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Manifold Approximate Fibrations Are Approximately Bundles

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Abstract. In [4], we gave a classification of manifold approximate fibrations in terms of the lifting problem for a certain bundle. Our description of this bundle in [4] is not particularly illuminating, and one purpose of this paper is to give a more transparent description of it (see Theorem 0.1). There are two obvious compatibility questions which we deal with in Theorems 0.2 and 0.3. As an application of these results we prove a topological tubular neighborhood theorem (Theorem 4).

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A manifold approximate fibration is a map $p\colon M \to B$, where B and M are manifolds and p is a proper map which has the approximate covering homotopy property. Recall from [4] that manifold approximate fibrations over an i-dimensional manifold B (with total space of dimension at least 5) are classified as follows. First select a "fibre germ": i.e., a manifold approximate fibration $q\colon F \to \mathbb{R}^i$. Then $\mathrm{MAF}(B)_q$ is roughly defined to be the simplicial set of manifold approximate fibrations over B such that the inverse image of $\mathbb{R}^i$ (where $\mathbb{R}^i \subset B$) mapping down to $\mathbb{R}^i$ is a manifold approximate fibration which is controlled homeomorphic to q or to the "conjugate" approximate fibration $\bar{q}\colon F \xrightarrow{q} \mathbb{R}^i \xrightarrow{-1} \mathbb{R}^i$ where -1 denotes multiplication by -1 in the first coordinate. In the sequel we shall call q *self-conjugate* iff q and $\bar{q}$ are controlled homeomorphic.

When q is self-conjugate we constructed a bundle $\eta\colon \mathrm{MAF}(q) \to BTop_i$ and a differential $d\colon \mathrm{MAF}(B)_q \to \mathrm{Lifts}(\tau\colon B \to BTop_i)$ where τ is the classifying map for the tangent bundle of B, and Lifts denotes the space of lifts of this map to the total

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space of our bundle η. The Classification Theorems of [4, Theorems (1.4) and (7.12)] imply that d is a homotopy equivalence. (If q is not self-conjugate, then we replace $BTop_i$ by $BSTop_i$ and the result still holds.)

In [4] the homotopy type of the fibre of η is identified with $BTop^c(q)$, the classifying space of the simplicial group of controlled homeomorphisms of q.

By [4, (12.2)], $Top^c(q)$ can be identified with the simplicial group of level-preserving homeomorphisms of the mapping cylinder of q, denoted $M(q)$, which are the identity on the range copy of $\mathbb{R}^i$. Let $Top^{level}(q)$ denote the simplicial group of level preserving homeomorphisms of $M(q)$ which leave the origin of the range copy of $\mathbb{R}^i$ fixed. By the covering isotopy theorem for manifold approximate fibrations, [4, 14.3] and [5, 10.1], the natural map $Top^{level}(q) \to Top_i$ is always onto $STop_i$ and is onto all of Top_i iff q is self-conjugate.

When q is self-conjugate we have the corresponding map of classifying spaces, $BTop^{level}(q) \to BTop_i$ whose fibre is $BTop^c(q)$. (If q is not self-conjugate, then we replace $BTop_i$ by $BSTop_i$ and the result still holds.) This suggests the main theorem of this paper, which will be proved in § 3.

Theorem 0.1. *Let* $\dim F \geq 5$. *When q is self-conjugate there exists a differential from* $\mathrm{MAF}(B)_q$ *to the simplicial set of lifts of* $B \xrightarrow{\tau} BTop_i$ *to* $BTop^{level}(q)$ *which is a homotopy equivalence. (If q is not self-conjugate, then we replace $BTop_i$ by $BSTop_i$ and the result still holds.)*

There are two compatibility questions. In [4] we constructed a map $\mathrm{MAF}(B) \to \mathrm{Hur}(B)$, which, roughly speaking, takes the associated Hurewicz fibration. We can of course restrict this to $\mathrm{MAF}(B)_q$. There is a monoid map $Top^{level}(q) \to Homeo(F) \to G(F)$, where $G(F)$ denotes the simplicial monoid of self-homotopy equivalences of F. This map is given by restricting the level preserving map of $M(q)$ to the copy of F. Since Hurewicz fibrations over B with fibre F are classified by maps into $BG(F)$, we get another map $\mathrm{MAF}(B)_q \to \mathrm{Hur}(B)$.

Theorem 0.2. *The above two maps* $\mathrm{MAF}(B)_q \to \mathrm{Hur}(B)$ *are homotopic.*

For the other compatibility question, let V be a compact manifold without boundary. The projection $p\colon V \times \mathbb{R}^i \to \mathbb{R}^i$ is a manifold approximate fibration, which is the fibre germ for any fibre bundle over B with fibre V. Hence we get a map $\mathrm{Bun}(B)_V \to \mathrm{MAF}(B)_p$, where $\mathrm{Bun}(B)_V$ is the simplicial set of fibre bundles over B with fibre V. On the other hand, fibre bundles over B are classified by maps of B into $BHomeo(V)$. There is a simplicial group homomorphism, $Homeo(V) \times Top_i \to Top^{level}(p)$, given by sending $f \times h$ to the level-preserving map of $M(p)$ which is $f \times h$ on $V \times \mathbb{R}^i$ and h on $\mathbb{R}^i$. Maps of B into $BHomeo(V)$ are the same thing as lifts of $B \xrightarrow{\tau} BTop_i$ to $BHomeo(V) \times BTop_i$, so we get another map $\mathrm{Bun}(B)_V \to \mathrm{MAF}(B)_p$.

Theorem 0.3. *The above two maps* $\mathrm{Bun}(B)_V \to \mathrm{MAF}(B)_p$ *are homotopic.*

This new description of our bundle is useful in describing maps between the space of manifold approximate fibrations and other spaces. As an example, we will prove that locally flat (topological) submanifolds have mapping cylinder neighborhoods (i.e., "topological tubular neighborhoods"), and that these neighborhoods are essentially unique. This result is originally due to R. D. Edwards [3]. E. K. Pedersen also proved the existence of mapping cylinder neighborhoods as an application of his regular neighborhood theory, [8, Theorem 15], and F. Quinn has obtained a significant generalization [9, Theorem 3.1.1]. We then go on to analyze the full space of embeddings of a given mapping cylinder neighborhood into the larger manifold relative to the submanifold.

Theorem 0.4. (The Topological Tubular Neighborhood Theorem) *Let M^n and N^{n+k} be closed manifolds where $n \geq 5$ and $k \geq 1$, and suppose that M is a (topologically) locally flat submanifold of N. Then*

(i) *there exists a manifold approximate fibration $f\colon P \to M$, where P is a closed $(n+k-1)$ – manifold, such that the mapping cylinder $M(f)$ is homeomorphic to a closed neighborhood of M in N by a homeomorphism which is the identity on M. Moreover, f is unique up to controlled homeomorphism and the fibre germ of f is projection $\pi\colon \mathbb{R}^n \times S^{k-1} \to \mathbb{R}^n$. In addition,*

(ii) *the simplicial set $\mathscr{E}mb_M(M(f), N)$ of embeddings of $M(f)$ into N which are the identity on M is homotopy equivalent to $Top^c(f \times \mathrm{Id}_{\mathbb{R}})$.*

For information on how to analyze the simplicial group of controlled homeomorphisms, see [4] and [11].

We thank Shmuel Weinberger for stimulating conversations on this material. In a future paper with Weinberger we intend to extend some of the results of this paper to topologically stratified spaces and give applications to topological group actions on manifolds.

§ 1. Notation and definitions

Notation. All spaces will have compactly generated topologies and in our applications are usually seperable metric. Let $q\colon F_1 \to F_2$ be a map between two spaces, and let $M(q)$ denote its mapping cylinder

$$M(q) = \frac{F_1 \times [0,1] \amalg F_2}{(x,0) \sim q(c)}$$

(topologized by making the quotient topology compactly generated). $Top^{level}(q)$ denotes the simplicial group of homeomorphisms of $M(q)$ to itself which preserve the mapping cylinder levels. That is, if $\pi\colon M(q) \to [0,1]$ is the standard projection, then a homeomorphism

$$h\colon M(q) \to M(q) \text{ is in } Top^{level}(q) \text{ iff } h\pi^{-1}(t) = \pi^{-1}(t) \quad \forall t \in [0,1]\,.$$

Let B denote a fixed space with B paracompact.

Definition of $\mathscr{B}_1$. Consider triples, $E_1, E_2, f\colon E_1 \to E_2$, where

(1) for $i = 1, 2$, $p_i\colon E_i \to B$ is a fibre bundle with fibre F_i and group $Top(F_i)$;

(2) we have a commutative diagram

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ p_1 \searrow & & \swarrow p_2 \\ & B & \end{array}$$

and

(3) (locally trivial) for each $x \in B$ there exists a neighborhood, $\mathscr{U}$ of x in B such that, if $f_{\mathscr{U}}$ denotes the restriction of f to $p_1^{-1}(\mathscr{U}) \to p_2^{-1}(\mathscr{U})$, then there is a homeomorphism $\mathscr{U} \times M(q) \to M(f_{\mathscr{U}})$ which is level preserving (in the sense that the levels of the mapping cylinders are preserved) and

$$\begin{array}{ccc} \mathscr{U} \times M(q) & \longrightarrow & M(f_{\mathscr{U}}) \\ \searrow & & \swarrow \\ & \mathscr{U} & \end{array}$$

commutes (where the maps to $\mathscr{U}$ are the natural projections).

Let

$$\begin{array}{ccc} E_1' & \xrightarrow{f'} & E_2' \\ p_1' \searrow & & \swarrow p_2' \\ & B & \end{array}$$

be another such triple. A *map* between them is a pair of bundle isomorphisms $h_1\colon E_1 \to E_1'$ and $h_2\colon E_2 \to E_2'$ and a level preserving homeomorphism $H\colon M(f) \to M(f')$ which is fibre preserving over B and which restricts to h_1 (resp. h_2) at the "top" (resp. "bottom") of the mapping cylinder. Maps can clearly be composed.

Triples and maps as above form a category and one can easily choose embeddings and extend to an ℓ^2-Top^{op} functor as in [4]. This means that for $B \subset \ell^2$ fixed and for any space, X, a subspace of ℓ_2 we associate the category of triples and maps over $B \times X$ with the additional conditions that $E_i \subset \ell^2 \times X$ so that the composite $E_i \subset \ell^2 \times X \to X$ is a bundle projection for $i = 1, 2$. It can be checked that a map, $f\colon X_1 \to X_2$, induces a functor between the category associated to X_2 and the category associated to X_1 by pulling the bundles back along f. By letting X run over the various simplicies and letting f run over the various inclusions and degeneracies, one can associate a simplicial set to an ℓ^2-Top^{op} functor. Note that this ℓ^2-Top^{op} functor satisfies the Amalgamation Property so the resulting simplicial set is Kan. We let $\mathscr{B}_1$ denote this ℓ^2-Top^{op} functor.

Definition of $\mathscr{B}_2$. This is just the ℓ^2-Top^{op} functor of bundles over B with fibre $M(q)$ and group $Top^{level}(q)$.

Definition of μ. There is a function $\mu\colon \mathscr{B}_1 \to \mathscr{B}_2$ called the *mapping cylinder construction* wich takes a triple

$$\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ & p_1 \searrow \quad \swarrow p_2 & \\ & B & \end{array} \qquad \text{to} \qquad \begin{array}{c} M(f) \\ \downarrow \\ B \end{array}.$$

Note, by the "local triviality" condition on f, that $\begin{array}{c} M(f) \\ \downarrow \\ B \end{array}$ is a bundle with fibre $M(q)$ and group $Top^{level}(q)$.

Given a map (h_1, h_2, H) as above, H defines a bundle map from $\mu(f)$ to $\mu(f')$.

Indeed, an equivalent definition of the morphisms in $\mathscr{B}_1$ is pairs of bundle maps $h_i\colon E_i \to E_i'$, $i = 1, 2$, and a bundle map $H\colon \mu(f) \to \mu(f')$ which restricts to h_1 at the top and h_2 at the bottom of the mapping cylinders.

In the language of [4], μ is a natural transformation of ℓ^2-Top^{op} functors.

§ 2. A construction

As we shall discuss in more detail shortly, Theorem 0.1 is equivalent to the statement that a suitable version of the μ map is a homotopy equivalence. We would therefore like to produce a natural transformation of ℓ^2-Top^{op} functors which would be an inverse to μ. We will not succeed, but we will come close enough for our purposes.

Let $p\colon E \to B$ be a fibre bundle with fibre $M(q)$ and structure group $Top^{level}(q)$. The projection $\pi\colon M(q) \to [0,1]$ induces a natural projection $E \to [0,1]$ also denoted by π (this uses the fact that the structure group is $Top^{level}(q)$). For each $t \in [0,1]$ let $E^t = \pi^{-1}(t) \subset E$. Then p restricts to a map $p^t\colon E^t \to B$ which is a fibre bundle for each $t \in [0,1]$. For $0 < t \leq 1$, $p^t\colon E^t \to B$ has fibre F_1 and group $Top(F_1)$ whereas $p^0\colon E^0 \to B$ has fibre F_2 and group $Top(F_2)$.

We begin by defining a deformation retraction $d\colon E \times [0,1] \to E$ which takes $E \times 1$ into E^0. [The idea is to push down along the mapping cylinder rays, but we must work locally, which means that things must be pieced together, ...].

Let $\mathscr{I}$ be an ordered set. Choose a locally-finite, open cover, $\{\mathscr{U}_i\}_{i \in \mathscr{I}}$ of B and local trivializations $\begin{array}{ccc} \mathscr{U}_i \times M(q) & \longrightarrow & E_i \\ & \searrow \quad \swarrow & \\ & \mathscr{U}_i & \end{array}$, where $E_i = p^{-1}(\mathscr{U}_i)$, which are $Top^{level}(q)$-related on overlaps.

Let $\{\kappa_i\}_{i \in \mathscr{I}}$ be a partition of unity subordinate to $\{\mathscr{U}_i\}$. Thus, $\kappa_i\colon B \to [0,1]$, $\operatorname{supp} \kappa_i \equiv \operatorname{cl}\{x \in B \mid \kappa_i(x) > 0\} \subset \mathscr{U}_i$, $\{\operatorname{supp} \kappa_i\}_{i \in \mathscr{I}}$ is neighborhood-finite, and for each $x \in B$, $\sum_{i \in \mathscr{I}} \kappa_i(x) = 1$.

For each $i \in \mathscr{I}$; $x \in B$ and $s \in [0,1]$, define $c_i^s(x)\colon [0,1] \to [0,1]$ by

$$c_i^s(x)(t) = \begin{cases} t - s\kappa_i(x) & \text{if } \kappa_i(x) \le t \le 1 \\ 0 & \text{if } 0 \le t \le \kappa_i(x) \end{cases}.$$

This in turn induces a map $\tilde{c}_i\colon B \times M(q) \times [0,1] \to B \times M(q) \times [0,1]$ defined by

$$\begin{aligned} &\tilde{c}_i(x, [y, t], s) = (x, [y, c_i^s(x)(t)], s) && \text{if } y \in F_1,\ t \in [0,1] \\ &\tilde{c}_i(x, [y], s) = (x, [y], s) && \text{if } y \in F_2 \end{aligned}$$

Since $c_i^s(x)(0) = 0$, this is well-defined and continuous.

Note the following:

(a) $\tilde{c}_i(x, z, s) = (x, z', s)$ where $\pi(z') = c_i^s(x)(\pi(z))$,

(b) if $x \in B$ and $\kappa_i(x) = 0$, then $\tilde{c}_i(x, z, s) = (x, z, s)$ for all $z \in M(q)$ and all $s \in [0,1]$.

Now for each $i \in \mathscr{I}$, define $d_i\colon E \times [0,1] \to E \times [0,1]$ by

(1) $d_i | E_i \times [0,1]$ is given as the composition

$$E_i \times [0,1] \xrightarrow{h_i^{-1} \times \mathrm{id}} \mathscr{U}_i \times M(q) \times [0,1] \xrightarrow{\tilde{c}_i} \mathscr{U}_i \times M(q) \times [0,1] \xrightarrow{h_i} E_i \times [0,1],$$

(2) $d_i | (E - E_i) \times [0,1] = \mathrm{id}$, and note

(i) $$\pi \circ d_i(z, s) = \begin{cases} t - s\kappa_i(p(z)) & \kappa_i(p(z)) \le t \le 1 \\ 0 & 0 \le t \le \kappa_i(p(z)) \end{cases};$$

(ii) $p \circ d_i(z, s) = p(z)$.

That d_i is well-defined and continuous follows from (b) above.

Define $d\colon E \times [0,1] \to E$ as follows. For each $x \in B$, there exists a finite set, $\mathscr{F}_x \subset \mathscr{I}$, and a neighborhood, $\mathscr{V}_x$ of x, such that $i \in \mathscr{F}_x$ iff $\kappa_i(y) = 0 \quad \forall y \in \mathscr{V}_x$. Let $i_1, \ldots, i_r$ denote the elements in $\mathscr{F}_x$ with ordering $i_1 < \ldots < i_r$. Let $d(z, s) = d_{i_r} \circ \ldots \circ d_{i_1}(z, s) \in E$, for some choice of $\mathscr{F}_{p(z)}, \mathscr{V}_{p(z)}$.

The pairs $\mathscr{F}_{p(z)}, \mathscr{V}_{p(z)}$ are not unique, but we want to prove that the map d is unique. Let $\mathscr{F}'_{p(z)}, \mathscr{V}'_{p(z)}$ be another such pair, and let $d'(z, s)$ denote the resulting element of E. Note $\mathscr{F}'_{p(z)} \cup \mathscr{F}_{p(z)}, \mathscr{V}'_{p(z)} \cap \mathscr{V}_{p(z)}$ is also a pair with the required properties, so it will suffice to prove $d'(z, s) = d(z, s)$ under the additional assumptions that $\mathscr{F}_{p(z)} \subset \mathscr{F}'_{p(z)}$ and $\mathscr{V}'_{p(z)} \subset \mathscr{V}_{p(z)}$. Now note that if $j \in \mathscr{F}'_x - \mathscr{F}_x$, then $d_j(z, s) = (z, s)$, so $d'(z, s) = d(z, s)$.

If $\mathscr{F}_{p(z)}, \mathscr{V}_{p(z)}$ are fixed, notice that we can take $\mathscr{F}_{p(y)} = \mathscr{F}_{p(z)}, \mathscr{V}_{p(y)} = \mathscr{V}_{p(z)}$ for any y such that $p(y) \in \mathscr{V}_{p(z)}$. Hence d is locally-continuous and therefore continuous.

Note that $d|(E^0 \times [0,1] \cup E \times 0) = \mathrm{id}$ and that $p \circ d(z,s) = p(z)$. Since $\sum_{i \in \mathscr{I}} \kappa_i(x) = 1$, it follows that $\pi \circ d(z,s) = \begin{cases} \pi(z) - s & s \leq \pi(z) \leq 1 \\ 0 & 0 \leq \pi(z) \leq s \end{cases}$; and that $d(z,1) \in E^0$ for each $z \in E$.

Now consider the composite

$$E^1 \times [0,1] \subset E \times [0,1] \xrightarrow{\mathrm{id}_E \times \{t \mapsto 1-t\}} E \times [0,1] \xrightarrow{d} E.$$

One can check that this map is level preserving and that restricted to $E^1 \times 0$ it lands in E^0. If we let $f'\colon E^1 \to E^0$ denote this map, we get a well-defined, continuous map, $m\colon M(f') \to E$ which is level preserving.

Suppose that we have a bundle map, $h\colon E_1 \to E_2$. Choose a local trivialization of E_2 and a subordinate partition of unity. Order the partition. The above discussion produces maps $d_i\colon E_i \to [0,1] \to E_i$ making the following square commute:

$$\begin{array}{ccc} E_1 \times [0,1] & \xrightarrow{d_1} & E_1 \\ h \times \mathrm{Id}_{[0,1]} \downarrow & & \downarrow h \\ E_2 \times [0,1] & \xrightarrow{d_2} & E_2 . \end{array}$$

The map h induces bundle maps, $h^t\colon E_1^t \to E_2^t$ and $h^0 \circ f_1' = f_2' \circ h^1$. Hence we get a bundle map, $H\colon M(f_1') \to M(f_2')$, and a commutative square

$$\begin{array}{ccc} M(f_1') & \xrightarrow{m_1} & E_1 \\ H \downarrow & & \downarrow h \\ M(f_2') & \xrightarrow{m_2} & E_2 . \end{array}$$

Now suppose that E is a trivial $M(q)$ bundle. Pick any ordered partition of unity and use the global trivialization to induce local ones. In this case, $d\colon B \times M(q) \times [0,1] \to B \times M(q)$ is defined by $d(x,[z,t],s) - \begin{cases} (x,[z,t-s]) & \text{if } s \leq t \leq 1 \\ (x,[z,0]) & \text{if } 0 \leq t \leq s \end{cases}$, and $f'\colon B \times F_1 \to B \times F_2$ is $\mathrm{id}_B \times q$. The map m is just the canonical map from $M(\mathrm{id}_B \times q)$ to $B \times M(q)$ and, since we have compactly generated topologies, is a homeomorphism.

The last paragraph shows several things. First of all, it shows that we can use the local trivialization of E to give a local trivialization of E^1, E^0 and f' so that this triple forms an object in $\mathscr{B}_1$. It also shows that the map $m\colon M(f') \to E$ is a local homeomorphism: indeed, m restricted over $\mathscr{U}_i$ is a homeomorphism. Since m is also fibre preserving, m is 1-1 and onto, so m is a homeomorphism.

One can also check that d (and hence m) depends only on the ordered partition of unity and the local trivializations.

Using this last remark, note that there is a relative version of our results. Suppose that $A \subset B$ is an NDR pair. Suppose that we are given an ordered set $\mathscr{J}$ indexing a partition of unity subordinate to a local trivialization of $E_A = p^{-1}(A)$. We can extend the partition of unity as follows. There is a set $\mathscr{J}$ with $\mathscr{I} \subset \mathscr{J}$ and $\mathscr{J}$ can be ordered so that the induced order on $\mathscr{I}$ is the given one. Furthermore, we can find local trivializations of E, extending the given ones over A, and a subordinate partition of unity, $\kappa_j'\colon B \to [0,1]$, $j \in \mathscr{J}$ so that $\kappa_j'(x) = \begin{cases} \kappa_j(x) & \text{if } x \in A \text{ and } j \in \mathscr{I} \\ 0 & \text{if } x \in A \text{ but } j \notin \mathscr{I} \end{cases}$.

We can perform the μ-construction, getting a bundle over B with structure group $Top^{level}(q)$ and fibre $M(q)$ and then we can restrict this bundle to A. We can also perform the μ-construction directly using our original data over A. It can be checked that the resulting two bundles over A are isomorphic. Put another way, we have the following relative result.

Proposition 2.1. *Let $A \subset B$ be an NDR subspace. Let* $\begin{array}{ccc} E_1 & \xrightarrow{f} & E_2 \\ p_1 \searrow & & \swarrow p_2 \\ & A & \end{array}$ *be a triple over A and let $M \to B$ be a $Top^{level}(q)$-bundle over B with fibre $M(q)$. Finally, suppose that there is a $Top^{level}(q)$ bundle isomorphism $m\colon \mu(E_1, E_2, f) \to M|_A$. Then, there exists a triple,* $\begin{array}{ccc} E_1' & \xrightarrow{f'} & E_2' \\ p_1 \searrow & & \swarrow p_2 \\ & B & \end{array}$ *; a bundle isomorphism $m'\colon \mu(E_1', E_2', f') \to M$; and an isomorphism of triples, (h_1, h_2, H) between (E_1, E_2, f) and $(E_1', E_2', f')|_A$ so that $m' \circ \mu(h_1, h_2, H) = m$.*

Remarks.

(1) There are two topological group homomorphisms, restriction to top and bottom, from $Top^{level}(q)$ to $Top(F_1)$ and $Top(F_2)$. Let $G_i \subset Top(F_i)$ be two subgroups, $i = 1, 2$, and suppose that the two subgroups $L_i = \{h \in Top^{level}(q) \mid h \mid F_i \in G_i\}$, $i = 1, 2$, are equal. Further suppose that the image of $L = L_1 = L_2$ in $Top(F_i)$ is all of G_i. Then the above definitions and constructions go through for pairs of G_i bundles with mapping cylinder bundles having structure group L. For example, this happens if $F_2 = \mathbb{R}^n$ and the structure group is Top_n and $Top^{level}(q)$ elements are required to preserve the origin at the 0-level. It also happens if F_1 and F_2 are oriented manifolds and all homeomorphisms preserve the orientations.

(2) If a bundle $\bar{p}\colon \bar{E} \to B$ is *fixed* with fibre F_2 and group $Top(F_2)$, and $\bar{\mathscr{B}}_1$ is the subset of $\mathscr{B}_1$ with second bundle $\bar{p}\colon \bar{E} \to B$, and $\bar{\mathscr{B}}_2$ is the subset of $\mathscr{B}_2$ whose natural F_2-subbundle is $\bar{p}\colon \bar{E} \to B$, then μ restricts to a map, $\bar{\mu}\colon \bar{\mathscr{B}}_1 \to \bar{\mathscr{B}}_2$, which is a bijection on homotopy groups.

§3. Proof of Theorem 0.1

The key to the proof is a special case of Remarks (2) and some results from [4]. We begin by recalling that $\mu\colon \mathscr{B}_1 \to \mathscr{B}_2$ is a homotopy equivalence of simplicial sets and that classical bundle theory identifies $\mathscr{B}_2$ with the simplicial set of maps $Maps(B, BTop^{level}(q))$. Indeed, in [4, Prop. 2.1], we construct a differential between $\mathscr{B}_2$ and $Maps(B, BTop^{level}(q))$ which we prove is a homotopy equivalence. Also recall that $\mathscr{B}_1$ is the simplicial set of pairs of bundles over B with a map between the two bundles.

Now restrict: let $\bar{\mathscr{B}}_1$ denote the sub-simplicial set of $\mathscr{B}_1$ where the range bundle is the tangent bundle of the manifold B^i; and let $\bar{\mathscr{B}}_2$ denote the sub-simplicial set of $\mathscr{B}_2$ where the "bottom" bundle associated to the mapping cylinder bundle is the tangent bundle. It can be checked that μ restricts to a homotopy equivalence $\mu\colon \bar{\mathscr{B}}_1 \to \bar{\mathscr{B}}_2$. A check of the definitions shows that $\bar{\mathscr{B}}_1$ is identical with what we called $\overline{\mathrm{MAF}}(B)_q$ in [4, 2.2]. Theorem 0.1 follows: it can be checked that our differential defines a homotopy equivalence between $\bar{\mathscr{B}}_2$ and $Lifts\begin{pmatrix} & BTop^{level}(q) \\ \nearrow & \downarrow \\ B \xrightarrow{\tau} & BTop_i \end{pmatrix}$; and it follows from [4, Th. 2.2.2] that we have a homotopy equivalence between $\mathrm{MAF}(B)_q$ and $\overline{\mathrm{MAF}}(B)_q$. These two results prove the theorem. □

One of the main results of [4] (and the starting point for this paper) was the construction of a simplicial bundle, $\mathrm{MAF}(\mathscr{U}_i) \to BTop_i$, and the construction of a homotopy equivalence, $\eta\colon \mathrm{MAF}(B)_q \to Lifts\begin{pmatrix} & \mathrm{MAF}(\mathscr{U}_i)_q \\ \nearrow & \downarrow \\ B \xrightarrow{\tau} & BTop_i \end{pmatrix}$. Furthermore, we have just finished describing $\mathrm{MAF}(B)_q$ as a space of lifts of the simplicial bundle $BTop^{level}(q) \to BTop_i$. A natural question is the relation between these two bundles. We construct a fibre map between them so that the induced map on the simplicial sets of lifts is compatible with our identifications.

To begin, recall the simplicial bundle $\mathrm{MAF}(\mathscr{U}_i) \to BTop_i$ from [4, Examples, p. 20]. The identity map, id: $\mathrm{MAF}(\mathscr{U}_i) \to \mathrm{MAF}(\mathscr{U}_i)$, is a lift of the bundle projection. If we take any simplex in $\mathrm{MAF}(\mathscr{U}_i)_q$ and apply the assembly map, [4, bottom p. 22], we get a bundle pair over our simplex. If we apply the μ map to this pair, we get a simplex of $BTop^{level}(q)$. It can be checked that this map preserves boundaries and so we get a simplicial map, $\hat{\mu}\colon \mathrm{MAF}(\mathscr{U}_i)_q \to BTop^{level}(q)$ such that

$$\begin{matrix} \mathrm{MAF}(\mathscr{U}_i)_q & \xrightarrow{\hat{\mu}} & BTop^{level}(q) \\ & \searrow \quad \swarrow & \\ & BTop_i & \end{matrix}$$

commutes. The result we want is

Proposition 3.1. *If B^i is an i-dimensional manifold, the composite*

$$MAF(B)_q \xrightarrow{\eta} Lifts\left(\begin{array}{ccc} & & \mathrm{MAF}(\mathscr{U}_i)_q \\ & \nearrow & \downarrow \\ B & \xrightarrow{\tau} & BTop_i \end{array}\right) \xrightarrow{\hat{\mu}^*} Lifts\left(\begin{array}{ccc} & & BTop^{level}(q) \\ & \nearrow & \downarrow \\ B & \xrightarrow{\tau} & BTop_i \end{array}\right)$$

agrees with the map defined in this paper, where $\hat{\mu}^*$ is the map induced by the fibre map $\hat{\mu}$ on the simplicial set of lifts.

The proof is a check of the definitions. □

§ 4. Proof of Theorem 0.2

Before discussing Theorem 0.2, we need a lemma, which in turn needs some material from [4] which we recall. The simplicial set Hur(B) is actually a piece of a bi-simplicial set, $\mathrm{Hur}(B)_{*,*}$, where it sits as the subset $\mathrm{Hur}(B)_{*,0}$. There is also a simplicial set, $\mathrm{Hur}(B)_{*,1}$, in which a k-simplex consists of two fibrations over $B \times \Delta^k$, say E_1 and E_2, and a fibre homotopy equivalence, $f\colon E_1 \to E_2$. There are two boundary maps, $\mathrm{Hur}(B)_{*,1} \to \mathrm{Hur}(B)_{*,0}$, denoted ∂_0 and ∂_1 with $\partial_i(f) = E_i$. Both ∂_0 and ∂_1 are simplicial maps. There is a simplicial map $s_0\colon \mathrm{Hur}(B)_{*,0} \to \mathrm{Hur}(B)_{*,1}$ with $s_0(E) = \mathrm{id}_E$. Note s_0 is also simplicial and that $\partial_i \circ s_0$ is the identity. In [4, p. 32 last paragraph of proof of Lemma 7.8] we showed that s_0 is a homotopy equivalence, so ∂_0 and ∂_1 are homotopic.

Lemma 4.1 *Let X be a simplicial set, and let f_0 and f_1 be two simplicial maps $X \to \mathrm{Hur}(B)$. Suppose that for each simplex $x \in X$, we have a fibre homotopy equivalence*

$$h_x\colon f_0(x) \to f_1(x)$$

so that the resulting map $X \to \mathrm{Hur}(B)_{*,1}$ is simplicial. Then f_0 and f_1 are homotopic.

The proof if clear from the above discussion.

Next we recall what we want to prove and introduce some notation. Let $\overline{\overline{\mathrm{MAF}}}(B)$ denote the simplicial set we called $\bar{\mathscr{B}}_2$ above, where we change notation to be able to display the manifold B. There is a map $\overline{\overline{\mathrm{MAF}}}(B) \to \mathrm{Bun}(B)$ which takes the mapping cylinder bundle and remembers the sub-bundle obtained by restricting to the top of the mapping cylinder (Bun(B) is the simplicial set of bundles over B). Of course we can forget the bundle and just remember that it is a fibration, so we get a map $\mathrm{MAF}(B)_q \to \mathrm{Hur}(B)$. We have another map of $\mathrm{MAF}(B)_q$ into Hur(B), namely take the associated fibrewise Hurewicz fibration of each simplex.

Theorem 0.2 states that these two maps are homotopic. The proof occupies the remainder of this section.

The differential from Theorem 0.1 gives a commutative square

$$(*)\qquad \begin{array}{ccc} M & \to & d(M) \\ \downarrow & & \downarrow \\ B\times\Delta^k & \to & B\times\Delta^k \end{array}$$

where $d(M) \to B \times \Delta^k$ is the sub-bundle of the mapping cylinder bundle given by restricting to the top. Let $\mathscr{E}(M)$ (resp. $\mathscr{E}(d(M))$) be the result of applying the fibrewise Hurewicz fibration construction, [4, 9.1] to the MAF $M \to B \times \Delta^k$ (resp. $d(M) \to B \times \Delta^k$). Then we have a commutative square of fibre maps

$$(**)\qquad \begin{array}{ccc} M & \to & d(M) \\ \downarrow & & \downarrow \\ \mathscr{E}(M) & \to & \mathscr{E}(d(M)) \end{array} .$$

All these maps are simplicial, and the top right object is the image of our first map into Hur(B). The bottom left object is the image of our second map into Hur(B). The right-hand vertical map is well-known to be a fibre homotopy equivalence. It follows from Lemma 4.1 that we are done if we can prove that the bottom map is also a fibre-homotopy equivalence. Note the square is a homotopy fibre square since both vertical maps are homotopy equivalences. Hence, to prove that the bottom map is a homotopy equivalence, it will suffice to prove that the top map, $i_M\colon M \to d(M)$, is a homotopy equivalence. The remainder of this section is devoted to proving this.

We begin by noting that $i_M\colon M \to d(M)$, has a left inverse. To see this, recall that

$$(***)\qquad \begin{array}{ccc} d(M) & \subset & B\times M\times\Delta^k \\ \downarrow & & \downarrow \\ \tau_B\times\Delta^k & \subset & B\times B\times\Delta^k \end{array}$$

is a pull-back, so we get a map $d(M) \to M$ by projection, which is easily seen to have the property that the composite $M \to d(M) \to M$ is the identity.

Next, consider the square obtained by restricting to the fibre germ:

$$\begin{array}{ccc} F\times\Delta^k & \to & M \\ \downarrow & & \downarrow \\ \mathbb{R}^i\times\Delta^k & \to & B\times\Delta^k \end{array} .$$

By [4, 12.15], the homotopy fibre of $F \times \Delta^k \to \mathbb{R}^i \times \Delta^k$ maps to the homotopy fibre of $M \to B \times \Delta^k$ by a homotopy equivalence. Hence, it suffices to prove the desired result for the special case in which B is $\mathbb{R}^i$. In this special case, diagram (***) has $B = \mathbb{R}^i$ and M some open manifold homotopy equivalent to F. The bottom horizontal map in the square is a homotopy equivalence and the square is a pull-back, so the top horizontal map is also a homotopy equivalence. By the definition of the map $d(M) \to M$, it is a homotopy equivalence. Since this map is a left inverse to the map $i_M: M \to d(M)$, i_M is also a homotopy equivalence, and we are done.

§ 5. The proof of Theorem 0.3

The proof is an easy chase through the definitions to identify the mapping cylinder bundle if one starts with a fibre bundle.

§ 6. The topological tubular neighborhood theorem

We will use the following notation for the remainder of the paper. Let M^n and N^{n+k} denote manifolds as in Theorem 0.4 and let $\pi: \mathbb{R}^n \times S^{k-1} \to \mathbb{R}^n$ be projection. Recall that Top_n^{n+k} is the simplicial group of homeomorphisms of $\mathbb{R}^{n+k}$ which leave the standard copy of $\mathbb{R}^n$ invariant and leave the origin fixed, whereas $Top_{n,n+k}$ is the simplicial subgroup of homeomorphisms which leave $\mathbb{R}^n$ fixed.

Since $M(\pi)$ is naturally homeomorphic to $\mathbb{R}^n \times B^k$, there is a restriction map

$$\varrho: Top^c(\pi) \to Top_{n,n+k}$$

which incorporates an identification of $\mathbb{R}^n \times \mathring{B}^k$ with $\mathbb{R}^{n+k}$. The first step in the proof of the theorem is to use a result of Anderson and Hsiang to identify the homotopy fibre of ϱ.

Lemma 6.1. *The homotopy fibre of ϱ is the simplicial group $C^b(\mathbb{R}^n \times S^{k-1})$ of bounded concordances on $\mathbb{R}^n \times S^{k-1}$.*

Proof. Anderson and Hsiang [2] have shown that the homotopy fibre of the stabilization map

$$\sigma: Top^b(\mathbb{R}^n \times S^{k-1}) \to Top^b(\mathbb{R}^{n+1} \times S^{k-1})$$

is $C^b(\mathbb{R}^n \times S^{k-1})$. Here Top^b denotes the simplicial group of bounded homeomorphisms. We will show that ϱ and σ are equivalent (up to homotopy) and hence have the same homotopy fibre.

Recall that a radial compactification argument due to Anderson and Hsiang [1] shows that $Top^b(\mathbb{R}^{n+1} \times S^{k-1})$ is homotopy equivalent to $Top(S^n * S^{k-1} \text{ rel } S^n)$ (see also [7]). Also, one-point compactification yields an identification of $Top_{n,n+k}$ with $Top(S^n * S^{k-1} \text{ rel } S^n)$.

Note that $Top^c(\pi) = Top^{level}(\mathbb{R}^n \times B^k \text{ rel } \mathbb{R}^n \times 0)$ where "level" refers to the invariance of the $\mathbb{R}^n \times tS^{k-1}$ for $0 \le t \le 1$. A homotopy equivalence $Top^b(\mathbb{R}^n \times S^{k-1}) \to Top^c(\pi)$ is explicitly constructed in [5, Remark 3.7]. On the other hand, restriction gives a homotopy equivalence $Top^{level}(S^{n-1} * B^k \text{ rel } S^{n-1} * 0) \to Top(S^{n-1} * S^{k-1} \text{ rel } S^{n-1})$ because the fibre is contractible by an Alexander trick. By Anderson-Hsiang again, there is a homotopy equivalence $Top^b(\mathbb{R}^n \times S^{k-1}) \to Top(S^{n-1} * S^{k-1} \operatorname{rel} S^{n-1})$. Putting these equivalences together gives $Top^c(\pi) \simeq Top^{level}(S^{n-1} * B^k \operatorname{rel} S^{n-1} * 0)$.

Using these facts we can give more useful descriptions of both ϱ and σ. After writing $B^k = S^{k-1} * 0$, there is a forgetful map (which forgets "level")

$$\lambda\colon Top^{level}(S^{n-1} * B^k \operatorname{rel} S^{n-1} * 0) \to Top(S^{n-1} * S^{k-1} * 0 \operatorname{rel} S^{n-1} * 0).$$

Let

$$r\colon Top(S^{n-1} * S^{k-1} * 0 \operatorname{rel} S^{n-1} * 0) \to Top(S^{n-1} * S^{k-1} \operatorname{rel} S^{n-1})$$

be the restriction map. Let $S^0 = \{0, 1\}$ and let

$$\Sigma\colon Top(S^{n-1} * S^{k-1} \operatorname{rel} S^{n-1}) \to Top(S^0 * S^{n-1} * S^{k-1} \operatorname{rel} S^0 * S^{n-1})$$

be the suspension map. Then σ is essentially (i.e., up to homotopy) $\Sigma \circ r \circ \lambda$.

Now factor Σ as the composition of two coning maps, $\Sigma = c_1 \circ c_0$, where

$$c_0\colon Top(S^{n-1} * S^{k-1} \operatorname{rel} S^{n-1}) \to Top(S^{n-1} * S^{k-1} * 0 \operatorname{rel} S^{n-1} * 0)$$

and

$$c_1\colon Top(S^{n-1} * S^{k-1} * 0 \operatorname{rel} S^{n-1} * 0) \to Top(S^0 * S^{n-1} * S^{k-1} \operatorname{rel} S^0 * S^{n-1}).$$

Then ϱ is essentially $c_1 \circ \lambda$.

Thus, we want to show $c_1 \lambda \simeq \Sigma r \lambda$. But the Alexander trick gives a homotopy $\mathrm{Id} \simeq c_0 r$ and hence, $c_1 \lambda \simeq c_1 c_0 r \lambda = \Sigma r \lambda$. □

We will also need the following result.

Lemma 6.2 *There exists a homotopy fibre square*

$$\begin{array}{ccc} BTop_n^{n+k} & \longrightarrow & BTop^{level}(\mathbb{R}^{n+1} \times B^k) \\ \downarrow & & \downarrow \\ BTop_n & \longrightarrow & BTop_{n+1} \end{array}$$

where the vertical maps are induced by restrictions and the lower horizontal map is induced by stabilization.

Proof. First note that the map $Top_n \to Top_{n+1}$ is homotopic to the composition

$$Top_n \xrightarrow{a} Top(S^n) \xrightarrow{c} Top_{n+1}.$$

where a is induced by one-point compactification and c is induced by coning.

Form the pull-back

$$\begin{array}{ccc} \mathscr{E} & \longrightarrow & Top^{level}(\mathbb{R}^{n+1} \times B^k) \\ \downarrow & & \downarrow \\ Top(S^n) & \xrightarrow{c} & Top_{n+1} \end{array}$$

Thus, a vertex of $\mathscr{E}$ consists of a vertex H of $Top^{level}(\mathbb{R}^{n+1} \times B^k)$ such that $H|\mathbb{R}^{n+1} \times 0 = \mathrm{cone}(h)$ for some (uniquely determined) homeomorphism $h: S^n \to S^n$.

Let $Top(S^n * S^{k-1} \,\mathrm{inv}\, S^n)$ be the simplicial group of homeomorphisms of $S^n * S^{k-1}$ which leave S^n invariant. Then there is a fibration

$$\begin{array}{ccc} Top(S^n * S^{k-1} \,\mathrm{rel}\, S^n) & \longrightarrow & Top(S^n * S^{k-1} \,\mathrm{inv}\, S^n) \\ & & \downarrow \\ & & Top(S^n) \end{array}$$

which we claim is fibre homotopically equivalent to $\mathscr{E} \to Top(S^n)$. To see this consider the fibration

$$\begin{array}{ccc} Top^b(\mathbb{R}^{n+1} \times S^{k-1}) & \longrightarrow & \mathscr{D} \\ & & \downarrow \\ & & Top\,(S^n) \end{array}$$

where $\mathscr{D}$ is the simplicial group whose typical vertex consists of a pair (H, h) where $H: \mathbb{R}^{n+1} \times S^{k-1} \to \mathbb{R}^{n+1} \times S^{k-1}$ and $h: S^n \to S^n$ are homeomorphisms such that

$$\begin{array}{ccc} \mathbb{R}^{n+1} \times S^{k-1} & \xrightarrow{H} & \mathbb{R}^{n+1} \times S^{k-1} \\ \downarrow & & \downarrow \\ \mathbb{R}^{n+1} & \xrightarrow{\mathrm{cone}(h)} & \mathbb{R}^{n+1} \end{array}$$

boundedly commutes (where the vertical maps are projections). Note that h is unique once it exists.

We claim that there are maps of fibrations

$$\begin{array}{ccccc} Top(S^n * S^{k-1} \,\mathrm{rel}\, S^n) & \leftarrow & Top^b(\mathbb{R}^{n+1} \times S^{k-1}) & \rightarrow & Top^{level}(\mathbb{R}^{n+1} \times B^k \,\mathrm{rel}\, \mathbb{R}^{n+1} \times 0) \\ \downarrow & & \downarrow & & \downarrow \\ Top(S^n * S^{k-1} \,\mathrm{inv}\, S^n) & \leftarrow & \mathscr{D} & \rightarrow & \mathscr{E} \\ & \searrow & \downarrow & \swarrow & \\ & & Top(S^n) & & \end{array}$$

The horizontal map coming out of $Top^b(\mathbb{R}^{n+1} \times S^{k-1})$ is explicitly constructed in [5, Remark 3.7]. The horizontal maps coming out of $\mathscr{D}$ are just extensions of the natural maps coming out of $Top^b(\mathbb{R}^{n+1} \times S^{k-1})$. As in the proof of Lemma 6.1 the maps coming out of $Top^b(\mathbb{R}^{n+1} \times S^{k-1})$ are homotopy equivalences by [1] and [5].

Therefore, we have a homotopy fibre square

$$\begin{array}{ccc} Top(S^n * S^{k-1} \operatorname{inv} S^n) & \longrightarrow & Top^{level}(\mathbb{R}^{n+1} \times B^k) \\ \downarrow & & \downarrow \\ Top(S^n) & \xrightarrow{c} & Top_{n+1} \end{array}$$

To finish the proof, note that there is a fibre square

$$\begin{array}{ccc} Top_n^{n+k} & \longrightarrow & Top(S^n * S^{k-1} \operatorname{inv} Sn) \\ \downarrow & & \downarrow \\ Top_n & \xrightarrow{a} & Top(S^n) \end{array}$$

where the upper horizontal map is also induced by one-point compactification. □

Proof (of Theorem 0.4 part i)). Let $\mathcal{N}_k(M)$ be the set of k-neighborhoods of M as studied by Rourke and Sanderson [10]. Thus, an element of $\mathcal{N}_k(M)$ is represented by a pair (i, Q) where Q is an $(n+k)$-manifold and $i: M \to Q$ is a locally flat embedding. Two such (i, Q) and (i', Q') are equivalent if there is an embedding $h: Q \to Q'$ defined in a neighborhood of $i(M)$ such that $hi = i'$.

The mapping cylinder construction defines a map $\mu: \pi_0 \mathrm{MAF}(M)_\pi \to \mathcal{N}_k(M)$ as follows. If $f: P \to M$ is a manifold approximate fibration with fibre germ π, then $\mu([f]) = [(i, M(f))]$ where $i: M \to M(f)$ is the natural inclusion. Our goal is to show that μ is a bijection.

The first step is to observe that there is a commuting diagram

$$\begin{array}{ccc} \pi_0 \mathrm{MAF}(M)_\pi & \xrightarrow{\mu} & \mathcal{N}_k(M) \\ d\downarrow & & \downarrow e \\ \pi_0 \overline{\overline{\mathrm{MAF}}}(M) & \xrightarrow{v} & \hat{\mathcal{N}}_k(M) \end{array}$$

with the following explanations. As in the proof of Theorem 0.1, $\overline{\overline{\mathrm{MAF}}}(M)$ is the simplicial set of certain bundle pairs over M with fibre $M(\pi)$ and d is the differential which is a homotopy equivalence. Here $\hat{\mathcal{N}}_k(M)$ is the set of equivalence classes of bundle pairs over M with fibre $\mathbb{R}^{n+k}$ which have τ_M as the subbundle. Rourke and Sanderson [10, Prop. 3.1] describe a map $e: \mathcal{N}_k(M) \to \hat{\mathcal{N}}_k(M)$ and use immersion theory to prove that it is a bijection. Recall the definition of e. If $i: M \to Q$ is a locally flat codimension k embedding, then $e([i, Q]) = [\tau_Q | M, \tau_M]$.

In order to define $v: \pi_0 \overline{\overline{\mathrm{MAF}}}(M) \to \hat{\mathcal{N}}_k(M)$, let $p: E \to M$ represent a class in $\pi_0 \overline{\overline{\mathrm{MAF}}}(M)$. Since the fibre of this bundle is homeomorphic to $\mathbb{R}^n \times B^k$, and the structure group is $Top^{level}(\mathbb{R}^n \times B^k)$, and the natural subbundle is τ_M, it follows that

$p\colon E \to M$ contains a bundle pair of the type occurring in $\hat{\mathcal{N}}_k(M)$. Then $v([p])$ is that bundle pair. It is not hard to see that the diagram commutes.

The second step is to observe that there is a commuting square

$$\begin{array}{ccc} \pi_0 \overline{\overline{\mathrm{MAF}}}(M) & \xrightarrow{v} & \hat{\mathcal{N}}_k(M) \\ \downarrow & & \downarrow \\ \pi_0 \mathit{Lifts}\left(\begin{array}{ccc} & \nearrow & BTop^{level}(\pi) \\ & & \downarrow \\ M & \xrightarrow{\tau} & BTop_n \end{array}\right) & \xrightarrow{r} & \pi_0 \mathit{Lifts}\left(\begin{array}{ccc} & \nearrow & BTop_n^{n+k} \\ & & \downarrow \\ M & \xrightarrow{\tau} & BTop_n \end{array}\right) \end{array}$$

where the vertical maps are the classification maps, and hence, equivalences. The map r is induced by restriction

$$Top^{level}(\pi) \to Top_n^{n+k}.$$

The fibres of the bundles

$$\begin{array}{ccc} BTop^{level}(\pi) & \text{and} & BTop_n^{n+k} \\ \downarrow & & \downarrow \\ BTop_n & & BTop_n \end{array}$$

are $BTop^c(\pi)$ and $BTop_{n,n+k}$, respectively. The map τ on total spaces induces the map $\varrho\colon Top^c(\pi) \to Top_{n,n+k}$ on loops of the fibres. This map was studied in Lemma 6.1. Since $\pi_i C^b(\mathbb{R}^n \times S^{k-1}) = 0$ for $0 \le i \le n-1$ by [1], it follows that the homotopy groups of the fibre of $BTop^{level}(\pi) \to BTop_n^{n+k}$ vanish through dimension n. It follows that r is a bijection.

That μ is a bijection follows by putting the two diagrams together. □

Proof (of Theorem 0.4 part ii)). First note that $\mathcal{E}mb_M(M(f), N)$ is homotopy equivalent to the simplicial set, $Imm_M(M(f), N)$, of immersions of $M(f)$ into $N \operatorname{rel} M$. This is because any such immersion is an embedding when restricted to a sufficiently small neighborhood of M in $M(f)$ and any such neighborhood contains a natural copy of $M(f)$. Now immersion theory [6] implies that $Imm_M(M(f), N)$ is homotopy equivalent to the simplicial set $Aut(\tau_N | M \operatorname{rel} \tau_M)$ of bundle automorphisms of $\tau_N | M \operatorname{rel} \tau_M$. Then standard bundle theory implies that $Aut(\tau_N | M \operatorname{rel} \tau_M)$ is homotopy equivalent to the loop space of

$$\mathit{Lifts}\left(\begin{array}{ccc} & \nearrow & BTop_n^{n+k} \\ & & \downarrow \\ M & \xrightarrow{\tau} & BTop_n \end{array}\right)$$

where the loops are based at the lift which classifies $\tau_N | M$.

Analogously, we have previously shown [4] that $Top^c(f \times \mathrm{id}_{\mathbb{R}})$ is homotopy equivalent to the loop space of

$$\mathit{Lifts}\left(\begin{array}{ccc} & & BTop^{level}(\mathbb{R}^{n+1} \times B^k) \\ & \nearrow & \downarrow \\ M & \xrightarrow{\tau} & BTop_{n+1} \end{array}\right)$$

where the loops are based at the lift which classifies $f \times \mathrm{id}_{\mathbb{R}}$.

To complete the proof, observe that Lemma 6.2 implies that these two spaces of lifts are homotopy equivalent. □

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