Annals of Mathematics, **156** (2002), 867–889

The approximate tubular neighborhood theorem

By Bruce Hughes*

In memory of James Robert Boyd Jr. (1921–2001)

Abstract

Skeleta and other pure subsets of manifold stratified spaces are shown to have neighborhoods which are teardrops of stratified approximate fibrations (under dimension and compactness assumptions). In general, the stratified approximate fibrations cannot be replaced by bundles, and the teardrops cannot be replaced by mapping cylinder neighborhoods. Thus, this is the best possible topological tubular neighborhood theorem in the stratified setting.

1. Introduction

One of the most striking differences between smooth and topological manifolds concerns neighborhoods of submanifolds. For smooth manifolds there is the classical Tubular Neighborhood Theorem of Whitney asserting that every smooth submanifold has a neighborhood which is the mapping cylinder of a smooth spherical fibre bundle. For locally flat topological submanifolds, the examples of Rourke and Sanderson [26] show that neighborhoods which are mapping cylinders of topological spherical fibre bundles need not exist. However, Edwards [7] proved that locally flat topological submanifolds of manifolds of dimension greater than five do have mapping cylinder neighborhoods, but the maps are a weak type of bundle now called a manifold approximate fibration (see [20]).

For stratified spaces, there is a similar, but even more pronounced, difference between the smooth and topological categories. On the one hand, there are the smoothly stratified spaces originally studied by Mather, Thom

*Supported in part by NSF Grant DMS-9971367.

1991 *Mathematics Subject Classification.* Primary 57N80, 57N40; Secondary 55R65, 58A35.

Key words and phrases. manifold stratified space, approximate tubular neighborhood, teardrop neighborhood, manifold stratified approximate fibration.

and Whitney (see [22], [29], [9]). Skeleta have mapping cylinder neighborhoods whose maps are systems of topological fibre bundles (see [23]). On the other hand, there are the topologically stratified spaces of Siebenmann [27] and Quinn [25]. In this setting skeleta (and even strata) may fail to have mapping cylinder neighborhoods, and even when they do (as is the case for locally flat submanifolds), the maps need not be fibre bundles.

The main result of this paper provides a substitute for the missing mapping cylinder neighborhoods in topologically stratified spaces.

We work with the manifold stratified spaces of Quinn [25]. These spaces are more general than the locally conelike spaces of Siebenmann [27] in that Quinn's spaces are only locally conelike up to stratified homotopy equivalence. In fact, the beauty of Quinn's spaces is that their defining conditions are homotopy-theoretical (from which geometric-topological properties can be deduced). One point compactifications of manifolds with a finite number of tame ends are examples of Quinn stratified spaces which are locally conelike if and only if the manifolds admit boundary completions. For another illustration of the ubiquity of stratified spaces in the sense of Quinn, Cappell and Shaneson [1] have shown that mapping cylinders of stratified maps between smoothly stratified spaces are manifold stratified spaces, even though they need not be smoothly stratified (cf. [13]). For a survey on the various types of stratifications, as well as related information, see Hughes and Weinberger [21].

Here is the main result of this paper.

APPROXIMATE TUBULAR NEIGHBORHOOD THEOREM 1.1. *Let X be a manifold stratified space with compact singular set X_{sing} such that all the non-minimal strata of X have dimension greater than or equal to five. If $Y \subseteq X_{\text{sing}}$ is a pure subset of X, then Y has an approximate tubular neighborhood in X.*

The terminology in the theorem is explained fully in the sections to follow, but here is a brief introduction. Pure subsets are closed unions of strata, an important example being skeleta. Approximate tubular neighborhoods are generalizations of mapping cylinder neighborhoods of fibre bundles. Both the mapping cylinder structure and the fibre bundle structure are weakened. The mapping cylinder of a map $p : E \to B$ is replaced by the teardrop of a map $q : U \to B \times \mathbb{R}$. A neighborhood of B is constructed from this data by gluing B to U using the map q. If q were a fibre bundle, then this neighborhood would be an open mapping cylinder of the desuspension of q (in which the $\mathbb{R}$ factor is split off). In general, q is just required to have a very weak homotopy lifting property, namely, q is a manifold stratified approximate fibration. Even though desuspension is unavailable for these maps, there is still quite a lot of geometry behind them.

The proof of Theorem 1.1 relies on both the statements and techniques of special cases which have already been worked out. First, there is the very important case of manifold stratified spaces with only two strata studied by Hughes, Taylor, Weinberger and Williams [18]. Hughes and Ranicki [17] specialized further by requiring the lower stratum to be a point. That single strata have approximate tubular neighborhoods was established in [15].

The converse of Theorem 1.1, namely, that the teardrop of a manifold stratified approximate fibration is a manifold stratified space, was proved in [14].

Quinn indicated in his address to the International Congress [24] that topology with control is critical for the study of singular and stratified spaces. Indeed, the basic tools used in this paper come from controlled topology. In particular, the geometric techniques have evolved from Chapman's controlled engulfing methods [3], [4]. Other stratified tools come from [12] and [13]. The main external input needed for the proof of Theorem 1.1 is Quinn's Isotopy Extension Theorem [25].

Some applications of the approximate tubular neighborhood theorem have already been outlined in the literature. Perhaps the most important is the alternative approach to Weinberger's surgery-theoretic classification of manifold stratified spaces offered by him in [30, p. 189] and [32, pp. 518–519]. The alternative approach applies to unstable classification directly whereas Weinberger's first proof involves stabilization-destabilization.

Weinberger, in his address to the International Congress [31], mentions applications to equivariant versions of local contractiblity of homeomorphism groups and cell-like approximation theorems. These results were first established by Siebenmann [27] and Steinberger and West [28], respectively, in the locally linear case. Another important application is to complete the realization part of Quinn's h-cobordism theorem [25]. This was done for two strata by Hughes, Taylor, Weinberger and Williams in [18]. Further applications, including multiparameter isotopy extension theorems and Thom's isotopy lemmas, are mentioned in [11]. Complete details of these applications, along with results concerning uniqueness, will be forthcoming.

This paper is organized as follows. Sections 2, 3 and 4 contain the basic definitions and background information on manifold stratified spaces, stratified approximate fibrations, teardrops and approximate tubular neighborhoods. Section 5 contains a special case which will be used in the proof of the main result: it is shown that collars of strata have approximate tubular neighborhoods. Section 6 establishes approximate tubular neighborhoods for certain subsets of the singular set, the singular up-sets. Finally, Section 7 contains the proof of the main result.

I want to thank Bruce Williams for the many stimulating discussions we have had about the results in this paper.

2. Manifold stratified spaces

This section contains the basic definitions from the theory of stratifications as presented in [11], [12], [13], [14], [18], [25].

Definition 2.1. A *stratification* of a space X consists of an index set $\mathcal{I}$ and a locally finite partition $\{X_i\}_{i\in\mathcal{I}}$ of locally closed subspaces of X (the X_i are pairwise disjoint and their union is X). For $i \in \mathcal{I}$, X_i is called the *i-stratum* and the closed set

$$X^i = \cup\{X_k \mid X_k \cap \mathrm{cl}(X_i) \neq \emptyset\}$$

is called the *i-skeleton*. We say X is a *space with a stratification.*

For a space X with a stratification $\{X_i\}_{i\in\mathcal{I}}$, define a relation $\leq$ on the index set $\mathcal{I}$ by $i \leq j$ if and only if $X_i \subseteq \mathrm{cl}(X_j)$. The *Frontier Condition* is satisfied if for every $i, j \in \mathcal{I}$, $X_i \cap \mathrm{cl}(X_j) \neq \emptyset$ implies $X_i \subseteq \mathrm{cl}(X_j)$, in which case $\leq$ is a partial ordering of $\mathcal{I}$ and $X^i = \mathrm{cl}(X_i)$ for each $i \in \mathcal{I}$.

If X is a space with a stratification satisfying the Frontier Condition and Y is a union of strata of X, then $\mathrm{cl}(Y) \setminus Y$ is closed in X.

If X is a space with a stratification, then a map $f : Z \times A \to X$ is *stratum preserving along* A if for each $z \in Z$, $f(\{z\} \times A)$ lies in a single stratum of X. In particular, a map $f : Z \times I \to X$ is a *stratum preserving homotopy* if f is stratum preserving along I. A homotopy $f : Z \times I \to X$ whose restriction to $Z \times [0, 1)$ is stratum preserving along $[0, 1)$ is said to be *nearly stratum preserving*.

Definition 2.2. Let X be a space with a stratification $\{X_i\}_{i\in\mathcal{I}}$ and $Y \subseteq X$.

(1) Y is *forward tame* in X if there exist a neighborhood U of Y in X and a homotopy $h : U \times I \to X$ such that $h_0 = \text{inclusion} : U \to X$, $h_t|Y = \text{inclusion} : Y \to X$ for each $t \in I, h_1(U) = Y$, and $h((U \setminus Y) \times [0, 1)) \subseteq X \setminus Y$.

(2) The *homotopy link* of Y in X is defined by

$$\mathrm{holink}(X, Y) = \{\omega \in X^I \mid \omega(t) \in Y \text{ if and only if } t = 0\}.$$

(3) Let $x_0 \in X_i \subseteq X$. The *local holink* (or *local homotopy link*) *at* x_0 is

$$\mathrm{holink}(X, x_0) = \{\omega \in \mathrm{holink}(X, X_i) \mid \omega(0) = x_0 \\ \text{and } \omega(t) \in X_j \text{ for some } j, \text{ for all } t \in (0, 1]\}.$$

All path spaces are given the compact-open topology. Evaluation at 0 defines a map $q : \mathrm{holink}(X, Y) \to Y$ called *holink evaluation*. There is a natural stratification of $\mathrm{holink}(X, x_0)$ into disjoint subspaces

$$\mathrm{holink}(X, x_0)_j = \{\omega \in \mathrm{holink}(X, x_0) \mid \omega(1) \in X_j\}.$$

Definition 2.3. A space X with a stratification satisfying the Frontier Condition is a *manifold stratified space* if the following four conditions are satisfied:

(1) *Forward tameness.* For each $k > i$, the stratum X_i is forward tame in $X_i \cup X_k$.

(2) *Normal fibrations.* For each $k > i$, the holink evaluation

$$q : \mathrm{holink}(X_i \cup X_k, X_i) \to X_i$$

is a fibration.

(3) *Compactly dominated local holinks.* For each $x_0 \in X$, there exist a compact subset C of the local homotopy link $\mathrm{holink}(X, x_0)$ and a stratum preserving homotopy

$$h : \mathrm{holink}(X, x_0) \times I \to \mathrm{holink}(X, x_0)$$

such that $h_0 = \mathrm{id}$ and $h_1(\mathrm{holink}(X, x_0)) \subseteq C$.

(4) *Manifold strata property.* X is a locally compact, separable metric space, each stratum X_i is a topological manifold (without boundary) and X has only finitely many nonempty strata.

If X is only required to satisfy conditions (1) and (2), then X is a *homotopically stratified space.*

Definition 2.4. The *singular set* of a space X with a stratification $\{X_i\}_{i\in\mathcal{I}}$ is

$$X_{\mathrm{sing}} = \quad \cup\{X_i \mid \text{ for some } j \in \mathcal{I}, j \neq i, \mathrm{cl}(X_j) \cap X_i \neq \emptyset\}.$$

In other words, X_{sing} is the union of all nonmaximal strata of X.

Definition 2.5. A subset A of a space X with a stratification is a *pure subset* if A is closed and is a union of strata of X.

3. Stratified approximate fibrations

We now give the definitions of the types of maps which are important for manifold stratified spaces.

Definition 3.1. Let X and Y be spaces with stratifications $\{X_i\}_{i\in\mathcal{I}}$ and $\{Y_j\}_{j\in\mathcal{J}}$, respectively, and let $p : X \to Y$ be a map.

(1) p is a *stratified fibration* provided that given any space Z and any commuting diagram

$$\begin{array}{ccc} Z & \xrightarrow{f} & X \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle p} \\ Z \times I & \xrightarrow{F} & Y \end{array}$$

with F a stratum preserving homotopy, there exists a *stratified solution*; i.e., a stratum preserving homotopy $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(z,0) = f(z)$ for each $z \in Z$ and $p\tilde{F} = F$. The diagram above is a *stratified homotopy lifting problem*.

(2) p is a *weak stratified approximate fibration* provided that given any stratified homotopy lifting problem, there exists a *weak stratified controlled solution*; i.e., a map $\tilde{F} : Z \times I \times [0,1) \to X$ which is stratum preserving along $I \times [0,1)$ such that $\tilde{F}(z,0,t) = f(z)$ for each $(z,t) \in Z \times [0,1)$ and the function $\bar{F} : Z \times I \times I \to Y$ defined by $\bar{F}|Z \times I \times [0,1) = p\tilde{F}$ and $\bar{F}|Z \times I \times \{1\} = F \times \mathrm{id}_{\{1\}}$ is continuous.

(3) p is a *manifold stratified approximate fibration* (MSAF) if X and Y are manifold stratified spaces and p is a proper weak stratified approximate fibration.

(4) If α is an open cover of Y, then p is a *stratified α-fibration* provided that given any stratified homotopy lifting problem, there exists a *stratified α-solution*; i.e., a stratum preserving homotopy $\tilde{F} : Z \times I \to X$ such that $\tilde{F}(z,0) = f(z)$ for each $z \in Z$ and $p\tilde{F}$ is α-close to F.

(5) p is a *manifold approximate fibration* (MAF) if p is a MSAF and X and Y have only one stratum each (i.e., they are manifolds).

See [16] for clarification about weak stratified approximate fibrations and how the definition above relates to definitions in previous papers.

4. Teardrops and approximate tubular neighborhoods

This section contains a review of the basic teardrop construction. Given spaces X, Y and a map $p : X \to Y \times \mathbb{R}$, the *teardrop* of p is the space denoted by $X \cup_p Y$ whose underlying set is the disjoint union $X \amalg Y$ with the minimal topology such that

(1) $X \subset X \cup_p Y$ is an open embedding, and
(2) the function $c : X \cup_p Y \to Y \times (-\infty, +\infty]$ defined by

$$c(x) = \begin{cases} p(x), & \text{if } x \in X \\ (x, +\infty), & \text{if } x \in Y \end{cases}$$

is continuous.

This is a generalization of the construction of the open mapping cylinder of a map $g : X \to Y$. Namely, $\overset{\circ}{\mathrm{cyl}}(g)$ is the teardrop $(X \times \mathbb{R}) \cup_{g\times \mathrm{id}} Y$. However, not all teardrops are open mapping cylinders because not all maps to $Y \times \mathbb{R}$ can be split as a product. See [18] for more about the teardrop construction.

If X is a space with a stratification and $A \subseteq X$, we say A has an *approximate tubular neighborhood in* X if there is an open neighborhood U of A and an MSAF

$$p : U \setminus A \to A \times \mathbb{R}$$

such that the natural function $(U \setminus A) \cup_p A \to U$ is a homeomorphism. This has previously been called an MSAF *teardrop neighborhood in* X. The condition is equivalent to saying that p is an MSAF and the natural extension

$$\tilde{p} : U \to A \times (-\infty, +\infty]$$

is continuous. In this case, $\tilde{p}$ is also an MSAF when $A \times (-\infty, +\infty]$ is given the natural stratification (see [14, Prop. 7.1], [18]).

If A does have an approximate tubular neighborhood in X, then it is usually more convenient to replace $\mathbb{R}$ by $(0, +\infty)$ with $\{0\} \in [0, +\infty)$ playing the role of $\{+\infty\} \in (-\infty, +\infty]$. Thus, there is a map of the form

$$\varphi : U \to A \times [0, +\infty)$$

where U is an open neighborhood of A in X, $\varphi^{-1}(A \times \{0\}) = A$, $\varphi| : A \to A \times \{0\}$ is the identity, and φ is an MSAF. This map φ is called an *approximate tubular neighborhood map for* A in X.

The following results show that the teardrop construction yields manifold stratified spaces and that strata have approximate tubular neighborhoods.

THEOREM 4.1 ([14]). *If X and Y are manifold stratified spaces each with only finitely many strata and $p : X \to Y \times (0, +\infty)$ is a manifold stratified approximate fibration, then the teardrop $X \cup_p Y$ with the natural stratification is a manifold stratified space.*

THEOREM 4.2 ([15]). *Let X be a manifold stratified space with a stratum A satisfying*:

(1) *A has compact closure* $\mathrm{cl}(A)$ *in X,*

(2) *if Z_1 and Z_2 are distinct strata of X with $Z_1 \subseteq \mathrm{cl}(A) \cap \mathrm{cl}(Z_2)$, then* $\dim(Z_2) \geq 5$.

Then A has an approximate tubular neighborhood in X.

5. Approximate tubular neighborhoods of collars of strata

In [15] there is a proof that strata in manifold stratified spaces have approximate tubular neighborhoods (under dimension and compactness assumptions). In this section we extend that result slightly to show that collars of strata have approximate tubular neighborhoods. This will be important in the next section.

The main result of this section is Proposition 5.2. Its proof uses variations of the results in [15] on "stratified sucking" and "homotopy near a stratum." The reader is required to be familiar with those proofs. Lemma 5.1 shows how to deal with the problem that limits of stratum preserving processes need not be stratum preserving.

Throughout this section let N denote a manifold (without boundary), possibly noncompact. Consider $N \times [0, +\infty)$ as a manifold stratified space with two strata: $N \times \{0\}$ and $N \times (0, +\infty)$. Let $\pi_1 : N \times [0, +\infty) \to N$ and $\pi_2 : N \times [0, +\infty) \to [0, +\infty)$ denote the two projections. We will assume that the one-point compactification of N is a manifold stratified space with two strata (N and the point at infinity) and that $\dim N \geq 5$ if N is noncompact. This allows us to assume that N has a metric with the property that for every $\varepsilon > 0$ there is a $\delta > 0$ such that any two maps into N which are δ-close are ε-homotopic (rel any subset where the two maps agree). Usually one would only have such a property for measurements made by open covers. However, under the assumptions, N has a cocompact open subset which is the infinite cyclic cover of a compact manifold [17]. It follows that the desired metric can be constructed. $N \times [0, +\infty)$ is given a standard product metric.

LEMMA 5.1. *Suppose W is a manifold stratified space and $p, p_n : W \to N \times [0, +\infty)$ are proper maps for $n = 1, 2, 3, \ldots$ such that*

(1) *p_n is stratum preserving $\frac{1}{2^n}$-homotopic to p_{n+1} for $n = 1, 2, 3, \ldots$,*

(2) *p_n is a stratified $\frac{1}{2^n}$-fibration for $n = 1, 2, 3, \ldots$,*

(3) *$p = \lim_{n\to\infty} p_n$ (uniformly).*

Define $q : N \times [0, +\infty) \to N \times [0, +\infty)$ by

$$q(x, s) = \begin{cases} (x, 0) & \text{if } 0 \leq s \leq 10 \\ (x, s - 10) & \text{if } s \geq 10. \end{cases}$$

Then $qp : W \to N \times [0, +\infty)$ is an MSAF.

Proof. We begin with a general construction which will be used in the proof. This so called $*$-construction is used to convert homotopies into stratum preserving homotopies. Let $F : Z \times I \to N \times [0, +\infty)$ be a homotopy such that $|\pi_2 F(z, t) - \pi_2 F(z, 0)| < 1$ for each $(z, t) \in Z \times I$. Define $F^* : Z \times I \to$

$N \times [0,+\infty)$ by $\pi_1 F^* = \pi_1 F$ and

$$\pi_2 F^*(z,t) = \begin{cases} \pi_2 F(z,0) & \text{if } \pi_2 F(z,0) \leq 5 \\ (\pi_2 F(z,0) - 5) \cdot \pi_2 F(z,t) & \\ \qquad\qquad +(6 - \pi_2 F(z,0)) \cdot \pi_2 F(z,0) & \text{if } 5 \leq \pi_2 F(z,0) \leq 6 \\ \pi_2 F(z,t) & \text{if } 6 \leq \pi_2 F(z,0). \end{cases}$$

One can verify the following properties:

(1) $\pi_1 F^* = \pi_1 F$,

(2) $F_0^* = F_0$,

(3) F^* is stratum preserving,

(4) $F^*(z,t) = F(z,t)$ if $\pi_2 F(z,0) \geq 6$,

(5) $qF^* = qF$.

In the course of the proof we will use the observation $\pi_1 qp = \pi_1 p$. Also, let $\hat{q} : N \times [10,+\infty) \to N \times [0,+\infty)$ be the restriction of q and note that $\hat{q}$ is a homeomorphism. Now let

$$\begin{array}{ccc} Z & \xrightarrow{\ f\ } & W \\ {\scriptstyle \times 0}\big\downarrow & & \big\downarrow{\scriptstyle qp} \\ Z \times I & \xrightarrow{\ F\ } & N \times [0,+\infty) \end{array}$$

be a stratified homotopy lifting problem for which we are required to find a weak stratified controlled solution. Since the problem is stratified,

$$\pi_2 F(z,t) > 0 \iff \pi_2 qpf(z) > 0 \iff \pi_2 pf(z) > 10.$$

Define $F' : Z \times I \to N \times [0,+\infty)$ by

$$F'(z,t) = \begin{cases} (\pi_1 F(z,t), \pi_2 pf(z)) & \text{if } \pi_2 pf(z) \leq 10 \\ (\pi_1 F(z,t), \pi_2 \hat{q}^{-1} F(z,t)) = \hat{q}^{-1} F(z,t) & \text{if } \pi_2 pf(z) \geq 10. \end{cases}$$

One can verify that F' is continuous, $qF' = F$ and

$$\begin{array}{ccc} Z & \xrightarrow{\ f\ } & W \\ {\scriptstyle \times 0}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ Z \times I & \xrightarrow{\ F'\ } & N \times [0,+\infty) \end{array}$$

is a stratified homotopy lifting problem.

We will first show that for every $\varepsilon > 0$, the original problem has a stratified ε-solution. For each $n = 1, 2, 3\ldots$ let $G^n : W \times [\frac{n-1}{n}, \frac{n}{n+1}] \to N \times [0,+\infty)$ be a stratum preserving $\frac{1}{2^n}$-homotopy from p_n to p_{n+1}. The G^n piece together to define a map $W \times [0,1) \to N \times [0,+\infty)$ which is stratum preserving along $[0,1)$. Moreover, $G : W \times I \to N \times [0,+\infty)$ defined by $G|(W \times [\frac{n-1}{n}, \frac{n}{n+1}]) = G^n$ and $G(w,1) = p(w)$ is a homotopy (continuous, but not necessarily stratum preserving). For each $n = 1, 2, 3, \ldots$ consider the homotopy $H^n : Z \times [-\frac{1}{n}, 1] \to N \times [0, +\infty)$ defined by

$$H^n(z,t) = \begin{cases} G(f(z), t+1) & \text{if } -\frac{1}{n} \le t \le 0 \\ F'(z,t) & \text{if } 0 \le t \le 1. \end{cases}$$

Note that $H^n(z, -\frac{1}{n}) = G(f(z), \frac{n-1}{n}) = p_n f(z)$. The $*$-construction yields a stratified homotopy lifting problem

$$\begin{array}{ccc} Z & \xrightarrow{f} & W \\ {\scriptstyle \times\{-\frac{1}{n}\}}\big\downarrow & & \big\downarrow{\scriptstyle p_n} \\ Z \times [-\frac{1}{n}, 1] & \xrightarrow{(H^n)^*} & N \times [0, +\infty) \end{array}$$

which therefore has a stratified $\frac{1}{2^n}$-solution $\tilde{H}^n : Z \times [-\frac{1}{n}, 1] \to W$. Given $\varepsilon > 0$, n can be chosen large and $\tilde{H}^n$ can be reparametrized (by covering the interval $[-\frac{1}{n}, 0]$ quite rapidly) to get a homotopy $\hat{H}^n : Z \times I \to W$ which is a stratified ε-solution of the original problem; that is, $qp\hat{H}^n$ is close to F with the closeness depending on n. This follows from the observations

$$p_n\tilde{H}^n \sim (H^n)^* \Rightarrow p\tilde{H}^n \sim (H^n)^* \Rightarrow qp\tilde{H}^n \sim q(H^n)^* = qH^n \sim qF' = F$$

where "$\sim$" denotes closeness which is small depending on n.

Now to get a stratified controlled solution from the existence of stratified ε-solutions (for every $\varepsilon > 0$) one follows the proof of the corresponding unstratified result [19, Lemma 12.11] using the $*$-construction as needed. □

For more notation, let X be a manifold stratified space containing $N \times [0, +\infty)$ so that $N \times \{0\}$ and $N \times (0, +\infty)$ are strata of X.

PROPOSITION 5.2. *Suppose*

(1) $N \times [0, +\infty)$ *has compact closure in* X,

(2) *if* Y *and* Z *are distinct strata of* X *with* $Z \subseteq \mathrm{cl}(N \times [0, +\infty)) \cap \mathrm{cl}(Y)$, *then* $\dim(Y) \ge 5$.

Then $N \times [0, +\infty)$ *has an approximate tubular neighborhood in* X.

Proof. From this point on we ask the reader to be familiar with the proofs in [17], [18] and, especially, [15] of the special cases of the approximate tubular neighborhood theorem. Let $Z = \mathrm{cl}(N \times [0, +\infty)) \setminus (N \times [0, +\infty))$. In order for the techniques of [15, §§6,7] to apply we need to assume that Z is a single point. As in [15] we can reduce to this case by passing to the quotient space X/Z. Since $N \times [0, +\infty)$ is stratified forward tame in X [12], there is a neighborhood U of $N \times [0, +\infty)$ in X and a nearly stratum preserving deformation of U to $N \times [0,+\infty)$ rel $N \times [0,+\infty)$. This deformation induces a map into the open mapping cylinder of the holink evaluation $\mathrm{holink}_s(X, N \times [0,+\infty)) \to N \times [0,+\infty)$ (see [15, §6]). The open mapping cylinder has natural $[0,+\infty)$-coordinates and there results a proper map $f : U \to N \times [0,+\infty) \times [0,+\infty)$ such that

(1) $f^{-1}(N\times[0,+\infty)\times\{0\}) = N\times[0,+\infty)$ and $f| : f^{-1}(N\times[0,+\infty)\times\{0\}) \to N \times [0,+\infty) \times \{0\}$ is the identity, and

(2) the map f by virtue of factoring through the mapping cylinder of a stratified fibration (via a homotopy equivalence with good control) has good enough lifting properties that for every open cover β of $N \times [0,+\infty) \times (0,+\infty)$, f is properly homotopic rel $N \times [0,+\infty)$ to a map $\tilde{f} : U \to N \times [0,+\infty) \times [0,+\infty)$ such that $\tilde{f}| : W \to N \times [0,+\infty) \times (0,+\infty)$ is a proper stratified β-fibration where $W = U \setminus (N \times [0,+\infty))$.

Given an open cover α of $N \times [0,+\infty) \times (0,+\infty)$, if β is fine enough, then the techniques of [15], which are consequences of engulfing, show that $\tilde{f}|$ is α-close to a map $p : W \to N\times[0,+\infty)\times(0,+\infty)$ with the property that there exists a sequence $\{p_n\}_{n=1}^{\infty}$ of proper maps $p_n : W \to N \times [0,+\infty) \times (0,+\infty)$ such that

(1) p_n is stratum preserving $\frac{1}{2^n}$-homotopic to p_{n+1} for $n = 1, 2, 3, \ldots$,

(2) p_n is a stratified $\frac{1}{2^n}$-fibration for $n = 1, 2, 3, \ldots$,

(3) $p = \lim_{n\to\infty} p_n$ (uniformly).

If α is chosen correctly, then p extends continuously to a map $\tilde{p} : U \to N \times [0,+\infty) \times [0,+\infty)$ via the identity $N \times [0,+\infty) \to N \times [0,+\infty) \times \{0\}$. Define $q : N \times [0,+\infty) \times (0,+\infty) \to N \times [0,+\infty) \times (0,+\infty)$ by

$$q(x,s,t) = \begin{cases} (x,0,t) & \text{if } 0 \le s \le 10 \\ (x, s-10, t) & \text{if } s \ge 10. \end{cases}$$

We now apply Lemma 5.1 to the current situation by incorporating the $(0,+\infty)$ factor into N. We conclude that $qp : W \to N \times [0,+\infty) \times (0,+\infty)$ is an MSAF. We want to apply Lemma 6.11 below in order to conclude that there exists a stratum preserving homeomorphism of X onto the teardrop $(X \setminus (N \times [0,+\infty))) \cup_{qp} (N \times [0,+\infty))$ which restricts to the identity on $N \times [0,+\infty)$. In order to use Lemma 6.11 we need to observe that $q| : N \times [0,+\infty) \to N \times [0,+\infty)$ extends to a stratum preserving map of U to itself which is a homeomorphism on the complement of $N \times [0,+\infty)$. This is a special case of how Quinn's Isotopy Extension Theorem [25] is used in the proof of Lemma 6.7 below. Since $N \times [0,+\infty)$ has an approximate tubular neighborhood in the teardrop, the proof is complete. $\square$

6. Approximate tubular neighborhoods for singular up-sets

This section establishes the result that certain subsets of the singular set have approximate tubular neighborhoods. This will be crucial in the inductive proof of the main result.

Definition 6.1. If X is a space with a stratification $\{X_i\}_{i\in\mathcal{I}}$, then a subset Y of X_{sing} is a *singular up-set* of X if Y is a union of strata of X_{sing} and if whenever X_j is a stratum of X_{sing} for which there exists a stratum Y_i of Y with $Y_i \subseteq \mathrm{cl}(X_j)$, then $X_j \subseteq Y$.

Note that a singular up-set Y in X need not be closed in X.

THEOREM 6.2. *Let X be a manifold stratified space with compact singular set X_{sing} and let $Y \subseteq X_{\text{sing}}$ be a singular up-set of X satisfying:*

if Z_1 and Z_2 are distinct strata of X with $Z_1 \subseteq \mathrm{cl}(Y) \cap \mathrm{cl}(Z_2)$, then $\dim(Z_2) \geq 5$.

Then Y has an approximate tubular neighborhood in X.

Proof. The proof is by induction on the number k of strata of Y. If $k = 1$, then Y is a stratum of X with compact closure and the Main Theorem of [15] implies that Y has an approximate tubular neighborhood in X.

Assume $k > 1$ and Theorem 6.2 holds in the case of fewer than k strata. Write $Y = A \cup B$ where A is a minimal stratum of Y and $B = Y \setminus A$. Using [15] again, A has an approximate tubular neighborhood in X. By the inductive hypothesis, B has an approximate tubular neighborhood in X. Of course, A also has an approximate tubular neighborhood in Y (by restricting the approximate tubular neighborhood map for A in X to Y).

Step 1 (*Notation for the approximate tubular neighborhoods*). Let U_B be an open neighborhood of B in X for which there is an approximate tubular neighborhood map $\varphi_B : U_B \to B \times [0, +\infty)$. Let V_A be an open neighborhood of A in Y for which there is an approximate tubular neighborhood map $\varphi_A : V_A \to A \times [0, +\infty)$. We need to show how to modify φ_A so that it has the additional property:

if $x \in \mathrm{cl}(A) \setminus A$ and U is an open neighborhood of x in X, then there exists an open neighborhood V of x in X such that $\varphi_A^{-1}(\{a\} \times [0, 11]) \subseteq U$ whenever $a \in V \cap A$.

To this end let $\rho : \mathrm{cl}(A) \to I$ be a map such that $\rho^{-1}(0) = \mathrm{cl}(A) \setminus A$ and $\operatorname{diam} \varphi_A^{-1}(\{a\} \times [0, \rho(a)])$ goes to 0 as $a \in A$ approaches $\mathrm{cl}(A) \setminus A$. Define $\rho' : A \times [0, +\infty) \to A \times [0, +\infty)$ by $\rho'(a, s) = (a, \frac{11s}{\rho(a)})$, and

$\varphi'_A : V_A \to A \times [0, +\infty)$ by $\varphi'_A = \rho' \circ \varphi_A$. Then φ'_A is the approximate tubular neighborhood map with the additional property (but we retain the notation φ_A for this map).

Step 2 (*Modifying X along V_A*). Let A' be the one-point compactification of $A \times [0, +\infty)$ with the point at infinity denoted ω.

CLAIM 6.3. *A' is a manifold stratified space with strata $\{\omega\}$, $A \times \{0\}$, $A \times (0, 10)$, $A \times \{10\}$ and $A \times (10, +\infty)$.*

Proof. Note that $\mathrm{cl}(A)$ is a compact manifold stratified space with $\mathrm{cl}(A) \setminus A$ as a closed manifold stratified subspace. Stratify $[0, +\infty]$ with strata $\{0\}$, $(0, 10)$, $\{10\}$, $(10, +\infty)$ and $\{+\infty\}$. Give $\mathrm{cl}(A) \times [0, +\infty]$ the product stratification (which makes it a manifold stratified space [16, 4.1]). Since $Z = ((\mathrm{cl}(A) \setminus A) \times [0, +\infty]) \cup (\mathrm{cl}(A) \times \{+\infty\})$ is a compact manifold stratified subspace and $A' = (\mathrm{cl}(A) \times [0, +\infty])/Z$, it follows from [15] that A' is a manifold stratified space. $\square$

Define $\varphi'_A : \mathrm{cl}(Y) \to A'$ by $\varphi'_A | V_A = \varphi_A : V_A \to A \times [0, +\infty) \subseteq A'$ and $\varphi'_A(\mathrm{cl}(Y) \setminus V_A) = \omega$.

CLAIM 6.4. *$\varphi'_A : \mathrm{cl}(Y) \to A'$ is a* MSAF.

Proof. Let

$$\begin{array}{ccc} Z & \xrightarrow{f} & \mathrm{cl}(Y) \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle \varphi'_A} \\ Z \times I & \xrightarrow{F} & A' \end{array}$$

be a stratified homotopy lifting problem. Let $Z_\omega = f^{-1}(\mathrm{cl}(Y) \setminus V_A)$. Then the problem above restricts to a stratified lifting problem

$$\begin{array}{ccc} Z \setminus Z_\omega & \xrightarrow{f|} & V_A \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle \varphi_A} \\ (Z \setminus Z_\omega) \times I & \xrightarrow{F|} & A \times [0, +\infty) \end{array}$$

which has a stratified controlled solution $\tilde{F} : (Z \setminus Z_\omega) \times I \times [0, 1) \to V_A$. It is not too hard to modify $\tilde{F}$ so that $\operatorname{diam} \tilde{F}(\{z\} \times I \times [0, 1))$ goes to zero as z gets close to Z_ω. If that modification is made, then $\tilde{F}$ will extend to a stratified controlled solution of the original problem by setting $\tilde{F}(z, s, t) = f(z)$ if $z \in Z_\omega$. $\square$

Define the attaching space $X' = X \cup_{\varphi'_A} A'$. It follows from [16, 6.2] that X' is a manifold stratified space. Let $q_X : X \amalg A' \to X'$ be the quotient map.

Step 3 (*A neighborhood of the collar* $A \times [0,10)$ *in* X'). Use Proposition 5.2 to get an open neighborhood W_A of $A \times [0,10)$ in X' and a proper map

$$\xi_A : W_A \to A \times [0,10) \times [0,+\infty)$$

such that $\xi_A^{-1}(A \times [0,10) \times \{0\}) = A \times [0,10)$ and $\xi_A| : A \times [0,10) \to A \times [0,10) \times \{0\}$ is the identity.

Step 4 (*Using uniqueness*).

LEMMA 6.5. *Let M and N be manifolds without boundary such that* $\dim(M) \geq 5$. *Suppose $p : M \times I \to N \times (0,10) \times I$ is a 1-parameter family of manifold approximate fibrations; that is, p is fiber preserving over I and $p_t : M \to N \times (0,10)$ is a manifold approximate fibration for each $t \in I$. Then there exists a manifold approximate fibration $\hat{p} : M \to N \times (0,10)$ such that $\hat{p} = p_0$ over $N \times (0,2)$ and $\hat{p} = p_1$ over $N \times (8,10)$.*

Proof. By the straightening principle [10] (cf. [19]) there is an isotopy $H : M \times I \to M \times I$ with $H_0 = \mathrm{id}_M$ such that pH is as close to $p_0 \times \mathrm{id}_I$ as desired. By the estimated homotopy extension theorem [5], there is a map $\tilde{p} : M \to N \times (0,10)$ such that $\tilde{p} = p_0$ over $N \times (0,4)$, $\tilde{p} = p_1 H_1$ over $N \times (6,10)$, and $\tilde{p}$ is close to p_0. By the sucking principle [10] (cf. [19]), we may additionally assume that $\tilde{p}$ is a manifold approximate fibration. By the isotopy extension theorem [8] there is a homeomorphism $h : M \to M$ such that $h =$ inclusion on $\tilde{p}^{-1}(N \times (0,3))$ and $h = H_1$ on $\tilde{p}^{-1}(N \times (7,10))$. Finally, $\hat{p} = \tilde{p}h^{-1}$ is the desired manifold approximate fibration. □

Returning to the proof of Theorem 6.2, let

$$U_{AB} = \varphi_B^{-1}(\varphi_A^{-1}(A \times (0,10)) \times [0,+\infty)) \subseteq U_B \subseteq X$$

and define the composition

$$\varphi_{AB} : U_{AB} \xrightarrow{\varphi_B} \varphi_A^{-1}(A \times (0,10)) \times [0,+\infty) \xrightarrow{\varphi_A \times \mathrm{id}_{[0,+\infty)}} A \times (0,10) \times [0,+\infty).$$

Note that φ_{AB} is an MSAF as follows. First, $\varphi_A \times \mathrm{id}_{[0,+\infty)}$ is an MSAF [16, 4.3]. Then [16, 7.4, 4.5] also implies that φ_{AB} is an MSAF. Now let $U'_{AB} = q_X(U_{AB}) \subseteq X'$. There is an induced map

$$\varphi'_{AB} : U'_{AB} \to A \times (0,10) \times [0,+\infty).$$

That is, $\varphi'_{AB} \circ q_X = \varphi_{AB} : U_{AB} \to A \times (0,10) \times [0,+\infty)$. Note that φ'_{AB} has the following three properties:

(1) $(\varphi'_{AB})^{-1}(A \times (0,10) \times \{0\}) = A \times (0,10)$,

(2) $\varphi'_{AB}| : A \times (0,10) \to A \times \{0\}$ is the identity,

(3) $\varphi'_{AB}| : U_{AB} \setminus (A \times (0,10)) \to A \times (0,10) \times (0,+\infty)$ is an MSAF.

The first two properties are obvious. For the third, note that since $q_X| : U_{AB} \setminus (A \times (0,10)) \to U'_{AB} \setminus (A \times (0,10))$ is a homeomorphism, we can express φ'_{AB} on $U_{AB} \setminus (A \times (0,10))$ as $\varphi_{AB} \circ q_X^{-1}$, and φ_{AB} is an MSAF. It follows from [14, Prop. 7.1] that $\varphi'_{AB} : U_{AB} \to A \times (0,10) \times [0,+\infty)$ is an MSAF.

Thus, we have two maps $\xi_A| : W_A \setminus (A \times \{0\}) \to A \times (0,10) \times [0,+\infty)$ and $\varphi'_{AB} : U'_{AB} \to A \times (0,10) \times [0,+\infty)$ which are MSAFs and give approximate tubular neighborhoods of $A \times (0,10)$ in X'. Moreover, over $A \times (0,10) \times (0,+\infty)$ these MSAFs are actually MAFs (because their inverse images miss X_{sing}). It follows from the uniqueness results of [18] and Lemma 6.5 that there exists a neighborhood W'_A of $A \times [0,10)$ in X' and a map

$$\xi'_A : W'_A \to A \times [0,10) \times [0,+\infty)$$

such that

(1) $W'_A \subseteq W_A \cup U'_{AB}$,

(2) $\xi'_A = \xi_A$ over $A \times (0,2) \times [0,+\infty)$,

(3) $\xi'_A = \varphi'_{AB}$ over $A \times (8,10) \times [0,+\infty)$,

(4) $(\xi'_A)^{-1}(A \times [0,10) \times \{0\}) = A \times [0,10)$ and $\xi'_A| : A \times [0,10) \to A \times [0,10) \times \{0\}$ is the identity,

(5) ξ'_A is an MSAF over $A \times (0,10) \times [0,+\infty)$.

Step 5 (*Shrinking and pushing*). Let $\hat{X}$ be the quotient space obtained from X with the equivalence relation generated by setting $x \sim y$ if $x, y \in V_A$, $\varphi_A(x) = (z,s)$ and $\varphi_A(y) = (z,t)$ for some $z \in A$ and $0 \leq s,t \leq 10$. Let $g : X \to \hat{X}$ be the quotient map. We identify A with its image under g so that $g| : A \to A$ is the identity. Because of the additional condition imposed on φ_A in Step 1, it follows that g is a closed map (which is to say that the induced decomposition of X is upper semicontinuous; cf. Daverman [6, p. 8]). Moreover, $\hat{X}$ is a locally compact, separable metric space and g is a proper map [6, pp. 13–17].

CLAIM 6.6. *There exists a homeomorphism $\tilde{g} : X \to \hat{X}$ such that*

(1) *$\tilde{g}| : A \to A$ is the identity,*

(2) *$\tilde{g}(Y) \subseteq g(Y)$, and, in fact, $\tilde{g}(S) \subseteq g(S)$ for each stratum S of Y,*

(3) *if S is a stratum of X missing Y, then $\tilde{g}(S) = g(S)$.*

The proof of Claim 6.6 is based on Bing's Shrinking Criterion (cf. [2], [6]) and the following lemma:

LEMMA 6.7. *There exist stratum preserving shrinking homeomorphisms for* g; *that is, for each open cover* $\mathcal{U}$ *of* X *and* $\mathcal{V}$ *of* $\hat{X}$, *there exists a stratum preserving homeomorphism* $H : X \to X$ (*that is, if* S *is a stratum of* X, *then* $H(S) = S$) *such that* gH *is* $\mathcal{V}$-*close to* g, *each* $Hg^{-1}(y)$ *lies in some element of* $\mathcal{U}$ *and* $H|\operatorname{cl}(A)$ *is the inclusion.*

Proof. An isotopy $h_t, t \in I$, of $A \times (0, +\infty)$ affecting only the $(0, +\infty)$-coordinates and moving $A \times (0, 10]$ close to $A \times \{0\}$ can be approximately lifted to a stratum preserving isotopy $\tilde{h}_t, t \in I$ of $V_A \setminus A$ so that $\varphi_A \tilde{h}_t$ is as close as needed to $h_t \varphi_A$. This comes from using the engulfing result [15, 4.3] together with Chapman's stacking technique [2, Lemma 3.5]. Now Quinn's Isotopy Extension Theorem [25] implies that $\tilde{h}_t$ can be extended to a stratum preserving isotopy $\hat{h}_t$ of all of X. This extension is done one stratum at a time in such a way that the desired control is retained. Then the $\hat{h}_1$ provide the required shrinking homeomorphisms for g. □

Proof of Claim 6.6. The proof of Bing's Shrinking Criterion given in [2] provides a proper map $k : X \to X$ constructed as a limit $k = \lim_{n\to\infty} H_1 \circ H_2 \circ \cdots \circ H_n : X \to X$ where the H_i's are shrinking homeomorphisms given by Lemma 6.7 so that $\tilde{g} = g \circ k^{-1}$ defines the desired homeomorphism. □

Let Y' be the quotient space obtained from Y with the equivalence relation generated by setting $x \sim y$ if $x, y \in V_A$ and $x, y \in \varphi_A^{-1}(z, s)$ for some $(z, s) \in A \times [0, 10]$. Note that Y' contains a natural copy of $A \times [0, 10]$. In fact, Y' is the attaching space

$$Y' = Y \cup_{\varphi_A|} (A \times [0, 10])$$

where $\varphi_A| : \varphi_A^{-1}(A \times [0, 10]) \to A \times [0, 10]$. Let

$$q_Y : Y \to Y'$$

be the quotient map. Write $Y' = (A \times [0, 10]) \cup B'$ where $(A \times [0, 10]) \cap B' = \emptyset$ and $(A \times [0, 10]) \cap \operatorname{cl}(B') = A \times \{10\}$.

Let Y'' be the quotient space obtained from Y' with the equivalence relation generated by setting $(x, s) \sim (x, t)$ for each $x \in A$ and $0 \leq s, t \leq 10$. Then $Y'' \subseteq \hat{X}$. Let $q_{Y'} : Y' \to \hat{X}$ be the composition of the quotient map $Y' \to Y''$ followed by the inclusion $Y'' \to \hat{X}$.

Let $\tilde{\beta} : X \to X$ be the map $\tilde{\beta} = \tilde{g}^{-1} \circ g$, let $\beta = \tilde{\beta}| : Y \to Y$, and let

$$\pi : Y' = (A \times [0, 10]) \cup B' \to Y = A \cup B$$

be the map $\pi = \tilde{g}^{-1} \circ q_{Y'}$. We call π *the push*; it is the key geometric move which allows the meshing of the two approximate tubular neighborhoods. Note that $\pi| : A \times [0,10] \to A \subseteq Y$ is the projection and $\pi| : \mathrm{cl}(B') \to Y$ is a stratum preserving homeomorphism. Essentially, π is the collapse of an external collar.

CLAIM 6.8. *Y' is a manifold stratified space with strata $A \times \{0\}$, $A \times (0,10)$, $A \times \{10\}$ and $q_Y(S) \cap B'$ for each stratum S of $B \subseteq Y$.*

Proof. First $\mathrm{cl}(B')$ is a manifold stratified space because $\pi| : \mathrm{cl}(B') \to Y$ is a stratum preserving homeomorphism. Then Y' is a manifold stratified space by the adjunction theorem of [16, 6.2]. □

CLAIM 6.9. *$\pi : Y' \to Y$ is a* MSAF.

Proof. Let

$$\begin{array}{ccc} Z & \xrightarrow{f} & Y' \\ {\scriptstyle \times 0}\downarrow & & \downarrow{\scriptstyle \pi} \\ Z \times I & \xrightarrow{F} & Y \end{array}$$

be a stratified homotopy lifting problem. Define $\tilde{F} : Z \times I \to Y'$ by $\tilde{F}(z,t) = (\pi|\,\mathrm{cl}(B'))^{-1}F(z,t)$ if $f(z) \in \mathrm{cl}(B')$. If $f(z) \in A \times [0,10]$, define $\tilde{F}(z,t)$ by setting $\pi_1\tilde{F}(z,t) = F(z,t)$ and $\pi_2\tilde{F}(z,t) = \pi_2 f(z)$ where $\pi_1 : A \times [0,10] \to A$ and $\pi_2 : A \times [0,10] \to [0,10]$ are the projections. Then $\tilde{F}$ is a stratified solution showing that π is actually a stratified fibration. □

CLAIM 6.10. *$q_Y : Y \to Y'$ is a* MSAF.

Proof. This follows from [16, 7.1]. □

Step 6 (*Recognizing a teardrop*). The plan is to define a neighborhood U of Y in X together with a MSAF $\varphi : U \setminus Y \to Y \times (0,+\infty)$. This map will not extend via the identity $Y \to Y \times \{0\}$ so that we will not be able to conclude immediately that this gives an approximate tubular neighborhood of Y in X. However, φ will extend via β and we will then be able to draw the necessary conclusions from the following lemma (with U playing the role of X so that Y is closed).

LEMMA 6.11. *Let X and $Y = A \cup B$ be as above, but now assume that Y is closed in X. Suppose $\tilde{\beta} : X \to X$ is a proper surjection such that:*

(1) *$\tilde{\beta}^{-1}(Y) = Y$ and $\beta : Y \to Y$ denotes the restriction of $\tilde{\beta}$,*

(2) *$\beta^{-1}(A) = N$ is a closed neighborhood of A in Y,*

(3) *$\beta| : A \to A$ is the identity,*

(4) *$\tilde{\beta}| : X \setminus N \to X \setminus A$ is a homeomorphism.*

Suppose further that $\varphi : X \to Y \times [0,+\infty)$ is a proper map such that $\varphi^{-1}(Y \times \{0\}) = Y$ and $\varphi(x) = (\beta(x), 0)$ for each $x \in Y$. Then there is a *homeomorphism* $h : X \to (X \setminus Y) \cup_{\varphi|(X\setminus Y)} Y$ *which restricts to the identity on* Y. *Moreover, if* $\tilde{\beta}$ *is stratum preserving in the sense that* $\tilde{\beta}(S) = S$ *for each stratum* S *of* $X \setminus Y$, *then* h *also has this property.*

Proof. Define $h : X \to (X \setminus Y) \cup_{\varphi|(X\setminus Y)} Y$ by

$$h(x) = \begin{cases} x & x \in Y, \\ \tilde{\beta}^{-1}(x) & x \in X \setminus Y. \end{cases}$$

Clearly, h is a bijection. The continuity criterion from [18] can be used to see that h is continuous as follows. First, one needs to check that $h| : X \setminus Y \to (X \setminus Y) \cup_{\varphi|(X\setminus Y)} Y$ is an open embedding. But this map is $\tilde{\beta}^{-1}$, so this is obvious. Second, letting $c : (X \setminus Y) \cup_{\varphi|(X\setminus Y)} \to Y \times [0, +\infty)$ be the teardrop collapse, one must check that $c \circ h : X \to Y \times [0, +\infty)$ is continuous. This map is seen to be

$$x \mapsto \begin{cases} (x, 0) & x \in Y, \\ \varphi\tilde{\beta}^{-1}(x) & x \in X \setminus Y. \end{cases}$$

Let $x_n \in X \setminus Y$ $n = 1, 2, 3, \ldots$ be a sequence with $x_n \to x_0 \in Y$ and show that $\varphi\tilde{\beta}^{-1}(x_n) \to (x_0, 0)$. If $x_0 \in B$, then $\tilde{\beta}^{-1}(x_n) \to \tilde{\beta}^{-1}(x_0) = \beta^{-1}(x_0)$. Thus, $\varphi\tilde{\beta}^{-1}(x_n) \to \varphi\beta^{-1}(x_0) = (x_0, 0)$. If, on the other hand, $x_0 \in A$, it follows from the local compactness of X and the properness of $\tilde{\beta}$ that after passing to a subsequence we may assume that $\tilde{\beta}^{-1}(x_n) \to x_0'$ for some $x_0' \in X$. Then $x_n \to \tilde{\beta}(x_0')$ and so $\tilde{\beta}(x_0') = x_0 \in Y$. Thus, $x_0' \in Y$ and $\varphi(x_0') = (\beta(x_0'), 0) = (x_0, 0)$. Finally, $\varphi\tilde{\beta}^{-1}(x_n) \to \varphi(x_0') = (x_0, 0)$ as desired. Hence, h is continuous.

To see that h^{-1} is continuous, first note that it is given by

$$x \mapsto \begin{cases} x & x \in Y, \\ \tilde{\beta}(x) & x \in X \setminus Y. \end{cases}$$

It suffices to consider a sequence $x_n \in X \setminus Y$, $n = 1, 2, 3, \ldots$ such that $x_n \to x_0 \in Y$ in the teardrop of φ and show that $\tilde{\beta}(x_n) \to x_0$ in X. We know that $\varphi(x_n) \to (x_0, 0)$ in $Y \times [0, +\infty)$ (because the teardrop collapse is continuous). By the local compactness of Y and the properness of φ, we may assume after passing to a subsequence that $x_n \to x_0'$ for some $x_0' \in X$. Thus, $\varphi(x_n) \to \varphi(x_0')$. So $\varphi(x_0') = (x_0, 0) \in Y \times \{0\}$ and so $\beta(x_0') = x_0$. Also $\varphi(x_n) \to (x_0, 0)$. Note $x_0' \in Y$. Now $x_n \to x_0'$ in X implies $\tilde{\beta}(x_n) \to \tilde{\beta}(x_0')$ in X which in turn implies $\tilde{\beta}(x_n) \to \beta(x_0') = x_0$ as desired. Hence, h^{-1} is continuous and h is a homeomorphism. □

Step 7 (*Completion of the proof*). We return to the completion of the proof of Theorem 6.2. Let $U_L = q_X^{-1}(W_A') \cap X$ and $U = U_L \cup U_B$. Thus, U is an open neighborhood of Y in X. Let $U_R = U \setminus q_X^{-1}((\xi_A')^{-1}(A \times [0, 8] \times [0, +\infty)))$. Note that $U = U_L \cup U_R$ and

$$U_L \cap U_R = q_X^{-1}((\xi_A')^{-1}(A \times (8, 10) \times [0, +\infty))).$$

Define

$$\varphi : U \to Y \times [0, +\infty)$$

as follows:

(1) $\varphi|U_L$ is the composition

$$U_L \xrightarrow{q_X} W'_A \xrightarrow{\xi'_A} A \times [0,10) \times [0,+\infty) \xrightarrow{\pi \times \mathrm{id}_{[0,+\infty)}} A \times [0,+\infty) \xrightarrow{\subseteq} Y \times [0,+\infty).$$

(2) $\varphi|U_R$ is the composition

$$U_R \xrightarrow{\varphi_B} B \times [0,+\infty) \xrightarrow{q_Y \times \mathrm{id}_{[0,+\infty)}} Y' \times [0,+\infty) \xrightarrow{\pi \times \mathrm{id}_{[0,+\infty)}} Y \times [0,+\infty).$$

In order to verify that these definitions of φ agree on the overlap, first note that

$$\begin{aligned} U_L \cap U_R = q_X^{-1}((\varphi'_{AB})^{-1}(A \times (8,10) \times [0,+\infty)) \\ = \varphi_B^{-1}(\varphi_A^{-1}(A \times (8,10)) \times [0,+\infty)) \subseteq U_{AB}. \end{aligned}$$

From the definition of q_Y it follows that the composition

$$U_L \cap U_R \xrightarrow{\varphi_B} \varphi_A^{-1}(A \times (8,10)) \times [0,+\infty) \xrightarrow{q_Y \times \mathrm{id}_{[0,+\infty)}} Y' \times [0,+\infty)$$

is the composition

$$\begin{aligned} U_L \cap U_R \xrightarrow{\varphi_B} \varphi_A^{-1}(A \times (8,10)) \times [0,+\infty) \xrightarrow{\varphi_A \times \mathrm{id}_{[0,+\infty)}} A \times (8,10) \times [0,+\infty) \\ \xrightarrow{\text{inclusion}} Y' \times [0,+\infty). \end{aligned}$$

In turn, by the definition of φ_{AB}, this is the composition

$$U_L \cap U_R \xrightarrow{\varphi_{AB}} A \times (8,10) \times [0,+\infty) \xrightarrow{\text{inclusion}} Y' \times [0,+\infty).$$

Use the definition of φ'_{AB} to express $\varphi_{AB} = \varphi'_{AB} \circ q_X$. It follows that $\varphi|U_R$ on $U_L \cap U_R$ is given by the composition

$$\begin{aligned} U_L \cap U_R \xrightarrow{\varphi'_{AB} \circ q_X} A \times (8,10) \times [0,+\infty) \xrightarrow{\pi \times \mathrm{id}_{[0,+\infty)}} A \times [0,+\infty) \\ \xrightarrow{\text{inclusion}} Y \times [0,+\infty). \end{aligned}$$

Using the properties of ξ'_A in Step 4, this is the composition

$$\begin{aligned} U_L \cap U_R \xrightarrow{\xi'_A \circ q_X} A \times (8,10) \times [0,+\infty) \xrightarrow{\pi \times \mathrm{id}_{[0,+\infty)}} A \times [0,+\infty) \\ \xrightarrow{\text{inclusion}} Y \times [0,+\infty) \end{aligned}$$

and, hence, we have the desired agreement.

In fact, we have shown that the compositions

$$U_L \xrightarrow{\xi'_A \circ q_X} A \times [0,10) \times [0,+\infty) \xrightarrow{\text{inclusion}} Y' \times [0,+\infty)$$

and

$$U_R \xrightarrow{(q_Y \times \mathrm{id}_{[0,+\infty)}) \circ \varphi_B} Y' \times [0,+\infty)$$

agree on $U_L \cap U_R$. Hence we have a map

$$\varphi' : U \to Y' \times [0,+\infty)$$

and our goal now is to show that

$$\varphi : U \xrightarrow{\varphi'} Y' \times [0,+\infty) \xrightarrow{\pi \times \mathrm{id}_{[0,+\infty)}} Y \times [0,+\infty)$$

has the property that its restriction

$$\varphi| : U \setminus Y \to Y \times (0,+\infty)$$

is an MSAF. This is accomplished by the following claims:

CLAIM 6.12. $\varphi'| : U \setminus Y \to Y' \times (0,+\infty)$ *is a* MSAF.

Proof. According to the characterization in [13] it suffices to show that the mapping cylinder $\mathrm{cyl}(\varphi'|)$ is a homotopically stratified space. Since this condition is a local one it follows from the fact that $\varphi'|$ is locally a MSAF. □

CLAIM 6.13. $\varphi| : U \setminus Y \to Y \times (0,+\infty)$ *is an* MSAF.

Proof. First $\pi \times \mathrm{id}_{[0,+\infty)}$ is a MSAF by Claim 6.9 and [16, 4.3]. Now combine Claim 6.12 and [16, 4.5]. □

Finally, note that $\varphi| : Y \to Y \times \{0\}$ is the map $\pi \circ q_X = \beta$ so that Lemma 6.11 can be applied to show that U is stratum preserving homeomorphic to the teardrop $(U \setminus Y) \cup_{\varphi|(U\setminus Y)} Y$. This shows that Y has an approximate tubular neighborhood in X and completes the proof of Theorem 6.2. □

COROLLARY 6.14. *Let X be a manifold stratified space with a compact singular set X_{sing} such that all nonminimal strata of X are of dimension greater than or equal to five. Then X_{sing} has an approximate tubular neighborhood in X.*

Proof. X_{sing} is a singular up-set satisfying the hypothesis of Theorem 6.2. □

7. Proof of the main result

In this section we restate and prove the main result.

THEOREM 7.1. *Let X be a manifold stratified space with compact singular set X_{sing} such that all nonminimal strata of X are of dimension greater than or equal to five. If $Y \subseteq X_{\text{sing}}$ is a pure subset of X, then Y has an approximate tubular neighborhood in X.*

Proof. The proof is by induction on the number n of strata of X. We may assume that $n > 0$ and that the result is true for manifold stratified spaces with fewer than n strata. Given X and Y as in the hypothesis, let Z be the union of the strata of $X_{\text{sing}} \setminus Y$ and let $A = Y \cap \text{cl}(Z)$. Then A is a pure subset of X_{sing} and $A \subseteq (X_{\text{sing}})_{\text{sing}}$. Since X_{sing} has fewer strata than X, it follows that A has an approximate tubular neighborhood in X_{sing}. Say U is an open neighborhood of A in X_{sing} for which there is an approximate tubular neighborhood map

$$p : U \to A \times [0, +\infty).$$

By Corollary 6.14, X_{sing} has an approximate tubular neighborhood in X. Say V is an open neighborhood of X_{sing} in X for which there is an approximate tubular neighborhood map

$$q : V \to X_{\text{sing}} \times [0, +\infty).$$

Note that $U \cup Y$ is open in X_{sing}. Let $U_1 = U \setminus (Y \setminus A)$ so that $U_1 \cap Y = A$. Let $W = q^{-1}((U \cup Y) \times [0, +\infty))$. Then $q| : W \to (U \cup Y) \times [0, +\infty)$ is still a MSAF [16, 7.4]. Define

$$\tilde{p} : U \cup Y \to (A \times [0, +\infty)) \cup (Y \times \{0\})$$

(where the range is a subset of $Y \times [0, +\infty)$) by $\tilde{p}|U_1 = p| : U_1 \to A \times [0, +\infty)$ and $\tilde{p}| : Y \to Y \times \{0\}$ is the identity. Note that $p| : U_1 \to A \times [0, +\infty)$ is a MSAF because U_1 is a pure subset of U. It follows from [16, 7.2] that $\tilde{p}$ is a MSAF. Define

$$r : A \times [0, +\infty) \times [0, +\infty) \to A \times [0, +\infty)$$

by $r(a, s, t) = (a, s + t)$. It follows from [16, 4.6] that r is a stratified fibration. Define

$$\tilde{r} : [(A \times [0, +\infty)) \cup Y \times \{0\}] \times [0, +\infty) \to Y \times [0, +\infty)$$

by

$$\tilde{r}(x, s, t) = \begin{cases} r(x, s, t) & \text{if } x \in A \\ (x, t) & \text{if } s = 0. \end{cases}$$

It follows from [16, 7.2] that $\tilde{r}$ is a MSAF. Consider the composition

$$f : W \xrightarrow{q|} (U \cup Y) \times [0, +\infty) \xrightarrow{\tilde{p} \times \mathrm{id}_{[0,+\infty)}} \\ [(A \times [0, +\infty)) \cup Y \times \{0\}] \times [0, +\infty) \xrightarrow{\tilde{r}} Y \times [0, +\infty).$$

It follows from [16, 4.3, 4.5] that f is a MSAF. It is then easy to check that f is an approximate tubular neighborhood map for Y in X. □

Remark 7.2. The theorem also applies to a manifold stratified space X with noncompact singular set provided, in addition, that all the noncompact strata are of dimension greater than or equal to five and the one-point compactification of X is a manifold stratified space with the point at infinity constituting a new stratum.

Vanderbilt University, Nashville, Tennessee
E-mail address: bruce.hughes@vanderbilt.edu

References

[1] S. Cappell and J. Shaneson, The mapping cone and cylinder of a stratified map, in *Prospects in Topology*, *Proc. Conference in Honor of William Browder* (F. Quinn, ed.), *Ann. of Math. Studies* **138**, 58–66, Princeton Univ. Press, Princeton, NJ, 1995.

[2] T. A. Chapman, *Lectures on Hilbert Cube Manifolds*, Conf. Board of the Math. Sci., *Conf. Series in Math.* **28**, Amer. Math. Soc., 1976 Providence, RI, 1976.

[3] ———, Approximation results in topological manifolds, *Trans. Amer. Math. Soc.* **262** (1980), 303–334.

[4] ———, *Approximation Results in Hilbert Cube Manifolds*, *Mem. Amer. Math. Soc.* **34**, No. 251, 1981.

[5] T. A. Chapman and S. Ferry, Approximating homotopy equivalences by homeomorphisms, *Amer. J. Math.* **101** (1979), 583–607.

[6] R. Daverman, *Decompositions of Manifolds*, Academic Press, Orlando, FL, 1986.

[7] R. D. Edwards, TOP regular neighborhoods, handwritten manuscript, 1973.

[8] R. D. Edwards and R. C. Kirby, Deformations of spaces of imbeddings, *Ann. of Math.* **93** (1971), 63–88.

[9] M. Goresky and R. MacPherson, *Stratified Morse Theory*, *Ergeb. Math. Grenzgeb.* **14**, Springer-Verlag, New York, 1988.

[10] B. Hughes, Approximate fibrations on topological manifolds, *Michigan Math. J.* **32** (1985), 167–183.

[11] ———, Geometric topology of stratified spaces, *Electron. Res. Announc. Amer. Math. Soc.* **2** (1996), 73–81.

[12] ———, Stratified path spaces and fibrations, *Proc. Roy. Soc. Edinburgh Sect. A* **129** (1999), 351–384.

[13] B. Hughes, Stratifications of mapping cylinders, *Topology Appl.* **94** (1999), 127–145.

[14] ———, Stratifications of teardrops, *Fund. Math.* **161** (1999), 305–324.

[15] ———, Neighborhoods of strata in manifold stratified spaces, Vanderbilt University preprint, July 2000.

[16] ———, Products and adjunctions of manifold stratified spaces, *Topology and its Appl.*, to appear.

[17] B. Hughes and A. Ranicki, *Ends of Complexes*, *Cambridge Tracts in Math.* **123**, Cambridge Univ. Press, Cambridge, 1996.

[18] B. Hughes, L. Taylor, S. Weinberger, and B. Williams, Neighborhoods in stratified spaces with two strata, *Topology* **39** (2000), 873–919.

[19] B. Hughes, L. Taylor, and B. Williams, Bundle theories for topological manifolds, *Trans. Amer. Math. Soc.* **319** (1990), 1–65.

[20] ———, Manifold approximate fibrations are approximately bundles, *Forum Math.* **3** (1991), 309–325.

[21] B. Hughes and S. Weinberger, Surgery and stratified spaces, in *Surveys on Surgery Theory*, Vol. 2 (S. Cappell, A. Ranicki, and J. Rosenberg, eds.), *Ann. of Math. Studies* **149**, Princeton Univ. Press, Princeton, NJ, 319–352.

[22] J. Mather, *Notes on Topological Stability*, photocopied notes, Harvard Univ., Cambridge, MA, 1970.

[23] ———, Stratifications and mappings, in *Dynamical Systems*, 195–232, *Proc. Sympos., Univ. Bahia* (Salvador, Brazil, 1971) (M. M. Peixoto, ed.), Academic Press, New York, 1973.

[24] F. Quinn, Applications of topology with control, 598–606, *Proc. Internat. Congress of Mathematicians* (Berkeley, CA, 1986), Amer. Math. Soc., Providence, RI, 1987.

[25] ———, Homotopically stratified sets, *J. Amer. Math. Soc.* **1** (1988), 441–499.

[26] C. Rourke and B. Sanderson, An embedding without a normal microbundle, *Invent. Math.* **3** (1967), 293–299.

[27] L. Siebenmann, Deformation of homeomorphisms on stratified sets. I, II, *Comment. Math. Helv.* **47** (1972), 123–163.

[28] M. Steinberger and J. West, Approximation by equivariant homeomorphisms, I, *Trans. Amer. Math. Soc.* **302** (1987), 297–317.

[29] R. Thom, Ensembles et morphismes stratifiés, *Bull. Amer. Math. Soc.* **75** (1969), 240–284.

[30] S. Weinberger, *The Topological Classification of Stratified Spaces*, *Chicago Lectures in Math.*, Univ. of Chicago Press, Chicago, IL, 1994.

[31] ———, Nonlocally linear manifolds and orbifolds, 637–647, in *Proc. Internat. Congress of Mathematicians* (Zürich, Switzerland, 1994), Birkhäuser, Basel, 1995.

[32] ———, Microsurgery on stratified spaces, 509–521, in *Geometric Topology* (Athens, GA, 1993), *AMS/IP Stud. Adv. Math.* **2.1** (William H. Kazez, ed.), Amer. Math. Soc., Providence, RI, 1997.

(Received October 23, 2000)