Local Similarities and the Haagerup Property

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Abstract

A new class of groups, the finitely determined groups of local similarities on compact ultrametric spaces, is introduced and it is proved that groups in that class have the Haagerup property (that is, they are a-T-menable in the sense of Gromov). The class includes Thompson’s groups, which have already been shown to have the Haagerup property by D. Farley, as well as many other groups acting on boundaries of trees.

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1 Introduction

This paper is motivated by D. Farley’s theorem [8] that R. Thompson’s famous infinite, finitely presented, simple group \( V \) has the Haagerup property. Farley’s result and method are extended here to a new class of countable, discrete groups, which includes many Thompson-like groups and groups of local similarities on compact, locally rigid ultrametric spaces.
A countable discrete group $\Gamma$ has the **Haagerup property** if there exists an isometric action $\Gamma \curvearrowright \mathcal{H}$ on some affine Hilbert space $\mathcal{H}$ such that the action is metrically proper, which means for every bounded subset $B$ of $\mathcal{H}$, the set $\{g \in \Gamma \mid gB \cap B \neq \emptyset\}$ is finite. The Haagerup property is also called Gromov’s a-T-menability property. We refer to Cherix, Cowling, Jolissaint, Julg, and Valette [5] for a detailed discussion of the Haagerup property.

One reason for interest in the Haagerup property is that Higson and Kasparov [10] proved that the Baum–Connes conjecture with coefficients is true for groups with the Haagerup property.

The groups for which we verify the Haagerup property come with actions on compact ultrametric spaces. Examples of such spaces are the end spaces, or boundaries, of rooted, locally finite simplicial trees. See Section 2, especially Remark 2.3, for more details.

The actions of the groups on compact ultrametric spaces are by local similarities. There is a finiteness condition on the local restrictions of these local similarities. See Section 3 for the precise definitions.

The following is the main result of this paper.

**Theorem 1.1** If $\Gamma$ is a finitely determined group of local similarities on a compact ultrametric space $X$, then $\Gamma$ has the Haagerup property.

Examples of groups satisfying the hypothesis of Theorem 1.1 are given in Section 4. These include Thompson’s groups ($F$, $T$, and $V$) as well as other Thompson-like groups. Moreover, if $X$ is a locally rigid, compact ultrametric space, then the full group $LS(X)$ of all local similarities on $X$ is shown to satisfy the hypothesis. Such spaces $X$ include the end spaces of rigid trees in the sense of Bass and Lubotzky [2] with many interesting examples constructed by Bass and Kulkarni [1] and Bass and Tits [3]. See Hughes [11] for more on locally rigid ultrametric spaces.

Theorem 1.1 is proved in Section 6 by showing that the given action of $\Gamma$ on $X$ induces a zipper action of $\Gamma$ on some set. Zipper actions are defined in Section 5. This concept is implicit in Farley [8] and is a special case of Valette’s characterization of the Haagerup property for countable, discrete groups [5, Proposition 7.5.1].

Zipper actions seem to be related to proper actions on spaces with walls, another sufficient condition for the Haagerup property (see Cherix et al. [5, Section 1.2.7]). In fact, Farley has a separate proof, using this criterion, that Thompson’s groups have the Haagerup property [7, 9]. See Cherix, Martin, and Valette [6] for a characterization of the Haagerup property for countable, discrete groups in terms of spaces of walls. Example 6.7 shows that zipper actions do not naively lead to spaces with walls. One should also note the similarity of zipper actions with the criterion of Sageev [17] for a group pair to be multi–ended.

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2 Ultrametric spaces and local similarities

This section contains some background on ultrametric spaces and local similarities.

Definition 2.1 An ultrametric space is a metric space \((X, d)\) such that \(d(x, y) \leq \max\{d(x, z), d(z, y)\}\) for all \(x, y, z \in X\).

Classical examples of ultrametrics arise from \(p\)-adic norms, where \(p\) is a prime. For example, the \(p\)-adic norm \(|·|_p\) on the integers \(\mathbb{Z}\) is defined by \(|x|_p := p^{-\max\{n \in \mathbb{N}_0 \mid p^n \text{ divides } x\}}\). The corresponding metric on \(\mathbb{Z}\) is an ultrametric. For more on the relationship between ultrametrics and \(p\)-adics, see Robert [15].

For the purposes of this paper, the most important examples of ultrametrics arise as end spaces of trees, which are recalled in the following example.

Example 2.2 Let \(T\) be a locally finite simplicial tree; that is, \(T\) is the (geometric realization of) a locally finite, one-dimensional, simply connected, simplicial complex. There is a natural unique metric \(d\) on \(T\) such that \((T, d)\) is an \(\mathbb{R}\)-tree, every edge is isometric to the closed unit interval \([0,1]\), and the distance between distinct vertices \(v_1, v_2 \in T\) is the minimum number of edges in a sequence of edges \(e_0, e_1, \ldots, e_n\) with \(v_1 \in e_0, v_2 \in e_n\) and \(e_i \cap e_{i+1} = \emptyset\) for \(0 \leq i \leq n - 1\). Whenever we refer to a locally finite simplicial tree \(T\), the metric \(d\) on \(T\) will be understood to be the natural one just described. Choose a root (i.e., a base vertex) \(v \in T\) and define the end space of \((T, v)\) by

\[
\text{end}(T, v) = \{x : [0, \infty) \to T \mid x(0) = v \text{ and } x \text{ is an isometric embedding}\}.
\]

For \(x, y \in \text{end}(T, v)\), define

\[
d_e(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1/e^t_0 & \text{if } x \neq y \text{ and } t_0 = \sup\{t \geq 0 \mid x(t) = y(t)\}. \end{cases}
\]

It follows that \((\text{end}(T, v), d_e)\) is a compact ultrametric space of diameter \(\leq 1\).

Remark 2.3 There is a well-known, close relationship between trees and ultrametrics. For example, if \((X, d)\) is a compact ultrametric space, then there exists a rooted, locally finite simplicial tree \((T, v)\) and a homeomorphism \(h : [0, \infty) \to [0, \infty)\) such that \((X, hd)\) is isometric to \(\text{end}(T, v)\). Moreover, every compact ultrametric space \((X, d)\) of diameter \(\leq 1\) is isometric to the endspace of a rooted, proper \(\mathbb{R}\)-tree \((T, v)\). See Hughes [12] and [11] for more details and further references.

1 An \(\mathbb{R}\)-tree is a metric space \((T, d)\) that is uniquely arcwise connected, and for any two points \(x, y \in T\) the unique arc from \(x\) to \(y\) is isometric to the subinterval \([0, d(x, y)]\) of \(\mathbb{R}\).
In an ultrametric space, if two balls intersect, then one must contain the other. Moreover, closed balls are open sets and open balls are closed sets. In the compact case, there is the following result, the proof of which is elementary and is left to the reader.

**Lemma 2.4** If $X$ is a compact ultrametric space and $Y \subseteq X$, then the following are equivalent:

1. $Y$ is open and closed.
2. $Y$ is a finite union of open balls in $X$.
3. $Y$ is a finite union of closed balls in $X$. $\square$

We conclude this section with the basic definitions concerning local similarities.

**Definition 2.5** If $\lambda > 0$, then a map $g : X \to Y$ between metric spaces $(X, d_X)$ and $(Y, d_Y)$ is a $\lambda$-similarity provided $d_Y(gx, gy) = \lambda d_X(x, y)$ for all $x, y \in X$.

**Definition 2.6** A homeomorphism $g : X \to X$ between metric spaces is a local similarity if for every $x \in X$ there exists $r, \lambda > 0$ such that $g$ restricts to a surjective $\lambda$-similarity $g| : B(x, r) \to B(gx, \lambda r)$.

**Definition 2.7** For a metric space $X$, $LS(X)$ denotes the group of all local similarities from $X$ onto $X$.

We will be concerned with the group $LS(X)$ only when $X$ is a compact ultrametric space. It has a natural topology (the compact-open topology), but in this paper we always endow subgroups of $LS(X)$ with the discrete topology.

## 3 Finitely determined groups of local similarities

In this section, we introduce the groups that are the object of study in this paper and establish some of their elementary properties. Throughout this section, let $X$ be a compact ultrametric space with ultrametric $d$. The groups are defined in terms of an extra structure on $X$, which we now define.

**Definition 3.1** Suppose for each pair $B_1, B_2$ of closed balls in $X$ there is given a (possibly empty) finite set $\text{Sim}(B_1, B_2)$ of surjective similarities $B_1 \to B_2$ such that whenever $B_1, B_2, B_3$ are closed balls in $X$, the following properties hold:

1. (Identities) $\text{id}_{B_1} \in \text{Sim}(B_1, B_1)$.
2. (Inverses) If $h \in \text{Sim}(B_1, B_2)$, then $h^{-1} \in \text{Sim}(B_2, B_1)$.
3. (Compositions) If $h_1 \in \text{Sim}(B_1, B_2)$ and $h_2 \in \text{Sim}(B_2, B_3)$, then $h_2 h_1 \in \text{Sim}(B_1, B_3)$.
4. (Restrictions) If \( h \in \text{Sim}(B_1, B_2) \) and \( B_3 \subseteq B_1 \), then \( h|B_3 \in \text{Sim}(B_3, h(B_3)) \).

The collection \( \text{Sim}(X) = \{ \text{Sim}(B_1, B_2) \mid B_1, B_2 \text{ are closed balls in } X \} \) is called a finite similarity structure for \( X \).

The word finite is used here to describe the similarity structure, not to imply that the collection \( \text{Sim}(X) \) is finite (in general, it will not be finite), but because \( \text{Sim}(X) \) is a collection of finite sets.

**Example 3.2** The trivial finite similarity structure on \( X \) is given by

\[
\text{Sim}(B_1, B_2) = \begin{cases} 
\{ \text{id}_{B_1} \} & \text{if } B_1 = B_2 \\
\emptyset & \text{otherwise}
\end{cases}
\]

for each pair of closed balls \( B_1, B_2 \) in \( X \).

More examples are given in the next section.

**Definition 3.3** A subgroup \( \Gamma \) of \( LS(X) \) is finitely determined by the finite similarity structure \( \text{Sim}(X) \) if for every \( g \in \Gamma \) and \( x \in X \) there exists \( r, \lambda > 0 \) such that \( g|\bar{B}(x, r) \in \text{Sim}(\bar{B}(x, r), \bar{B}(gx, \lambda r)) \). In this case, the group \( \Gamma \) is said to be a finitely determined group of local similarities on \( X \).

Throughout the rest of this section, let \( \Gamma \leq LS(X) \) be a group finitely determined by the finite similarity structure \( \text{Sim}(X) \). Recall that \( \Gamma \) is given the discrete topology.

**Definition 3.4** A region for \( g \in \Gamma \) is a closed ball \( B \) in \( X \) such that \( g(B) \) is a ball and \( g|B \in \text{Sim}(B, g(B)) \). A region \( B \) for \( g \in \Gamma \) is a maximum region for \( g \) if it is not properly contained in any region for \( g \).

**Lemma 3.5** For each \( g \in \Gamma \) and for each \( x \in X \) there exists a unique maximum region \( B \) for \( g \) such that \( x \in B \).

**Proof.** By definition, \( x \) is contained in some region \( R \) for \( g \). Compactness of \( X \) implies \( R \) is contained in only finitely many closed balls of \( X \). There is a largest such ball \( B \) that is a region for \( g \), and it must be a maximum region. It is the unique maximum region for \( g \) containing \( x \) because any two intersecting balls of \( X \) have the property that one contains the other. \( \square \)

It follows that for each \( g \in \Gamma \), the maximum regions of \( g \) form a partition of \( X \) (that is, the maximum regions of \( g \) cover \( X \) and are mutually disjoint), and any closed ball in \( X \) contains, or is contained in, a maximum region of \( g \).

**Definition 3.6** If \( g \in \Gamma \), then the maximum partition for \( g \) is the partition of \( X \) into the maximum regions of \( g \).

Thus, any partition of \( X \) into regions for an element \( g \in \Gamma \) refines the maximum partition for \( g \).

The following lemma follows immediately from the definitions and the Inverses Property.
Lemma 3.7 If \( g \in \Gamma \) and \( R \) is a region for \( g \), then \( g(R) \) is a region for \( g^{-1} \). In addition, if \( R \) is a maximum region for \( g \), then \( g(R) \) is a maximum region for \( g^{-1} \). □

Lemma 3.8 Let \( \mathcal{P}_+ \) and \( \mathcal{P}_- \) be two partitions of \( X \) into closed balls. The set

\[
\Gamma(\mathcal{P}_\pm) = \{ g \in \Gamma \mid \mathcal{P}_+ \text{ is the maximum partition for } g \text{ and } \mathcal{P}_- \text{ is the maximum partition for } g^{-1} \}
\]

is finite.

Proof. Say \( \mathcal{P}_+ = \{B_1, \ldots, B_n\} \) where \( n = |\mathcal{P}_+| \) is the cardinality of \( \mathcal{P}_+ \). Let \( \text{Bi}(\mathcal{P}_+, \mathcal{P}_-) \) denote the finite set of bijections from \( \mathcal{P}_+ \) to \( \mathcal{P}_- \). For \( h \in \text{Bi}(\mathcal{P}_+, \mathcal{P}_-) \), let \( S_h := \prod_{i=1}^n \text{Sim}(B_i, h(B_i)) \) and note that \( S_h \) is finite. Define the finite set \( F \) to be the disjoint union \( F := \bigsqcup_{h \in \text{Bi}(\mathcal{P}_+, \mathcal{P}_-)} S_h \), which we prefer to write as \( F = \bigcup_{h \in \text{Bi}(\mathcal{P}_+, \mathcal{P}_-)} S_h \). If \( g \in \Gamma(\mathcal{P}_\pm) \), then \( g_* \in \text{Bi}(\mathcal{P}_+, \mathcal{P}_-) \) is defined by \( g_*(B) = g(B) \) for all \( B \in \mathcal{P}_+ \). Clearly, there is an injection \( \Gamma(\mathcal{P}_\pm) \to F \) given by \( g \mapsto (g_*, (g|B_1, \ldots, g|B_n)) \). □

Recall that if \( \mathcal{P} \) and \( \mathcal{Q} \) are two collections, then \( \mathcal{P} \text{ refines } \mathcal{Q} \) means for every \( P \in \mathcal{P} \) there exists \( Q \in \mathcal{Q} \) such that \( P \subseteq Q \).

Lemma 3.9 Let \( \mathcal{P}_+ \) and \( \mathcal{P}_- \) be two partitions of \( X \) into closed balls. The set

\[
\Gamma_{\text{ref}}(\mathcal{P}_\pm) = \{ g \in \Gamma \mid \mathcal{P}_+ \text{ refines the maximum partition for } g \text{ and } \mathcal{P}_- \text{ refines the maximum partition for } g^{-1} \}
\]

is finite.

Proof. Given any closed ball in \( X \), there exist only finitely many distinct closed balls of \( X \) containing \( B \). Hence, any partition of \( X \) into closed balls refines only finitely many other partitions of \( X \) into closed balls. Thus, there exist only finitely many pairs, say \( (\mathcal{P}_+^i, \mathcal{P}_-^i) \) for \( i = 1, \ldots, n \), of partitions of \( X \) into closed balls such that \( \mathcal{P}_+ \) refines \( \mathcal{P}_+^i \) and \( \mathcal{P}_- \) refines \( \mathcal{P}_-^i \) for all \( i = 1, \ldots, n \). Clearly, \( \Gamma_{\text{ref}}(\mathcal{P}_\pm) = \bigcup_{i=1}^n \Gamma(\mathcal{P}_\pm^i) \) and the result follows from Lemma 3.8. □

Lemma 3.10 \( \Gamma \) is countable.

Proof. \( X \) has only countably many closed balls; hence, \( X \) has only countably many partitions into closed balls and only countably many pairs, say \( (\mathcal{P}_+^i, \mathcal{P}_-^i) \) for \( i = 1, 2, 3, \ldots \), of partitions of \( X \) into closed balls. Clearly, \( \Gamma = \bigcup_{i=1}^\infty \Gamma(\mathcal{P}_\pm^i) \) and the result follows from Lemma 3.8. □

4 Examples

In this section we give examples of finitely determined groups of local similarities on compact ultrametric spaces. The examples include Thompson’s groups so
that Farley’s result [8] is recovered from Theorem 1.1. Many other Thompson-like groups are finitely determined, as well as the full local similarity groups of end spaces of certain trees constructed by Bass and Kulkarni [1] and Bass and Tits [3].

We begin by recalling standard alphabet language and notation. An excellent reference is Nekrashevych [13]. An alphabet is a non-empty finite set $A$ and finite (perhaps empty), ordered subsets of $A$ are words. The set of all words is denoted $A^*$ and the set of infinite words is denoted $A^\omega$; that is,

$$A^* = \prod_{n=0}^{\infty} A^n \quad \text{and} \quad A^\omega = \prod_{1}^{\infty} A.$$ 

Let $T_A$ be the tree associated to $A$. The vertices of $T_A$ are words in $A$; two words $v$, $w$ are connected by an edge if and only if there exists $x \in A$ such that $v = wx$ or $vx = w$. The root of $T_A$ is $\emptyset$. Thus, $A^\omega = \text{end}(T_A, \emptyset)$ and so comes with a natural ultrametric as described in Example 2.2 making $A^\omega$ compact.

We may assume that $A$ is totally ordered. There is then an induced total order (the lexicographic order) on $A^\omega$.

Example 4.1 (The Higman–Thompson groups $G_{d,1}$) Let $\Gamma = LS_{l.o.p}(A^\omega)$ be the subgroup of $LS(A^\omega)$ consisting of locally order preserving local similarities on $A^\omega$, where a map $h: A^\omega \to A^\omega$ is locally order preserving if for each $x \in A^\omega$ there exists $\epsilon > 0$ such that $h|_{B(x, \epsilon)}: B(x, \epsilon) \to A^\omega$ is order preserving. We denote $\text{id}_{A^\omega} = e$; it is the unique order preserving isometry $A^\omega \to A^\omega$. Any closed ball in $A^\omega$ has a unique order preserving similarity onto $A^\omega$; hence, if $B_1$ and $B_2$ are two closed balls in $A^\omega$, then there is a unique order preserving similarity of $B_1$ onto $B_2$. Let $\text{Sim}(B_1, B_2)$ consist of that unique order preserving similarity. This can be described using alphabet language quite easily as follows. A closed ball in $A^\omega$ is given by $vA^\omega$, where $v \in A^*$. For $v, w \in A^*$, $\text{Sim}(vA^\omega, wA^\omega)$ consists of the unique order preserving similarity $vA^\omega \to wA^\omega$; $vx \mapsto wx$. Clearly, this defines a finite similarity structure $\text{Sim}(A^\omega)$ and $\Gamma$ is a finitely determined group of local similarities on $A^\omega$.

When the alphabet is $A = \{0, 1\}$, we get Thompson’s group $V = LS_{l.o.p}(A^*)$. The subgroups $F \leq T \leq V$ are also finitely determined by the same finite similarity structure $\text{Sim}(A^*)$ (elements of $T$ are further required to be cyclicly order preserving; those of $F$, to be order preserving). In general, $LS_{l.o.p}(A^\omega)$ is the Higman–Thompson group $G_{d,1}$, where $d = |A|$. For background on these groups, see Cannon, Floyd, and Parry [4] and for other references, see Hughes [11, Section 12.3].

Example 4.2 (Generalized Higman–Thompson groups $LS_{l.o.p}(X)$) The previous example can easily be extended to end spaces of rooted, ordered, proper $\mathbb{R}$-trees $(T, v)$ so that the groups $LS_{l.o.p}(X)$, where $X = \text{end}(T, v)$, defined in Hughes [11, Section 12.3], become finitely determined groups of local similarities on $X$. In particular, it is easy to see that the Higman-Thompson groups $G_{d,n}$, $n \geq 1$, fit into this framework.
Example 4.3 (Subgroups) A subgroup $H$ of a group $\Gamma$ of local similarities finitely determined by the finite similarity structure $\text{Sim}(X)$ is also finitely determined by $\text{Sim}(X)$. This is clear because Definition 3.3 is a condition on elements of $\Gamma$, which therefore holds for each element of $H$.

Example 4.4 (Nekrashevych-Röver groups $V_d(H)$, $H$ finite) Suppose $H$ is a finite, self-similar group over the alphabet $A$, with $d = |A|$ (see Nekrashevych [13]). Nekrashevych [14] defines a group $V_d(H) \leq \text{LS}(A^\omega)$ generalizing a construction of Röver [16]. To describe these groups note that there is a natural similarity from $A^\omega$ onto any closed ball of $A^\omega$; thus, any surjective similarity $h: B_1 \to B_2$ between closed balls gives rise to an isometry $h_\ast$ of $A^\omega$:

$$h_\ast: A^\omega \to B_1 \xrightarrow{h} B_2 \to A^\omega.$$  

Then an element $g \in \text{LS}(A^\omega)$ is in $V_d(H)$ if and only if for each $x \in A^\omega$ there exists $\epsilon, \lambda > 0$ such that $g|: B(x, \epsilon) \to B(gx, \lambda \epsilon)$ is a $\lambda$-similarity and $(g|)_x \in H$. For his general construction, Nekrashevych does not require $H$ to be finite, but we require it in order to define the following finite similarity structure $\text{Sim}(A^\omega)$: if $B_1, B_2$ are closed balls of $A^\omega$, then $\text{Sim}(B_1, B_2)$ consists of all surjective similarities $h: B_1 \to B_2$ such that $h_\ast \in H$. It is the self-similarity of $H$ that gives the Restrictions Property. Clearly, $V_d(H)$ is finitely determined by $\text{Sim}(A^\omega)$.

For example, note that in the special case $H = \{1\}, V_d(H) = G_{d,1}$.

For a nontrivial example, let $\Sigma_d$ be the symmetric group on $A$. The action of $\Sigma_d$ on $A^\ast$ given by $\sigma(a_1 \ldots a_n) = \sigma(a_1) \ldots \sigma(a_n)$ induces an action of $\Sigma_d$ on the tree $T_A$ and we let $H \cong \Sigma_d$ be the corresponding self-similar subgroup of $\text{Aut}(T_A)$. Note that $G_{d,1} \leq V_d(\Sigma_d)$ and that $\Gamma := V_d(\Sigma_d) \cap \text{Aut}(T_A)$ is a contracting self-similar group with nucleus $\Sigma_d$ (see Nekrashevych [13, Section 2.11] for the definitions).

Generalizing this last observation, let $\Gamma$ be any contracting self-similar subgroup of $\text{Aut}(T_A)$ whose nucleus $\mathcal{N}$ is a finite group (in general, contracting self-similar groups have nuclei that are finite sets—the condition that the nucleus be a group is quite restrictive). It follows that $\mathcal{N}$ is a finite self-similar group and we can form the finitely determined group $V_d(\mathcal{N})$. For each pair $B_1, B_2$ of closed balls in $A^\omega$, $\text{Sim}(B_1, B_2)$ is naturally identified with $\mathcal{N}$. Note that $\Gamma \leq V_d(\mathcal{N}) \cap \text{Aut}(T_A)$.

Example 4.5 (Groups acting on trees with finite vertex stabilizers) Let $(T, v)$ be a geodesically complete, rooted, locally finite simplicial tree, where geodesically complete means no vertex, except possibly the root, has valency 1. Let $\Gamma$ be a subgroup of the isometry group $\text{Isom}(T)$ such that $\Gamma$ has finite vertex stabilizers (that is, for each vertex $w \in T$, the isotropy group $\Gamma_w$ is finite). There is a well-known homomorphism $\epsilon: \text{Isom}(T) \to \text{LS}(X)$, where $X = \text{end}(T, v)$, explicitly described in Hughes [11, Section 12.1]. We will show that $\epsilon(\Gamma)$ is finitely determined. If $B$ is a closed ball in $X$, then there exists a vertex $w_B \in T$ such that $B = \{x \in X \mid x(d(v, w_B)) = w_B\}$ and $T_B = \{x(t) \mid x \in$
B and \( t \geq d(v, w_B) \) is a subtree of \( T \) with \( B \) similar to \( \text{end}(T_B, w_B) \). Define a finite similarity structure \( \text{Sim}(X) \) as follows. If \( B_1, B_2 \) are closed balls in \( X \), let

\[
\text{Sim}(B_1, B_2) = \{ \epsilon(g) : B_1 \to B_2 \mid g \in \Gamma, \ g(w_{B_1}) = w_{B_2}, \text{ and } g(T_{B_1}) = T_{B_2} \}.
\]

The finite vertex stabilizers assumption implies that \( \text{Sim}(B_1, B_2) \) is finite. The other properties of a similarity structure are easy to verify. Moreover, \( \epsilon(\Gamma) \) is finitely determined by \( \text{Sim}(X) \). Note that \( \epsilon \) is an injection except when \( T \) is isometric to \( \mathbb{R} \). In particular, finitely generated free groups are finitely determined. Of course, it is well-known that discrete groups acting on trees with finite vertex stabilizers have the Haagerup property (see Cherix et al. [5, Section 1.2.3].

Example 4.6 (Local similarity groups of locally rigid, compact ultrametric spaces) Let \( X \) be a locally rigid, compact ultrametric space as defined in Hughes [11]. It is proved there that a compact ultrametric space \( X \) is locally rigid if and only if the isometry group \( \text{Isom}(X) \) is finite. In particular, the isometry group of any closed ball in \( X \) is also finite. From this it follows easily that for any two closed balls \( B_1, B_2 \) in \( X \), the set of all surjective similarities from \( B_1 \) to \( B_2 \) is finite. We can therefore define a finite similarity structure \( \text{Sim}(X) \) by letting \( \text{Sim}(B_1, B_2) \) be the set of all similarities from \( B_1 \) onto \( B_2 \). Then the group \( \Gamma = \text{LS}(X) \) of all local similarites of \( X \) onto itself is finitely determined by \( \text{Sim}(X) \).

Example 4.7 (Local similarity groups of end spaces of rigid trees) Let \( T \) be a locally finite simplicial tree that is rigid; that is, the group of automorphisms \( \text{Aut}(T) \) is discrete; see Bass and Lubotzky [2]. Let \( X = \text{end}(T, v) \), where \( v \) is a chosen vertex of \( T \). Assuming that \( (T, v) \) is geodesically complete, the rigidity of \( T \) is equivalent to local rigidity of \( X \); see Hughes [11, Section 12.2]. Hence, \( \Gamma := \text{LS}(X) \) is finitely determined as described in the preceding example. An interesting source of examples of rigid trees come from \( \pi \)-rigid graphs of Bass and Kulkarni [1] and Bass and Tits [3]. These are finite, connected, simplicial graphs \( G \) with the property that if \( \tilde{G} \) is the universal covering tree of \( G \), then \( \text{Aut}(\tilde{G}) = \pi_1(G) \). In particular, \( \tilde{G} \) is rigid and \( \text{LS}(\text{end}(\tilde{G}, v)) \) is finitely determined.

5 Zipper action: A criterion for the Haagerup property

In this section we discuss a sufficient condition, called a zipper action, for a discrete group to have the Haagerup property. This condition is implicit in Farley [8] and is a special case of the necessary and sufficient condition due to Valette [5, Proposition 7.5.1]. Apart from the terminology, there is nothing original in this section.

Definition 5.1 A discrete group \( \Gamma \) has a zipper action if there is a left action \( \Gamma \curvearrowright \mathcal{E} \) of \( \Gamma \) on a set \( \mathcal{E} \) and a subset \( Z \subseteq \mathcal{E} \) such that
1. for every $g \in \Gamma$, the symmetric difference $gZ \Delta Z$ is finite, and

2. for every $r > 0$, \{ $g \in \Gamma \mid |gZ \Delta Z| \leq r$ \} is finite.

Note that if $\Gamma$ is an infinite group then condition 2 implies $Z$ must also be infinite.

The terminology arises as follows. We think of $Z$ as being an infinite zipper in $E$ that is unzipped by the action of $\Gamma$. Only a finite portion is unzipped by any finite subset of $\Gamma$, but as one takes larger finite subsets of $\Gamma$, more of $Z$ is unzipped.

Example 5.2 We show that the group $Z$ has a zipper action. Let

$$\mathcal{E} = \{(−∞, n] \subset \mathbb{R} \mid n \in \mathbb{Z}\} \quad \text{and} \quad Z = \{(−∞, n] \in \mathcal{E} \mid n \leq 0\}.$$ 

An action $Z \curvearrowright \mathcal{E}$ is defined by $g \cdot (−∞, n] = (−∞, g + n]$. If $g \in \mathbb{Z}$ and $g \geq 0$, then $Z \subseteq gZ$ and $gZ \Delta Z = \{(−∞, n] \in \mathcal{E} \mid 0 < n \leq g\}$. If $g \in \mathbb{Z}$ and $g \leq 0$, then $gZ \subseteq Z$ and $gZ \Delta Z = \{(−∞, n] \in \mathcal{E} \mid g < n \leq 0\}$. One may say “$Z$ is taken further off itself as $g \to +\infty$ in $\mathbb{Z}$” and “$Z$ is taken deeper into itself as $g \to -\infty$ in $\mathbb{Z}$.” Thus, $|gZ \Delta Z| = |g|$ for all $g \in \mathbb{Z}$. If $r \geq 0$, then \{ $g \in \mathbb{Z} \mid |gZ \Delta Z| = |g| \leq r$ \} = \{ $g \in \mathbb{Z} \mid |g| \leq r$ \}, which is finite.

The proof of the following theorem, which is a special case of Valette [5, Proposition 7.5.1], is implicit in Farley [8], but is included for completeness.

Theorem 5.3 If the discrete group $\Gamma$ has a zipper action, then $\Gamma$ has the Haagerup property.

Proof. Define $\pi : \Gamma \to \ell^\infty(\mathcal{E})$ by $\pi(g) = \chi_{gZ} - \chi_Z$ (where $\chi_Y$ denotes the characteristic function of $Y \subseteq \mathcal{E}$).

1. The support of $\pi(g)$ is $gZ \Delta Z$; hence, $\pi(g)$ is finitely supported and $\pi(g)$ is in the Hilbert space $\ell^2(\mathcal{E})$ for all $g \in \Gamma$.

2. The square of the $\ell^2$-norm $\|\pi(g)\|_2^2 = |gZ \Delta Z|$ for all $g \in \Gamma$.

3. For every $r > 0$, \{ $g \in \Gamma \mid \|\pi(g)\|_2 \leq r$ \} is finite.

The action of $\Gamma$ on $\mathcal{E}$ induces a unitary left action of $\Gamma$ on $\ell^2(\mathcal{E})$, $\rho : \Gamma \to \mathcal{B}(\ell^2(\mathcal{E}))$, where $\mathcal{B}(\ell^2(\mathcal{E}))$ is the space of bounded linear operators on $\ell^2(\mathcal{E})$. Namely, $\rho(g)(f)(e) = f(g^{-1}e)$ for $g \in \Gamma$, $f : \mathcal{E} \to \mathbb{C}$ in $\ell^2(\mathcal{E})$, and $e \in \mathcal{E}$.

One checks that $\pi$ is a 1-cocycle for $\rho$; that is, $\pi(g_1g_2) = \rho(g_1)\pi(g_2) + \pi(g_1)$ for all $g_1, g_2 \in \Gamma$. For this it is useful to observe that $g\chi_Y = \chi_{gY}$ in $\ell^\infty(\mathcal{E})$ for any $Y \subseteq \mathcal{E}$. It follows that $A : \Gamma \to Isom(\ell^2(\mathcal{E}))$ defined by $A(g)(f) = \rho(g)(f) + \pi(g)$ is an affine isometric action of $\Gamma$ on $\ell^2(\mathcal{E})$. Moreover, property 3 above guarantees that $A$ is metrically proper. □

Remark 5.4 The existence of a zipper action is preserved by direct sums of groups. For let $\Gamma_i$ ($i = 1, 2$) be discrete groups having left actions $\Gamma_i \curvearrowright \mathcal{E}_i$ and subsets $Z_i \subset \mathcal{E}_i$ as in Definition 5.1. Let $\Gamma := \Gamma_1 \oplus \Gamma_2$, $\mathcal{E} := \mathcal{E}_1 \uplus \mathcal{E}_2$, $Z := Z_1 \downarrow Z_2$, and define a left action $\Gamma \curvearrowright \mathcal{E}$ in the obvious way: $(g_1, g_2) \cdot e_i = g_ie_i$ where $e_i \in \mathcal{E}_i$ and $i \in \{1, 2\}$. The conditions are readily checked.
6 The main construction

In this section we prove the following theorem.

**Theorem 6.1** If $\Gamma$ is a finitely determined group of local similarities on a compact ultrametric space $X$, then $\Gamma$ has a zipper action.

Clearly, Theorem 1.1 follows from Theorems 5.3 and 6.1.

Throughout this section, $X$ will denote a compact ultrametric space and $\Gamma \leq LS(X)$ will be a group finitely determined by a finite similarity structure $Sim(X)$.

Before defining a set $E$ with a zipper action $\Gamma \bowtie E$, another definition is required.

**Definition 6.2** Let $B$ be a closed ball in $X$. A function $f : B \to X$ is a local similarity embedding if for each $x \in B$ there exist $r, \lambda > 0$ such that $\overline{B}(x, r) \subseteq B$ and $f| : \overline{B}(x, r) \to B(fx, \lambda r)$ is a surjective $\lambda$-similarity. If the choices can be made so that $f| \in Sim(\overline{B}(x, r), B(fx, \lambda r))$, then $f$ locally comes from $Sim(X)$.

It follows from Lemma 2.4 that the image of a local similarity embedding $f : B \to X$ is a finite union of mutually disjoint closed balls in $X$.

Let $E$ be the set of equivalence classes $[f, B]$ where $B$ is a closed ball in $X$ and $f : B \to X$ is a local similarity embedding locally coming from $Sim(X)$. Two such $(f_1, B_1)$ and $(f_2, B_2)$ are equivalent provided there exists $h \in Sim(B_1, B_2)$ such that $f_2 h = f_1$ (in particular, $f_1(B_1) = f_2(B_2)$). The verification that this is an equivalence relation requires the Identities, Compositions, and Inverses Properties of the similarity structure.

Let $Z = \{[f, B] \in E \mid f(B) \text{ is a closed ball in } X \text{ and } f \in Sim(B, f(B))\}$.

Note that an element $[f, B] \in Z$ is uniquely determined by the closed ball $f(B)$. In fact, $[f, B] = [incl_Y, f(B)]$, where $incl_Y : Y \to X$ denotes the inclusion map. Thus,

$$Z = \{[incl_B, B] \in E \mid B \text{ is a closed ball in } X\}.$$  

In particular, $Z$ can be identified with the collection of all closed balls in $X$.

There is a left action $\Gamma \bowtie E$ defined by $g[f, B] = [gf, B]$. The fact that $[gf, B] \in E$ follows from the Compositions and Restrictions Properties of the similarity structure.

It follows from the description of $Z$ above that for all $g \in \Gamma$,

$$gZ = \{[g|B, B] \in E \mid B \text{ is a closed ball of } X\}.$$  

The next part of this section is devoted to establishing the two properties required of a zipper action in Corollary 6.5 and Lemma 6.6 below.

**Lemma 6.3** Let $B$ be a closed ball in $X$ and $g \in \Gamma$. Then $[incl_B, B] \in Z \setminus gZ$ if and only if $B$ properly contains a maximum region of $g^{-1}$.
Proof. Suppose first that $[\text{incl}_B, B] \in Z \setminus g Z$ and by way of contradiction, that there exists a maximum region $R$ for $g^{-1}$ containing $B$. Then $g^{-1}R$ is a ball and $g^{-1}|R \in \Sim(R, g^{-1}R)$. The Restrictions Property implies $g^{-1}|B \in \Sim(B, g^{-1}B)$ and $[g^{-1}|B, B] \in Z$. Clearly, $[\text{incl}_B, B] = g[g^{-1}|B, B] \in g Z$, a contradiction.

Conversely, let $R$ be a maximum region of $g^{-1}$ properly contained in $B$. If $[\text{incl}_B, B] \in g Z$, then there exists $[\text{incl}_{B_1}, B_1] \in Z$ such that $[g] B_1, B_1] = g[\text{incl}_{B_1}, B_1] = [\text{incl}_B, B]$, which is to say $g(B_1) = B$. Moreover, $[g] B_1, B_1] = [\text{incl}_B, B]$ implies that $g: B_1 \to B$ is in $\Sim(B_1, B)$. The Inverses Property implies $g^{-1}|: B \to B_1$ is in $\Sim(B, B_1)$. In particular, $B$ is a region for $g^{-1}$, contradicting the maximality of $R$. Thus, $[\text{incl}_B, B] \notin g Z$. □

**Lemma 6.4** For each $g \in \Gamma$, the function $[\text{incl}_B, B] \mapsto B$ is a bijection from $Z \setminus g Z$ to the set of closed balls of $X$ properly containing maximum regions of $g^{-1}$. Moreover, the function $g[\text{incl}_B, B] \mapsto B$ is a bijection from $g Z \setminus Z$ to the set of closed balls of $X$ properly containing maximum regions of $g$.

Proof. The first statement follows immediately from the preceding lemma. The second follows from the first together with the observation that $g[\text{incl}_B, B] \mapsto [\text{incl}_B, B]$ is a bijection from $g Z \setminus Z$ to $Z \setminus g^{-1}Z$. □

**Corollary 6.5** For each $g \in \Gamma$, the symmetric difference $g Z \triangle Z$ is finite.

Proof. This follows immediately from the preceding lemma because there are only a finite number of closed balls of $X$ containing a maximum region of $g$ or $g^{-1}$. □

**Lemma 6.6** For each $r > 0$, $\{g \in \Gamma \mid |g Z \triangle Z| \leq r\}$ is finite.

Proof. Let $\Gamma_r = \{g \in \Gamma \mid |g Z \triangle Z| \leq r\}$. Since $|g Z \triangle Z| = |g^{-1}Z \triangle Z|$, $g \in \Gamma_r$ if and only if $g^{-1} \in \Gamma_r$. For each $x \in X$, let

$$M_{r,x} = \{R \mid R \text{ is a maximum region for some } g \in \Gamma_r \text{ and } x \in R\}.$$  

By Lemma 6.4 if $g \in \Gamma_r$, then the number of closed balls of $X$ properly containing a maximum region of $g$ is less than or equal to $r$. In particular, if $R \in M_{r,x}$, there are at most $r$ closed balls of $X$ properly containing $R$. Since $M_{r,x}$ is totally ordered by inclusion, it follows that $M_{r,x}$ is finite and there exists $R_{r,x} \in M_{r,x}$ such that $R_{r,x} \subseteq R$ for all $R \in M_{r,x}$. The set $\mathcal{P}_r := \{R_{r,x} \mid x \in X\}$ is a partition of $X$ and each $R_{r,x}$ is a region for $g$ and for $g^{-1}$, for all $g \in \Gamma_r$. That is to say $\mathcal{P}_r$ refines the maximum partitions of both $g$ and $g^{-1}$ for all $g \in \Gamma_r$. Setting $\mathcal{P}_+ = \mathcal{P}_r = \mathcal{P}_-$, this means $\Gamma_r \subset \Gamma_{\ref{|\mathcal{P}_+|}}$ and the result follows from Lemma 3.9. □

The proof of Theorem 6.1 is now complete.

We close with an example showing that a zipper action does not necessarily give rise to a space with walls, at least in the most naive way. See Cherix et al. [5, Section 1.2.7] for definitions and references.
Example 6.7 Consider the alphabet $A = \{0, 1\}$ and Thompson’s group $V \leq LS(A^\omega)$ as in Example 4.1. The construction above gives sets $Z \subseteq E$ and a zipper action $V \curvearrowright E$. It might be expected that $W := \{(gZ, E \setminus gZ) \mid g \in V\}$ is a set of walls for $E$. However, this is not the case. Specifically, we show there exists $[f_1, B_1], [f_2, B_2] \in E$ and an infinite subset $G \subseteq V$ such that $[f_1, B_1] \in gZ$ and $[f_2, B_2] \notin gZ$ for every $g \in G$ (that is, there are two elements of $E$ separated by infinitely many walls). Let $B_1 = 0A^\omega$ and $B_2 = 1A^\omega$. Let $f_1: B_1 \to A^\omega$ be the inclusion and let $f_2: B_2 \to A^\omega$ be defined by

$$
\begin{align*}
  f_2(10^w) &= 10w, \\
  f_2(11^w) &= 111w,
\end{align*}
$$

for all $w \in A^\omega$.

Let $G = \{g \in V \mid g \text{ is a local isometry and } g|B_1 = f_1\}$. The required conditions are readily checked. (Note also that $[f_2, B_2] \in VZ$ so that a set of walls will not result by reducing the size of $E$.)

References


