{S} := \left\{ u \in \mathbb{R}^3 : u = \rho(v)v, \quad v \in S \right\}. \tag{4.1}$$

The simplest example is provided by $\rho \equiv 1$, in which case $\mathcal{S} = S$. We require that ρ be arbitrarily often differentiable in order to simplify subsequent arguments about the smoothness of functions defined on sphere-like surfaces. Depending on the application, it would suffice to require that ρ only be sufficiently often differentiable.

Definition 4.2. *Let $V = \{v_1, v_2, v_3\}$ be a set of points on a sphere-like surface $\mathcal{S}$ so that considered as vectors, they form a basis for $\mathbb{R}^3$. Then we define the surface triangle with vertices v_1, v_2, and v_3 to be the intersection of $\mathcal{S}$ with the trihedron generated by V.*

A surface triangle is a piece of surface in $\mathbb{R}^3$ with three boundary curves, each of which is the intersection of $\mathcal{S}$ with a plane through the origin. For example, the edge $\overline{v_2 v_3}$ is obtained by intersecting $\mathcal{S}$ with the plane passing through $0, v_2, v_3$.

In the remainder of this section we discuss properties of HBB-polynomials restricted to the surface $\mathcal{S}$.

Theorem 4.3. *The polynomials $\{B^d_{ijk}\}_{i+j+k=d}$ restricted to $\mathcal{S}$ are linearly independent.*

Proof: Suppose

$$p(v) = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}(v) = 0 \qquad \text{for all } v \in \mathcal{S}.$$

Then by the homogeneity of p, it must be identically zero on $\mathbb{R}^3$. The result now follows from the linear independence of the B^d_{ijk}'s on $\mathbb{R}^3$. ■

It is clear that both the de Casteljau and the subdivision algorithms developed in Sect. 3 can be applied to HBB-polynomials restricted to $\mathcal{S}$. We now consider the question of when two polynomials on adjoining surface triangles join smoothly across a common edge e. What we want is that for every curve c crossing e obtained by intersecting $\mathcal{S}$ with a plane, derivatives up to a given order m with respect to the arc length of c agree along e.

Theorem 4.4. *Suppose p and $\tilde{p}$ are polynomials as in (3.6) and (3.7) and let T and $\tilde{T}$ be the surface triangles obtained by intersecting the corresponding V and $\tilde{V}$ with a sphere-like surface $\mathcal{S}$. Then the restrictions of p and $\tilde{p}$ to $\mathcal{S}$ along with their derivatives up to order m join continuously along the common edge e between the two triangles, i.e., for every point $v \in e$ and every curve $c \in \mathcal{S}$ crossing e at v,*

$$D^j_c p(v) = D^j_c \tilde{p}(v), \qquad j = 0, \ldots, m, \tag{4.2}$$

if and only if (3.8) holds.

Proof: The fact that (3.8) implies (4.2) is immediate by Theorem 3.6 and the chain rule. The converse assertion follows from the fact that any derivative of a homogeneous function is itself homogeneous. ■

5. Spherical Barycentric Coordinates

Let S be the unit sphere in $\mathbb{R}^3$ with center at the origin obtained by setting $\rho \equiv 1$ in (4.1). In this case, the surface triangle generated by three unit vectors v_1, v_2, v_3 (which span $\mathbb{R}^3$) becomes the *spherical triangle*

$$T = \{v \in S \;:\; v = b_1 v_1 + b_2 v_2 + b_3 v_3, \quad b_i \geq 0\}.$$

It is clear that the boundary of T consists of the three circular arcs $\overline{v_1 v_2}$, $\overline{v_2 v_3}$, $\overline{v_3 v_1}$. Each of these arcs lies on a great circle, and is thus a geodesic curve on S.

Definition 5.1. *Let T be a spherical triangle with vertices v_1, v_2, v_3, and let v be a point on S. The (spherical) barycentric coordinates of v relative to T are the unique real numbers b_1, b_2, b_3 such that*

$$v = b_1 v_1 + b_2 v_2 + b_3 v_3. \tag{5.1}$$

It is clear from (5.1) that the spherical barycentric coordinates of a point v on the sphere S are exactly the same as the trihedral coordinates of v with respect to the trihedron generated by $\{v_1, v_2, v_3\}$. This implies they have the following properties (among others):

1) At the vertices of T,
$$b_i(v_j) = \delta_{ij}, \quad i, j = 1, 2, 3.$$
2) For all v in the interior of T, $b_i(v) > 0$.
3) In contrast to the usual barycentric coordinates on planar triangles (which always sum to 1),
$$b_1(v) + b_2(v) + b_3(v) > 1, \qquad \text{if } v \in T \text{ and } v \neq v_1, v_2, v_3.$$
4) If the edges of a spherical triangle T are extended to great circles, the sphere S is divided into eight regions. The spherical barycentric coordinates b_1, b_2, b_3 have constant signs on each of these eight regions.
5) If a point v lies on an edge of T, then one of its spherical barycentric coordinates vanishes. The remaining two spherical barycentric coordinates are ratios of sines of geodesic distances, rather than ratios of geodesic distances (see [Alfeld et al '94c]).
6) Spherical barycentric coordinates are infinitely often differentiable functions of v (since the determinant in the denominators of (2.3) never vanishes).
7) The spherical barycentric coordinates of a point v on the sphere relative to one spherical triangle T can be computed from those relative to another spherical triangle $\widetilde{T}$ by matrix multiplication.
8) The b_i are ratios of volumes of tetrahedra (cf. (2.3)). (They are *not* the volumes of the spherical wedges whose top faces are spherical triangles).
9) The spherical barycentric coordinates of a point v are invariant under rotation, i.e., they depend only on the relative positions of v and v_1, v_2, v_3 to each other. (This is important because the sphere S itself is rotationally invariant).
10) The span of the spherical barycentric coordinates $b_1(v), b_2(v), b_3(v)$ relative to any triangle is always the three-dimensional linear space obtained by restricting the space $\mathcal{L}$ of linear homogeneous polynomials on $\mathbb{R}^3$ to the sphere S, and

is thus independent of the triangle. For convenience, we will use $\mathcal{L}$ to denote both of these spaces (even though they correspond to different domains).

We now show that spherical barycentric coordinates can also be expressed in terms of certain natural angles associated with the geometry, just as in the planar case, see [Farin '88]. Again assuming that the vertices of a triangle T are the points in the set $V := \{v_1, v_2, v_3\}$, let n_i denote the unit normal vectors to the planes $P_i := \mathrm{span}(V \backslash \{v_i\})$, $i = 1, 2, 3$. The orientation of these vectors is chosen to be consistent with the orientation of the vertices v_i relative to P_i, *i.e.*,

$$\begin{aligned} \mathrm{sgn}\ \det\,(v_1, v_2, v_3) &= \mathrm{sgn}\ \det\,(n_1, v_2, v_3) \\ &= \mathrm{sgn}\ \det\,(v_1, n_2, v_3) = \mathrm{sgn}\ \det\,(v_1, v_2, n_3). \end{aligned}$$

For a point $v \in S$, let the angles α_i, β_i, be defined by the dot products

$$\sin\alpha_i := v \cdot n_i, \quad \sin\beta_i := v_i \cdot n_i, \qquad i = 1, 2, 3.$$

The α_i represent *oriented angles* between the vector v and the planes P_i, while the β_i are the analogous angles between v_i and the P_i (see Fig. 1). For nontrivial spherical triangles, $\det\,(v_1, v_2, v_3) \neq 0$, and therefore $\sin\beta_i \neq 0$, $i = 1, 2, 3$.

Theorem 5.2. *The spherical barycentric coordinates of the vector $v \in S$ with respect to the triangle T are given by*

$$b_i(v) = \frac{\sin\alpha_i}{\sin\beta_i}, \quad i = 1, 2, 3. \tag{5.2}$$

Proof: The proof follows immediately from (5.1) since $n_i = d_i / \|d_i\|$, where

$$d_1 := \begin{vmatrix} e_1 & v_2^x & v_3^x \\ e_2 & v_2^y & v_3^y \\ e_3 & v_2^z & v_3^z \end{vmatrix}, \quad d_2 := \begin{vmatrix} v_1^x & e_1 & v_3^x \\ v_1^y & e_2 & v_3^y \\ v_1^z & e_3 & v_3^z \end{vmatrix}, \quad d_3 := \begin{vmatrix} v_1^x & v_2^x & e_1 \\ v_1^y & v_2^y & e_2 \\ v_1^z & v_2^z & e_3 \end{vmatrix}.$$

Here $\|\cdot\|$ is the usual Euclidean norm, and the e_i are the unit coordinate vectors in $\mathbb{R}^3$. ■

Theorem 5.3. *For each $i = 1, 2, 3$, let C_i be the great circle passing through the points $v \in S$ and $v_i \in V$, and let y_i denote the intersection of C_i with the edge of T opposite to v_i. Then the spherical barycentric coordinates of v can be computed as*

$$b_i = \frac{\sin\delta_i}{\sin(\delta_i + \gamma_i)}, \quad i = 1, 2, 3, \tag{5.3}$$

where δ_i is the signed geodesic distance (measured along C_i) from y_i to v, and γ_i is the signed geodesic distance from v to v_i (see Fig. 1).

Proof: It suffices to consider the case $i = 1$. Clearly

$$v = \frac{\sin\delta_1}{\sin(\delta_1 + \gamma_1)} v_1 + \frac{\sin\gamma_1}{\sin(\delta_1 + \gamma_1)} y_1.$$

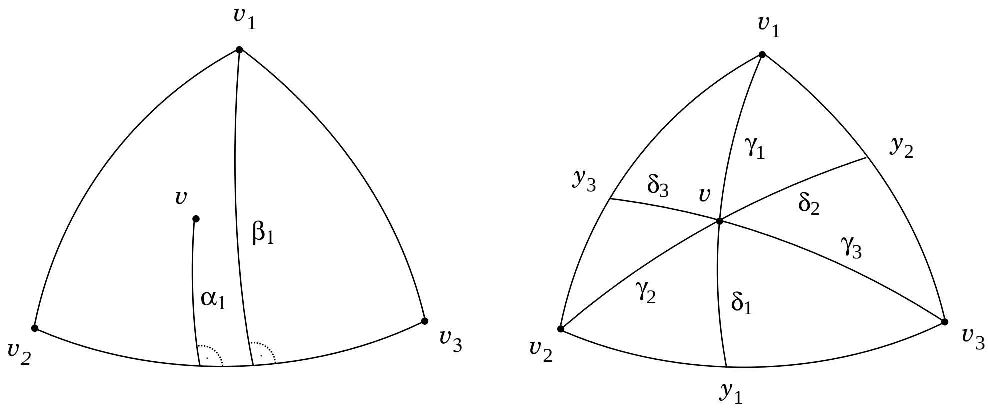

Fig.1. Computing spherical barycentric coordinates by (5.2) and (5.3).

Then since y_1 can be written as a linear combination of v_2 and v_3 (not involving v_1), (5.3) follows for $i = 1$. ■

Figure 1 illustrates both Theorems 5.2 and 5.3.

6. Uniqueness of Spherical Barycentric Coordinates

In the planar case, barycentric coordinates relative to a triangle are linear functions. The problem of defining barycentric coordinates relative to spherical triangles reduces to finding a natural generalization of linear functions for the sphere. Linear bivariate functions exhibit the important property that they vanish along lines in $\mathbb{R}^2$. This property seems to us also appropriate for defining a spherical analog $\mathcal{M}$ of the space of bivariate linear functions. Therefore, we require that

(i) $\mathcal{M}$ is a three dimensional space of continuous functions on the sphere,
(ii) for every great circle on S there exists a nonzero function $f \in \mathcal{M}$ vanishing identically along that circle,
(iii) $\mathcal{M}$ is rotation invariant, *i.e.*, if R is a rotation matrix then $f(\,\cdot\,) \in \mathcal{M}$ implies $f(R\,\cdot\,) \in \mathcal{M}$.

Note that the space $\mathcal{L}$ spanned by the spherical barycentric coordinate functions associated with any given spherical triangle satisfies all three requirements. We now show that this is the only space which does so.

Theorem 6.1. *The space $\mathcal{L}$ is the unique space of functions on S satisfying conditions (i)–(iii).*

Proof: Let C be a great circle on S. Suppose $\mathcal{M}$ satisfies (i)–(iii). Then there exists a nonzero function $f \in \mathcal{M}$ which vanishes identically on C. This means that the dimension of the space $\mathcal{M}|_C$ of functions from $\mathcal{M}$ restricted to C is at most two. We now show that it is equal to two. Suppose that the dimension is equal to one. By rotational invariance, this means that $\mathcal{M}|_C$ is the space of constant functions on C. Since $\mathcal{M}$ is rotation invariant, this must be true for all great circles C, and so $\mathcal{M}$ itself is the one dimensional space of constant functions. This contradicts our assumption that the dimension of $\mathcal{M}$ is 3. Therefore, we conclude that the dimension of $\mathcal{M}|_C$ is precisely two for all great circles C. By Theorem 3 of [Alfeld et al '95a], $\mathcal{M}|_C$ must be one of the spaces

$$\mathcal{L}_k := \operatorname{span}\{\sin k\theta, \cos k\theta\} \tag{6.1}$$

for some positive k. We next show that $\mathcal{M}$ cannot satisfy all three conditions (i)–(iii) unless $k = 1$.

Let $k \geq 2$ and let T be a spherical triangle with vertices v_1, v_2, v_3 which are chosen such that each of the three angles between pairs of the vectors v_1, v_2, v_3 is equal to π/k. Observe that this choice still makes it possible to place one of the vertices, say v_1, arbitrarily on S. Restricted to the edge $\overline{v_1 v_2}$, any nonzero function $f \in \mathcal{M}$ belongs to $\mathcal{L}_k$, and thus can be represented as

$$f|_{\overline{v_1 v_2}}(\theta) = a\cos(k\theta) + b\sin(k\theta), \quad a, b \in \mathbb{R},$$

where θ is a local angle variable in the plane containing the origin and v_1, v_2. Moreover, note that

$$f|_{\overline{v_1 v_2}}(\theta + \pi/k) = -f|_{\overline{v_1 v_2}}(\theta),$$

and therefore

$$f(v_1) + f(v_2) = 0. \tag{6.2}$$

Similarly, for the other two edges we obtain

$$f(v_2) + f(v_3) = 0, \quad f(v_3) + f(v_1) = 0. \tag{6.3}$$

The homogeneous system of equations (6.2)–(6.3) implies

$$f(v_1) = f(v_2) = f(v_3) = 0.$$

However, since v_1 can be chosen arbitrarily, this implies that f vanishes identically on S, which contradicts our assumption. Thus, k must equal 1, and $\mathcal{L}$ is the unique space satisfying (i)–(iii) since its restriction to C is $\mathcal{L}_1$. ■

It is instructive to discuss the geometric nature of the space $\mathcal{L}$. For $p \in \mathcal{L}$ we consider the surface

$$\mathcal{P} := \{p(v)v \;:\; v \in S\}.$$

Theorem 6.2. *The surface $\mathcal{P}$ represents a sphere passing through the origin.*

Proof: As pointed out earlier, the space $\mathcal{L}$ does not depend on the triangle with respect to which the spherical barycentric coordinates are defined. Therefore, for simplicity we will consider the quadrantal triangle T with vertices $v_i = e_i, i = 1,2,3$. A point $u \in \mathbb{R}^3$ with Cartesian coordinates u_1, u_2, u_3 can also be expressed in terms of its *symmetric polar coordinates* $r, \alpha_1, \alpha_2, \alpha_3$. Here, $r := \|u\|$, and $\alpha_1, \alpha_2, \alpha_3$ are the angles defined in Theorem 5.2. This terminology is justified by the identities

$$u_i = r \sin \alpha_i, \quad i = 1,2,3, \quad r^2 := \sum_{i=1}^{3} u_i^2, \quad \sum_{i=1}^{3} \sin^2 \alpha_i = 1, \tag{6.4}$$

which are reminiscent of identities for polar coordinates in $\mathbb{R}^2$. Also note that by (5.2), the barycentric coordinates b_1, b_2, b_3 of the point $v := u/r$ with respect to T are equal to the sines of the angles $\alpha_1, \alpha_2, \alpha_3$, respectively. A function $p \in \mathcal{L}$ can be uniquely represented as

$$p(v) = \sum_{i=1}^{3} a_i b_i(v), \quad v \in S, \quad a_i \in \mathbb{R}, \quad i = 1,2,3.$$

Hence, for every point u on the surface $\mathcal{P}$,

$$r = \sum_{i=1}^{3} a_i b_i = \sum_{i=1}^{3} a_i \sin \alpha_i.$$

Then by (6.4), this equation can be rewritten as

$$r^2 = \sum_{i=1}^{3} a_i r \sin \alpha_i = \sum_{i=1}^{3} a_i u_i,$$

or as

$$\sum_{i=1}^{3} (u_i^2 - a_i u_i) = 0,$$

which leads to

$$\sum_{i=1}^{3} \left(u_i - \frac{a_i}{2}\right)^2 = \sum_{i=1}^{3} \left(\frac{a_i}{2}\right)^2.$$

Thus, $\mathcal{P}$ indeed represents a sphere, centered at point $(a_1/2, a_2/2, a_3/2)$ and passing through the origin. ■

Theorem 6.2 shows that spheres passing through the origin are natural analogs of linear functions in the planar case. In fact, there is a one-to-one correspondence between planes in $\mathbb{R}^3$ and spheres passing through the origin. This is easily seen by considering the *inversion* centered at the origin, *i.e.*, the map $\Gamma : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}^3 \setminus \{0\}$ given by

$$\Gamma u := \frac{u}{\|u\|^2}, \quad u \in \mathbb{R}^3 \setminus \{0\}.$$

Any plane $\mathcal{H}$ in $\mathbb{R}^3$ not passing through the origin can be described uniquely by the equation $a^T u = 1$ for some vector $a \in \mathbb{R}^3$. The image of $\mathcal{H}$ under Γ is the sphere $\mathcal{P}$ of radius $\|a\|/2$ centered at $a/2$ (minus the origin). This follows immediately from the identity

$$\left\|\Gamma u - \frac{a}{2}\right\|^2 = \frac{u^T u - a^T u \|u\|^2 + \frac{1}{4} a^T a \|u\|^4}{\|u\|^4} = \frac{1}{4}\|a\|^2.$$

Conversely, the image of $\mathcal{P}$ under Γ is $\mathcal{H}$ because $\Gamma = \Gamma^{-1}$.

7. SBB-polynomials and SBB-patches

Given an HBB-polynomial

$$p(v) = \sum_{i+j+k=d} c_{ijk} B^d_{ijk}(v), \tag{7.1}$$

we refer to its restriction to a spherical triangle T as a *spherical Bernstein-Bézier* (SBB-) *polynomial.* SBB-polynomials inherit all of the properties of the HBB-polynomials discussed in Sect. 3. In particular, we can use the de Casteljau algorithm described in Theorem 3.4 to evaluate p or to subdivide it. Moreover, Theorem 3.6 shows that two SBB-polynomials (3.6) and (3.7) defined on neighboring spherical triangles join smoothly of order m across the common edge if and only if (3.8) holds.

Since we are now on the sphere, we can say more about the nature of HBB-polynomials. In particular, if we restrict such a polynomial to a great circle, it becomes a trigonometric polynomial of the geodesic distance along the circle – see [Alfeld et al '95a].

We are now in a position to define a surface patch in $\mathbb{R}^3$ associated with a SBB-polynomial.

Definition 7.1. *We call the surface*

$$\mathcal{P} := \{p(v)v \; : \; v \in T\}$$

a spherical Bernstein-Bézier (SBB-) *patch.*

A number of properties of SBB-patches follow immediately from properties of SBB-polynomials. For example, to compute points lying on the surface of an

SBB-patch, we can use the de Casteljau algorithm. To join patches smoothly, we only need to be sure that the corresponding SBB-polynomials join smoothly by enforcing the conditions of Theorem 3.6.

In CAGD applications, it is convenient to use *control nets* to construct and manipulate patches. Clearly, the natural way to define *control points* is to choose

$$C_{ijk} := c_{ijk} v_{ijk}, \qquad i+j+k=d,$$

where $v_{ijk} \in S$ are the vectors corresponding to appropriate points in the spherical triangle T. Moreover, in analogy with the planar case, we should choose

$$v_{d00} = v_1, \quad v_{0d0} = v_2, \quad v_{00d} = v_3,$$

where v_i are the vertices of T. The question now is how to choose the remaining v_{ijk}.

One choice is to take the usual Bézier sites on the planar triangle with vertices at v_1, v_2 and v_3, and project them upward onto the unit sphere. This leads to the points

$$v_{ijk} := \frac{iv_1 + jv_2 + kv_3}{\|iv_1 + jv_2 + kv_3\|}, \quad i+j+k=d. \tag{7.2}$$

A different choice is to take v_{ijk} to be some point on S where

$$B^d_{ijk}(v_{ijk}) = \max_{v \in T} B^d_{ijk}(v), \qquad i+j+k=d.$$

Using these points has the advantage that moving a particular control point C_{ijk} has the maximal effect at v_{ijk}.

Either of these choices could be used to provide a user with *design handles* for manipulating SBB-patches. Both are natural generalizations of the planar case. However, we note:

1) In contrast to the planar case, these choices do not lend themselves to a convenient geometric interpretation of C^1 smoothness conditions.
2) In the planar case, if the control points all lie on a plane, then the corresponding patch lies in this plane. We do not have an analog of this result for SBB-patches. This is because here the analog of a plane is a surface corresponding to a function from the space $\mathcal{L}$, while on great circles the analog of the space of linear functions is the space $\mathcal{L}_d$ defined in (6.1), see [Alfeld et al '95a]. But, unless $d=1$, the restriction of $\mathcal{L}$ to a great circle is not the space $\mathcal{L}_d$.

We now discuss the question of when it is possible to construct an SBB-patch which has a constant height above a spherical triangle T.

Theorem 7.2. *Let T be a spherical triangle, and suppose that d is even. Then there exists a unique SBB-polynomial p of degree d defined on T such that*

$$p(v) = 1, \qquad \text{for all } v \in T. \tag{7.3}$$

If d is odd, no such p exists.

Proof: If such a p exists, we can extend it to all of $\mathbb{R}^3$ by homogeneity. But by definition, any homogeneous polynomial p of degree d satisfies

$$p(-v) = (-1)^d p(v).$$

This means that (7.3) cannot hold when d is odd. Now suppose d is even. Then for $v = (v_1, v_2, v_3)$ on the unit sphere it is clear that the polynomial

$$p(v) = (v_1^2 + v_2^2 + v_3^2)^{d/2}, \qquad v = (v_1, v_2, v_3),$$

is of degree d and satisfies (7.3). The uniqueness assertion follows from the linear independence of the Bernstein-Bézier basis polynomials. ■

The fact that constants can be represented exactly by homogeneous polynomials of even degree on the sphere depends critically on the geometry of the sphere. For a general sphere-like surface it is not possible to represent constants exactly. For the purpose of applications, it is clearly desirable to be able to represent constants exactly. This suggests that the spaces where d is even may be more useful. However, if one wants to use a space where d is odd, this shortcoming can be remedied by adding constants to the space, and imposing additional conditions that ensure that if the underlying data have the same function value at all data sites, then the interpolant or approximant is identically equal to that value. For an illustration of this idea, see *e.g.* [Dyn '87].

We conclude this section by establishing the following degree-raising formula which is a direct analog of the corresponding degree-raising formula for the planar case, except that now (in view of Theorem 7.2), we restrict ourselves to raising the degree by two.

Theorem 7.3. *Let p be a SBB-polynomial as in (7.1). Then*

$$p = \sum_{i+j+k=d} c_{ijk} B_{ijk}^d = \sum_{i+j+k=d+2} \bar{c}_{ijk} B_{ijk}^{d+2},$$

where

$$\begin{aligned}\bar{c}_{ijk} = \frac{1}{(d+1)(d+2)} \Big[& i(i-1)c_{i-2,j,k} + \beta_{110} ij c_{i-1,j-1,k} \\ & + j(j-1)c_{i,j-2,k} + \beta_{101} ik c_{i-1,j,k-1} \\ & + k(k-1)c_{i,j,k-2} + \beta_{011} jk c_{i,j-1,k-1} \Big].\end{aligned}$$

Here

$$\beta_{011} = \frac{\sin^2 h_1}{\sin^2 \frac{h_1}{2}} - 2, \quad \beta_{101} = \frac{\sin^2 h_2}{\sin^2 \frac{h_2}{2}} - 2, \quad \textit{and} \quad \beta_{110} = \frac{\sin^2 h_3}{\sin^2 \frac{h_3}{2}} - 2,$$

where h_i is the arc length of the edge opposite vertex v_i, $i = 1, 2, 3$.

Proof: By Theorem 7.2, the constant function 1 can be written as a linear combination of the spherical Bernstein basis functions $\{B^2_{ijk}\}_{i+j+k=2}$. The coefficients can be found by interpolating at the vertices and the midpoints of the edges of T. This leads to

$$1 = b_1^2 + b_2^2 + b_3^2 + \beta_{011} b_2 b_3 + \beta_{101} b_1 b_3 + \beta_{110} b_1 b_2.$$

To complete the proof, we simply multiply p by this expression and collect terms. ■

8. Remarks

Remark 1. Let $\Delta := \{T_i\}_1^N$ be a *triangulation* of the sphere (see e.g. [Schumaker '93]). Given integers r and d, we define the *space of spherical splines* $\mathcal{S}_d^r(\Delta)$ to be the set

$$\mathcal{S}_d^r(\Delta) := \{s \in C^r(S) \ : \ s|_{T_i} \text{ is a SBB-polynomial of degree } d \text{ on } T_i,\ i = 1, \ldots, N\}.$$

This is the direct analog of the classical polynomial splines defined on planar triangulations, and clearly should have an analogous constructive theory, including results on dimension, local bases, approximation power, etc. We discuss these matters in [Alfeld et al '95b].

Remark 2. As in the classical univariate and planar Bernstein-Bézier theory, it is possible to give formulae for derivatives of SBB polynomials. We leave these to our paper [Alfeld et al '95c], where we also discuss several practical methods for interpolating scattered data on the surface of the sphere (or a sphere-like surface).

Remark 3. As in the planar case (see [Farin '86]), using our spherical barycentric coordinates, we can define *rational* spherical Bézier surfaces which can be used for interpolation. For an application to data fitting on the sphere, see [Liu & Schumaker '95].

Remark 4. Using the theory developed here, it is straightforward to define spherical analogs of *simplex splines*. In fact, our spaces of SBB-polynomials are related to certain multivariate trigonometric simplex splines defined in [Koch '88].

Remark 5. The spherical polynomials defined here are closely related to spherical harmonic functions. It is well known that *spherical harmonics* are restrictions of homogeneous harmonic polynomials to the unit sphere [Müller '66]. Thus, spherical polynomials are linear combinations of spherical harmonics. In particular, spherical barycentric coordinates are spherical harmonics of degree one.

Remark 6. Our construction of barycentric coordinates and BB-polynomials generalizes easily to spheres in higher dimension.

Remark 7. For other definitions of barycentric coordinates on the sphere, see [Baumgardner & Frederickson '85, Brown & Worsey '92, Lawson '84]. Of these three, only Lawson's coordinates are similar to ours, and in fact differ only in that they are normalized to sum to one. In all three papers, the barycentric coordinates are required to form a partition of unity, and therefore they inevitably fail to have many of the important properties which ours have.

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