NILPOTENCE $=\ TORSION$

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Nilpotent endomorphisms

- Let $A$ be an associative ring with 1.
- An endomorphism $\nu : P \to P$ of an $A$-module $P$ is **nilpotent** if $\nu^N = 0 : P \to P$ for some $N \geq 0$.
- If $\nu$ is nilpotent then $1 + \nu : P \to P$ is an isomorphism with
  \[
  (1 + \nu)^{-1} = 1 - \nu + \nu^2 - \cdots + (-1)^{N-1}\nu^{N-1} : P \to P.
  \]
- For an indeterminate $z$ let $A[z]$ be the polynomial extension, and let $A[[z]]$ be the ring of formal power series.
- **Proposition 1** Let $f, g : P \to Q$ be morphisms of f.g. projective $A$-modules. The $A[z]$-module morphism
  $f + gz : P[z] \to Q[z]$ is an isomorphism if and only if
  $f : P \to Q$ is an isomorphism and $f^{-1}g : P \to P$ is nilpotent.
- **Remark 1** Proposition 1 is false if $P$ is not f.g., for example if
  \[
  f = 1, \ g = y : P = A[[y]] \to P = A[[y]]
  \]
  with $(f + gz)^{-1} = \sum_{j=0}^{\infty} (-)^j g^j z^j : P[z] \to P[z]$. 
Near-projections

- Let $A[z, z^{-1}]$ be the Laurent polynomial extension of $A$.
- An endomorphism $\rho: P \to P$ of an $A$-module $P$ is a near-projection if $\rho(1 - \rho): P \to P$ is nilpotent.
- **Example 1** If $\nu$ is nilpotent then $\nu$ is a near-projection.
- **Example 2** If $\nu$ is nilpotent then $1 - \nu$ is a near-projection.
- **Proposition 2** Let $f, g: P \to Q$ be morphisms of f.g. projective $A$-modules. The $A[z, z^{-1}]$-module morphism $f + gz: P[z, z^{-1}] \to Q[z, z^{-1}]$ is an isomorphism if and only if $f + g: P \to Q$ is an isomorphism and $(f + g)^{-1}g: P \to P$ is a near-projection.
- **Remark 2** Proposition 2 is false if $P$ is not f.g. – same counterexample as in Remark 1.
Why is $1 - \rho + \rho z$ an isomorphism for a near-projection $\rho$?

Given a near-projection $\rho : P \to P$ let $\nu = \rho(1 - \rho) : P \to P$, so that $\nu^N = 0$ for some $N \geq 0$. Define the projection

$$
\pi = (\rho^N + (1 - \rho)^N)^{-1}\rho^N
= \rho + (1/2)(2\rho - 1)((1 - 4\nu)^{-1/2} - 1)
= \rho + (2\rho - 1)(\nu + 3\nu^2 + 10\nu^3 + \ldots) : P \to P
$$

The near-projection splits as

$$
\rho = \rho_+ \oplus \rho_- : P = P_+ \oplus P_- \to P = P_+ \oplus P_-
$$

with $P_+ = (1 - \pi)(P)$, $P_- = \pi(P)$ and the endomorphisms

$$
\rho_+ = \rho| : P_+ \to P_+ , \quad 1 - \rho_- = (1 - \rho)| : P_- \to P_-
$$

nilpotent.

The endomorphism of $(P_+ \oplus P_-)[z, z^{-1}]$

$$
1 - \rho + \rho z = (1 + \rho_+(z - 1)) \oplus z(1 + (1 - \rho_-)(z^{-1} - 1))
$$

is an isomorphism, by a double application of Proposition 1.
Algebraic $K$-theory

The **algebraic $K$-groups** of $A$ are the algebraic $K$-groups of the exact category $\text{Proj}(A)$ of f.g. projective $A$-modules

$$K_*(A) = K_*(\text{Proj}(A)) .$$

The **nilpotent $K$-groups** of $A$ are the algebraic $K$-groups of the exact category $\text{Nil}(A)$ of f.g. projective $A$-modules $P$ with a nilpotent endomorphism $\nu : P \to P$

$$\text{Nil}_*(A) = K_*(\text{Nil}(A)) = K_*(A) \oplus \widetilde{\text{Nil}}_*(A) .$$

**Proposition 3** Let $\text{Near}(A)$ be the exact category of f.g. projective $A$-modules $P$ with a near-projection $\rho : P \to P$. The equivalence of exact categories

$$\text{Near}(A) \xrightarrow{\sim} \text{Nil}(A) \times \text{Nil}(A) ; (P, \rho) \mapsto (P_+, \rho_+) \times (P_-, 1 - \rho_-)$$

induces an isomorphism of algebraic $K$-groups
The Bass-Heller-Swan Theorem

**Theorem** (B-H-S 1965 for $n \leq 1$, Quillen 1972 for $n \geq 2$)

For any ring $A$ there are natural splittings

\[
K_n(A[z]) = K_n(A) \oplus \tilde{\text{Nil}}_{n-1}(A),
\]

\[
K_n(A[z, z^{-1}]) = K_n(A) \oplus K_{n-1}(A) \oplus \tilde{\text{Nil}}_{n-1}(A) \oplus \tilde{\text{Nil}}_{n-1}(A).
\]

**Original proof**

(i) Use Higman linearization to represent every $\tau \in K_1(A[z])$ by a linear invertible $k \times k$ matrix

\[
B = B_0 + zB_1 \in GL_k(A[z])
\]

with $B_0 \in M_k(A)$ invertible and $(B_0)^{-1}B_1 \in M_k(A)$ nilpotent.

(ii) Represent every $\tau \in K_1(A[z, z^{-1}])$ by

\[
B = B_0 + zB_1 \in GL_k(A[z, z^{-1}])
\]

with $B_0 + B_1 \in M_k(A)$ invertible and $(B_0 + B_1)^{-1}B_1 \in M_k(A)$ a near-projection.

(iii) For $n \in \mathbb{Z}$ apply the algebraic $K$-theory commutative localization exact sequence for $A[z] \to \{z\}^{-1}A[z] = A[z, z^{-1}]$. 
The Farrell-Hsiang splitting theorem

- **Theorem** (1968)
  A homotopy equivalence \( h : M^n \to X^{n-1} \times S^1 \) with \( M \) an \( n \)-dimensional manifold and \( X \) an \((n - 1)\)-dimensional manifold has a **splitting obstruction**

  \[
  \Phi(h) \in \text{Nil}_0(\mathbb{Z}[\pi_1(X)]) / \text{Nil}_0(\mathbb{Z}) = \tilde{K}_0(\mathbb{Z}[\pi_1(X)]) \oplus \tilde{\text{Nil}}_0(\mathbb{Z}[\pi_1(X)]).
  \]

- \( \Phi(h) = 0 \) if (and for \( n \geq 6 \) only if) \( h \) is \( h \)-cobordant to a split homotopy equivalence \( h : M \to X \times S^1 \), with the restriction

  \[
  h\mid : V^{n-1} = h^{-1}(X \times \{\ast\}) \to X
  \]

  also a homotopy equivalence.

- \( \Phi(h) \) is a component of the Whitehead torsion

  \[
  \tau(h) = (-)^{n-1} \tau(h)^* \in \text{Wh}(\pi_1(X) \times \mathbb{Z})
  \]

  \[
  = \text{Wh}(\pi_1(X)) \oplus \tilde{K}_0(\mathbb{Z}[\pi_1(X)]) \oplus \tilde{\text{Nil}}_0(\mathbb{Z}[\pi_1(X)]) \oplus \tilde{\text{Nil}}_0(\mathbb{Z}[\pi_1(X)]).
  \]
Geometric transversality over $S^1$

Given a map $h: M \to X \times S^1$ let $\overline{M} = h^*(X \times \mathbb{R})$ be the pullback infinite cyclic cover of $M$, with $z: \overline{M} \to \overline{M}$ a generating covering translation.

Assuming $M$ is an $n$-dimensional manifold make $h$ transverse regular at $X \times \{*\} \subset X \times S^1$, with

$$V^{n-1} = h^{-1}(X \times \{*\}) \subset M^n$$

a 2-sided codimension 1 submanifold. Cutting $M$ at $V \subset M$ there is obtained a fundamental domain $(W; z^{-1}V, V)$ for $\overline{M}$

$$\overline{M} = \bigcup_{k=-\infty}^{\infty} z^k(W; z^{-1}V, V).$$

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
\overline{M} & z^{-1}V & W & V & zW & zV & z^2W & z^2V \\
\hline
\end{array}
\]
Algebraic transversality over $S^1$

Let $C(V), C(W)$ denote the cellular finite based f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complexes of the pullbacks to $V, W$ of the universal cover $\tilde{X}$ of $X$.

Identify $\mathbb{Z}[\pi_1(X \times S^1)] = \mathbb{Z}[\pi_1(X)][z, z^{-1}]$ and let $C(\overline{M})$ denote the cellular finite based f.g. free $\mathbb{Z}[\pi_1(X)][z, z^{-1}]$-module chain complex of the pullback to $M$ of the universal cover $\tilde{X} \times \mathbb{R}$ of $X \times S^1$.

The decomposition $\overline{M} = \bigcup_{k=-\infty}^{\infty} z^k W$ determines a Mayer-Vietoris presentation of $C(\overline{M})$

$$0 \longrightarrow C(V)[z, z^{-1}] \xrightarrow{f - zg} C(W)[z, z^{-1}] \longrightarrow C(\overline{M}) \longrightarrow 0$$

with $f, g : C(V) \to C(W)$ the left and right inclusions.

For any ring $A$ every finite f.g. free $A[z, z^{-1}]$-module chain complex $C$ has a Mayer-Vietoris presentation.
The two ends of $\overline{M}$

- Everything has an end, except a sausage which has two!
- The infinite cyclic cover of $M$ is a union

$$\overline{M} = \overline{M}^+ \cup \mathcal{V} \overline{M}^-$$

with

$$\overline{M}^+ = \bigcup_{k=1}^{\infty} z^k \mathcal{W}, \quad \overline{M}^- = \bigcup_{k=-\infty}^{0} z^k \mathcal{W}.$$
Chain homotopy nilpotence

- An $A$-module chain complex $C$ is **finitely dominated** if it is chain equivalent to a finite f.g. projective $A$-module chain complex.
- An $A$-module chain map $\nu : C \to C$ is **chain homotopy nilpotent** if $\nu^N \simeq 0 : C \to C$ for some $N \geq 0$.
- If $h : M^n \to X \times S^1$ is a homotopy equivalence then
  \[ C(M^+, V) \oplus C(M^-, V) \to C(V \to X) \]
  is a chain equivalence with $C(V \to X)$ a finite f.g. free $\mathbb{Z}[\pi_1(X)]$-module chain complex.
- The free $\mathbb{Z}[\pi_1(X)]$-module chain complex $C(M^+, V)$ is finitely dominated.
- The $\mathbb{Z}[\pi_1(X)]$-module chain map
  \[ \nu^+ : C(M^+, V) \to C(M^+, zW) \cong C(zM^+, zV) \cong C(M^+, V) \]
  is chain homotopy nilpotent.
\[ \overline{M}^+ = zW \cup z\overline{M}^+ \]
The F-H splitting obstruction from the chain complex point of view

- For a homotopy equivalence \( h : M^n \to X \times S^1 \) the contractible finite based f.g. free \( \mathbb{Z}[\pi_1(X)][z, z^{-1}] \)-module chain complex \( C(\overline{h} : \overline{M} \to X \times \mathbb{R}) \) fits into a short exact sequence

\[
0 \to C(V, X)[z, z^{-1}] \xrightarrow{f - zg} C(W, X \times I)[z, z^{-1}] \to C(\overline{h}) \to 0
\]

- The splitting obstruction of \( h \) is the nilpotent class

\[
\Phi(h) = (C(\overline{M}^+, V), \nu^+) \in \text{Nil}_0(\mathbb{Z}[\pi_1(X)])/\text{Nil}_0(\mathbb{Z})
\]

where

\[
C(\overline{M}^+, V) = \text{coker}(f - zg : zC(V, X)[z] \to C(W, X \times I)[z]).
\]

- \( \Phi(h) = 0 \) if and only if \( (C(\overline{M}^+, V), \nu^+) \) is equivalent to 0 by a finite sequence of algebraic handle exchanges.

- For \( n \geq 6 \) can realize algebraic handle exchanges by geometric handle exchanges.
Universal localization

(P.M.Cohn, 1971) Given a ring $R$ and a set $\Sigma$ of morphisms $\sigma : P \to Q$ of f.g. projective $R$-modules there exists a **universal localization** $\Sigma^{-1}R$, a ring with a morphism $R \to \Sigma^{-1}R$ universally inverting each $\sigma$.

**Universal property** For any ring morphism $R \to S$ such that $1 \otimes \sigma : S \otimes_R P \to S \otimes_R Q$ is an $S$-module isomorphism for each $\sigma \in \Sigma$ there is a unique factorization $R \to \Sigma^{-1}R \to S$.

**Warning 1** $R \to \Sigma^{-1}R$ need not be injective.

**Warning 2** $\Sigma^{-1}R$ could be 0.

**Gerasimov-Malcolmson normal form** An element $q\sigma^{-1}p \in \Sigma^{-1}R$ is an equivalence class of triples $((\sigma : P \to Q) \in \Sigma, p \in P, q \in Q^* = \text{Hom}_R(Q, R))$. 

The algebraic $K$-theory localization exact sequence

- Assume $R \to \Sigma^{-1}R$ is injective.
- An $(R, \Sigma)$-torsion module is an $R$-module $T$ such that

$$0 \to P_1 \xrightarrow{d} P_0 \xrightarrow{} T \xrightarrow{} 0$$

with $P_0, P_1$ f.g. projective $R$-modules and $1 \otimes d : \Sigma^{-1}P_1 \to \Sigma^{-1}P_0$ a $\Sigma^{-1}R$-module isomorphism.

- **Theorem** (Neeman+R., 2004) For an injective universal localization $R \to \Sigma^{-1}R$ such that

$$\text{Tor}_*^R(\Sigma^{-1}R, \Sigma^{-1}R) = 0 \text{ (stable flatness)}$$

there is a long exact sequence of algebraic $K$-groups

$$\ldots \to K_n(R) \to K_n(\Sigma^{-1}R) \to K_{n-1}(H(R, \Sigma)) \to K_{n-1}(R) \to \ldots$$

with $H(R, \Sigma)$ the exact category of $(R, \Sigma)$-torsion modules.
Triangular matrix rings

- Given rings $R_1, R_2$ and an $(R_2, R_1)$-bimodule $Q$ define the triangular matrix ring

$$R = \begin{pmatrix} R_1 & 0 \\ Q & R_2 \end{pmatrix}.$$

- **Proposition 4** (i) The category of $R$-modules is equivalent to the category of triples

$$M = (M_1, M_2, \mu : Q \otimes_{R_1} M_1 \to M_2)$$

with $M_i$ $R_i$-modules ($i = 1, 2$), $\mu$ an $R_2$-module morphism.

- (ii) An $R$-module $M$ is f.g. projective if and only if $M_1$ is a f.g. projective $R_1$-module, $\mu$ is injective, and $\text{coker}(\mu)$ is a f.g. projective $R_2$-module.

- (iii) $K_\ast(R) = K_\ast(R_1) \oplus K_\ast(R_2)$. 
**Full matrix rings**

- Let \( R = \begin{pmatrix} R_1 & 0 \\ Q & R_2 \end{pmatrix}, \) \( P_1 = \begin{pmatrix} R_1 \\ Q \end{pmatrix}, \) \( P_2 = \begin{pmatrix} 0 \\ R_2 \end{pmatrix}. \)

  The \( R \)-modules \( P_1, P_2 \) are f.g. projective, since \( P_1 \oplus P_2 = R. \)

- If \( R \to S \) is a ring morphism with \( S \otimes_R P_1 \cong S \otimes_R P_2 \) then
  \[
  S = M_2(T)
  \]
  with \( T = \text{End}_S(S \otimes_R P_1) = \text{End}_S(S \otimes_R P_2). \)

- Morita equivalence

  \[
  \{ S \text{-modules} \} \xrightarrow{\cong} \{ T \text{-modules} \} ; \ N \mapsto (T \ T) \otimes_S N.
  \]

- The induced functor

  \[
  \{ R \text{-modules} \} \to \{ S \text{-modules} \} \xrightarrow{\cong} \{ T \text{-modules} \} ;
  \]

  \[
  M = (M_1, M_2, \mu : Q \otimes_{R_1} M_1 \to M_2) \mapsto
  (T \ T) \otimes_R M = \text{coker}(T \otimes_{R_2} Q \otimes_{R_1} M_1 \to T \otimes_{R_1} M_1 \oplus T \otimes_{R_2} M_2)
  \]

  is an **assembly** map, i.e. local-to-global.
\((R, \Sigma)\)-torsion modules

- **Proposition 5** The universal localization of

\[
R = \begin{pmatrix} R_1 & 0 \\ Q & R_2 \end{pmatrix} = P_1 \oplus P_2
\]

inverting a set \(\Sigma\) of \(R\)-module morphisms \(\sigma : P_2 \to P_1\) is \(\Sigma^{-1}R = M_2(T)\) with \(T = \text{End}_{\Sigma^{-1}R}(\Sigma^{-1}P_1)\).

- **Proposition 6** Assume that \(R \to \Sigma^{-1}R = M_2(T)\) is injective, and that \(Q\) is a flat right \(R_1\)-module.

An \(R\)-module \(M = (M_1, M_2, \mu)\) is \((R, \Sigma)\)-torsion if and only if

1. \(\cdots \to 0 \to Q \otimes_{R_1} M_1 \xrightarrow{\mu} M_2\) is homology equivalent to a 1-dimensional f.g.projective \(R_1\)-module chain complex,

2. \(M_2\) is an h.d. 1 \(R_2\)-module,

3. the assembly

\[
T \otimes_{R_2} Q \otimes_{R_1} M_1 \to T \otimes_{R_1} M_1 \oplus T \otimes_{R_2} M_2
\]

is a \(T\)-module isomorphism.
Polynomial extensions as universal localizations

For any ring $A$ let

$$R = \begin{pmatrix} A & 0 \\ A \oplus A & A \end{pmatrix}, \quad P_1 = \begin{pmatrix} A \\ A \oplus A \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 \\ A \end{pmatrix}$$

and let $\sigma_+, \sigma_- : P_2 \to P_1$ be the two inclusions.

**Proposition 7** (Schofield, 1985)

(i) The universal localization of $R$ inverting $\Sigma_+ = \{\sigma_+\}$ is

$$\Sigma_+^{-1}R = M_2(A[z]) .$$

(ii) The universal localization of $R$ inverting $\Sigma = \{\sigma_+, \sigma_-\}$ is

$$\Sigma^{-1}R = M_2(A[z, z^{-1}]) .$$
Let $R = \begin{pmatrix} A & 0 \\ A \oplus A & A \end{pmatrix}$. An $R$-module $M = (P, Q, f, g)$ is defined by $A$-modules $P, Q$ and $A$-module morphisms $f, g : P \to Q$.

**Proposition 8** (i) The assembly of $M = (P, Q, f, g)$ with respect to $\Sigma^{-1} R = M_2(A[z])$ is the $A[z]$-module

$$(A[z] \ A[z]) \otimes_R M = \text{coker}(f + gz : P[z] \to Q[z]).$$

$M$ is an $(R, \Sigma_+)$-module if and only if $P, Q$ are f.g. projective $A$-modules and $f + gz$ is an $A[z]$-module isomorphism. Thus

$$\text{Nil}(A) \to H(A[z], \Sigma_+) \ ; \ (P, \nu) \mapsto (P, P, 1, \nu)$$

is an equivalence of exact categories, by Proposition 1.

(ii) Likewise for $\Sigma^{-1} A[z] = M_2(A[z, z^{-1}])$, with

$$\text{Near}(A) \to H(A[z], \Sigma) \ ; \ (P, \rho) \mapsto (P, P, \rho, 1 - \rho)$$

an equivalence of exact categories by Proposition 2.
Universal localization proof of B-H-S theorem

Apply the universal localization exact sequence
\[ \cdots \to K_n(R) \to K_n(\Sigma^{-1}R) \to K_{n-1}(H(R, \Sigma)) \to K_{n-1}(R) \to \cdots \]
to the stably flat universal localizations of \( R = \begin{pmatrix} A & 0 \\ A \oplus A & A \end{pmatrix} \)
\[ \Sigma_+^{-1}R = M_2(A[z]), \Sigma^{-1}R = M_2(A[z, z^{-1}]). \]

Identify
\[
K_*(R) = K_*(A) \oplus K_*(A), \\
K_*(\Sigma_+^{-1}R) = K_*(A[z]), H(R, \Sigma_+) = \text{Nil}(A), \\
K_*(\Sigma^{-1}R) = K_*(A[z, z^{-1}]), \\
H(R, \Sigma) = \text{Near}(A) = \text{Nil}(A) \times \text{Nil}(A)
\]
to recover
\[
K_n(A[z]) = K_n(A) \oplus \tilde{\text{Nil}}_{n-1}(A), \\
K_n(A[z, z^{-1}]) = K_n(A) \oplus K_{n-1}(A) \oplus \tilde{\text{Nil}}_{n-1}(A) \oplus \tilde{\text{Nil}}_{n-1}(A).
\]
Generalized free products

- A group \( \pi \) is a **generalized free product** if it is
  - either amalgamated free product \( \pi = \pi_1 * \rho \pi_2 \),
  - or an HNN extension \( \pi = \pi_1 * \rho \{ t \} \).

- (Bass-Serre, 1970) A group \( \pi \) is a generalized free product if and only if \( \pi \) acts on a tree \( T \) with \( T/\pi = [0, 1] \) or \( S^1 \).


- Nilpotence = torsion also in the generalized free product case.

- Also in algebraic \( L \)-theory, with the Cappell (1974) UNil-groups.
“There -- now I've taught you everything I know about codimension 1 splitting”