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DELOOPING CONTROLLED PSEUDO-ISOTOPIES OF HILBERT CUBE MANIFOLDS

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This paper is concerned with the controlled simple homotopy theory and controlled pseudo-isotopy theory of a Hurewicz fibration $p: E \to B$ from a compact Hilbert cube manifold E to a compact polyhedron B. The main result is that the controlled pseudo-isotopy space is homotopy equivalent to the loop space of the controlled Whitehead space.

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Introduction

Let $p: E \to B$ be a Hurewicz fibration from a compact Hilbert cube manifold E to a compact polyhedron B. In this paper we construct the controlled Whitehead space $\mathrm{Wh}(p: E \to B)$ and the space of controlled pseudo-isotopies $\mathscr{P}(p: E \to B)$ which turns out to be homotopy equivalent to the loop space of $\mathrm{Wh}(p: E \to B)$. The homotopy groups of $\mathrm{Wh}(p: E \to B)$ are the domain of controlled simple homotopy theory. In particular, $\pi_0\mathrm{Wh}(p: E \to B)$ is the controlled Whitehead group. This is explained as follows.

Recall that the problem of simple homotopy theory is to decide when a homotopy equivalence $f: K \to L$ between compact polyhedra is a simple homotopy equivalence. The Whitehead torsion $\tau(f)$ of f lies in the Whitehead group $\mathrm{Wh}(\mathscr{X}\pi_1 L)$ and is equal to zero if and only if f is a simple homotopy equivalence (see [9]). Results of Chapman [1] and West [19] relate this to Hilbert cube manifolds as follows: $\tau(f) = 0$ if and only if $f \times \mathrm{id}: K \times Q \to L \times Q$ is homotopic to a homeomorphism (Q denotes the Hilbert cube). It follows that the problem of simple homotopy theory is to decide when a homotopy equivalence between compact Hilbert cube manifolds is homotopic to a homeomorphism.

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If we are given a fibration $p: E \to B$ as above, then the problem of controlled simple homotopy theory is to decide when a controlled homotopy equivalence $f: M \to E$ (where M is a compact Hilbert cube manifold) is homotopic with control to a homeomorphism. By 'control' we mean arbitrarily small ε control in B. This is explained further in the body of the paper. We can now state our first main result.

Theorem 1. *If M is a compact Hilbert cube manifold and $f: M \to E$ is a controlled homotopy equivalence, then there is a well-defined element $\tau(f)$ in $\pi_0 \mathrm{Wh}(p: E \to B)$ which vanishes if and only if f is $p^{-1}(\varepsilon)$-homotopic to a homeomorphism for every $\varepsilon > 0$.*

A *pseudo-isotopy* on E is a homeomorphism $h: E \times [0, 1] \to E \times [0, 1]$ such that $h | E \times \{0\}$ is the identity. Pseudo-isotopies which move points an arbitrarily small amount when measured in B are formed into the controlled pseudo-isotopy space $\mathscr{P}(p: E \to B)$ in Section 4. Here is our second main result.

Theorem 2. *$\mathscr{P}(p: E \to B)$ is homotopy equivalent to the loop space $\Omega \mathrm{Wh}(p: E \to B)$.*

For the special case that p is a bundle projection over euclidean space with compact Hilbert cube manifold fiber, Theorems 1 and 2 are essentially contained in [14].

Our approach to controlled simple homotopy theory differs from the theories of Chapman [5] and Quinn [17] in two aspects. First, we use Hilbert cube manifolds in our definition rather than polyhedra. Second, we require that our controlling map p be a Hurewicz fibration. The setting in [5] and [17] is much more general Chapman [6] and Quinn [17] have also studied controlled pseudo-isotopies, again in a more general setting.

In the uncontrolled setting, Hatcher [12] defined a Whitehead space $\mathrm{Wh}(K)$ for a compact polyhedron K and showed (using a result of Chapman [3]) that $\Omega \mathrm{Wh}(K)$ is homotopy equivalent to the space of pseudo-isotopies on $K \times Q$.

In a future paper we will develop a method due to W.C. Hsiang in order to study the homotopy groups of $\mathrm{Wh}(p: E \to B)$ and $\mathscr{P}(p: E \to B)$ where p is the projection map of a locally trivial fiber bundle and B is a closed manifold. It will be shown that $\mathrm{Wh}(p: E \to B)$ is homotopy equivalent to the space of cross-sections of a certain bundle whose fiber was the object of study in [14].

2. Approximate fibrations and other preliminaries

The Hilbert cube will be denoted by Q. Hilbert cube manifolds, or Q-manifolds, are locally compact, separable metric spaces which are locally homeomorphic to Q. For Q-manifold basics, including the notion of Z-sets, see [2].

In this section and throughout the rest of the paper we fix a Hurewicz fibration $p: E \to B$ (i.e., a map with the homotopy lifting property for all spaces) where E

is a compact Q-manifold and B is either a compact polyhedron or a compact topological manifold with a handle decomposition.

Let X and Y be compact metric ANRs, let $f: X \to Y$ be a map, and let $\varepsilon > 0$. We say that f is an *ε-fibration* provided that given any Z and maps $G: Z \times [0,1] \to Y$ and $g: Z \to X$ for which $G(z,0) = fg(z)$, then there exists a map $\tilde{G}: Z \times [0,1] \to X$ such that $\tilde{G}(z,0) = g(z)$ and $f\tilde{G}$ is ε-close to G. If f is an ε-fibration for every $\varepsilon > 0$, then f is an *approximate fibration*. Approximate fibrations were introduced in [10].

We will use Δ to denote the standard n-simplex for a given n. A *fiber preserving* (f.p.) map is a map which preserves the obvious fiber over Δ (or over some other n-cell). Specifically, if $\rho: X \to \Delta$, $\sigma: Y \to \Delta$, and $f: X \to Y$ are maps, then f is f.p. if $\sigma f = \rho$. Usually the maps ρ and σ will be understood to be some natural projections and will not be explicitly mentioned. For fiber preserving Q-manifold results, including sliced Z-set unknotting, see [7], [8] or [11].

We will need to use results about approximate fibrations from [15] and [16]. Here is the main result from [16] (see [4] for the $n = 0$ case).

Theorem 2.1. *Let Δ and $\varepsilon > 0$ be given. There exists a $\delta > 0$ such that if M is a compact Q-manifold and $f: M \times \Delta \to B \times \Delta$ is an f.p. map so that $f_t: M \to B$ is a δ-fibration for each t in Δ and an approximate fibration for each t in $\partial\Delta$, then there exists an f.p. map $\tilde{f}: M \times \Delta \to B \times \Delta$ such that $\tilde{f}$ is ε-close to f, $\tilde{f} \mid M \times \partial\Delta = f \mid M \times \partial\Delta$, and $\tilde{f}_t: M \to B$ is an approximate fibration for each t in Δ.*

We will also need a relative version of this result which we now state.

Addendum 2.2. *If E is a Z-set in M and $f \mid E \times \Delta = p \times \mathrm{id}$, then we can further require that $\tilde{f} \mid E \times \Delta = p \times \mathrm{id}$.*

Proof. There is a relative version of Theorem 2.1 in [16]. As stated, it would imply our Addendum if $E = B \times F$ where F is a compact Q-manifold and $p =$ projection. However, the proof in [16] works equally well if $p: E \to B$ is the projection map of a locally trivial fiber bundle with compact Q-manifold fiber. To pass to the more general case where $p: E \to B$ is only assumed to be fibration, use Q-manifold theory to find an f.p. map $h: M \times Q \times \Delta \to M \times \Delta$ close to projection such that $h \mid M \times Q \times \partial\Delta$ is projection and $h \mid: M \times Q \times \mathrm{int}\,\Delta \to M \times \mathrm{int}\,\Delta$ is a homeomorphism. By sliced Z-set unknotting we may further assume that $h \mid: E \times \{0\} \times \Delta \to M \times \Delta$ is the inclusion.

It follows from [7] that $p(\mathrm{proj}): E \times Q \to B$ is a locally trivial fiber bundle projection with Q-manifold fiber. Let $\hat{f}: M \times Q \times \Delta \to B \times \Delta$ be an f.p. map such that $\hat{f}$ is close to $f(\mathrm{proj})$, $\hat{f}_t$ is an approximate fibration for each t in Δ, $\hat{f} \mid M \times Q \times \partial\Delta = f(\mathrm{proj}) \mid$, and $\hat{f} \mid E \times Q \times \Delta = (p \times \mathrm{id})(\mathrm{proj})$. Then set $\tilde{f} = \hat{f}h^{-1}$. □

A special case of the following theorem is the main result of [13]. That result, together with a significant improvement, appears in [15] for finite dimensional manifolds. The proof given in [15] also works for Q-manifolds and we now state the result without further proof.

Theorem 2.3. *Let Δ and $\varepsilon > 0$ be given. If M is a compact Q-manifold and $f: M \times \Delta \times [0, 1] \to B \times \Delta \times [0, 1]$ is an f.p. (over $\Delta \times [0, 1]$) map so that $f_t: M \to B$ is an approximate fibration for each t in $\Delta \times [0, 1]$, then there exists an f.p. homeomorphism $H: M \times \Delta \times [0, 1] \to M \times \Delta \times [0, 1]$ such that $H | M \times \Delta \times \{0\} = \mathrm{id}$ and fH is ε-close to $f_0 \times \mathrm{id}$ where $f_0 = f | M \times \Delta \times \{0\}$.*

There are also two addenda which we will need. The proof of the first follows from the proof of Theorem 2.3. The second follows from Theorem 2.3 and sliced Z-set unknotting.

Addendum 2.4. *There exists a $\delta > 0$, $\delta = \delta(\varepsilon, n)$, such that if we are additionally given an f.p. homeomorphism $G: M \times \partial\Delta \times [0, 1] \to M \times \partial\Delta \times [0, 1]$ such that $G | M \times \partial\Delta \times \{0\} = \mathrm{id}$ and fG is δ-close to $f_0 | \times \mathrm{id}$, then we can further require H to satisfy $H | M \times \partial\Delta \times [0, 1] = G$.*

Addendum 2.5. *If E is a Z-set in M and $f | E \times \Delta \times [0, 1] = p \times \mathrm{id}$, then we can further require that $H | E \times \Delta \times [0, 1] = \mathrm{id}$.*

3. Controlled simple homotopy theory

We continue to let $p: E \to B$ denote a fixed Hurewicz fibration where E is a compact Q-manifold and B is either a compact polyhedron on a compact topological manifold with a handle decomposition. In this section we define the controlled Whitehead space $\mathrm{Wh}(p: E \to B)$ as a semi-simplicial complex, study its homotopy relation from a geometrical point of view (Theorem 3.2), and define the torsions of certain controlled homotopy equivalences to E to be in the homotopy groups of $\mathrm{Wh}(p: E \to B)$. As a result (Corollary 3.4) we obtain the proof of Theorem 1. In addition, we define two more 'spaces' homotopy equivalent to $\mathrm{Wh}(p: E \to B)$ which will be useful in proving Theorem 2.

We first need some definitions. Let $\rho: X \to \Delta$ be a map (where X is compact ANR), let $f: X \to E \times \Delta$ be an f.p. map and let $\varepsilon > 0$. We say f is an f.p. $(p \times \mathrm{id})^{-1}(\varepsilon)$-*equivalence* provided there exist an f.p. map $g: E \times \Delta \to X$ and f.p. homotopies $F: X \times [0, 1] \to X$ and $G: E \times \Delta \times [0, 1] \to E \times \Delta$ such that $F_0 = \mathrm{id}$, $F_1 = gf$, $G_0 = \mathrm{id}$, $G_1 = fg$, and the diameters of $(p \times \mathrm{id}) f\{F(\{x\} \times [0, 1])\}$ and $(p \times \mathrm{id})\{G(\{y\} \times [0, 1])\}$ are less than ε for each x in X and y in $E \times \Delta$. If in addition X contains $E \times \Delta$, g is the inclusion, $f | E \times \Delta = \mathrm{id}$, and F is rel $E \times \Delta$, then we say f is an f.p. $(p \times \mathrm{id})^{-1}(\varepsilon)$-sdr. In the case that $n = 0$ (so that we can drop the f.p. requirement) and f is a $p^{-1}(\varepsilon)$-equivalence for every $\varepsilon > 0$, then we say that f is a *controlled homotopy equivalence.*

Approximate fibrations enter into this work because of the following fact: a homotopy equivalence $f: X \to E$ is a controlled homotopy equivalence if and only if $pf:X \to B$ is an approximate fibration [4, Proposition 2.3]. See [14, Lemma 2.1] for an f.p. sdr variation.

We can now define the controlled Whitehead space. Fix a Z-embedding of $E\times[0,1]$ in Q and identify E with $E\times\{0\}$. Then $\mathrm{Wh}(p:E\to B)$ is the semi-simplicial complex having typical n-simplices of the form $f:M\to E\times\Delta$ where

(1) We are given a projection $\rho:M\to\Delta$ of a locally trivial fiber bundle with compact Q-manifold fiber,

(2) M is embedded in $Q\times\Delta$ as a sliced Z-set,

(3) M contains $E\times\Delta$ as a sliced Z-set,

(4) f is an f.p. $(p\times\mathrm{id})^{-1}(\varepsilon)$-sdr for every $\varepsilon>0$.

As mentioned above, condition (4) can be reworded to state that f is an f.p. sdr and $(p\times\mathrm{id})f:M\to B\times\Delta$ is an approximate fibration.

It is clear that $\mathrm{Wh}(p:E\to B)$ satifies the Kan condition so that we may talk about its homotopy groups. These groups are based at the projection $E\times[0,1]\times\Delta\to E\times\Delta$ which we will always denote by π.

We now prove a technical lemma which will be useful in proving the results of this section. Note that condition (4) is the only part which does not already follow from the definitions.

Lemma 3.1. *Let $f:M\to E\times\Delta$ represent a class $[f]$ in $\pi_n\mathrm{Wh}(p:E\to B)$. For every $\varepsilon>0$ there exists an f.p. homotopy $F_t:M\to M$, $0\leq t\leq 1$, such that*

(1) $F_0=\mathrm{id}$ *and* $F_1=f$,

(2) $F_t|E\times\Delta=\mathrm{id}$ *for each* t,

(3) *the homotopy* $fF_t:M\to E\times\Delta$, $0\leq t\leq 1$, *is a* $(p\times\mathrm{id})^{-1}(\varepsilon)$-*homotopy*,

(4) $fF_t|:E\times[0,1]\times\partial\Delta\to E\times\partial\Delta$ *is* π *(i.e., projection) for each* t.

Proof. Recall that $M\subset Q\times\Delta$, $M\cap(Q\times\partial(\Delta)=E\times[0,1]\times\partial\Delta$, and $f=\pi$ on $E\times[0,1]\times\partial\Delta$. This is because f represents a class in $\pi_n\mathrm{Wh}(p:E\to B)$. Let $\rho:M\to\Delta$ be the given fiber bundle projection. We can trivialize ρ by finding an f.p. homeomorphism $k:E\times[0,1]\times\Delta\to M$ such that $k=\mathrm{id}$ on $E\times[0,1]\times\partial_n\Delta$ where $\partial_n\Delta$ is an $(n-1)$-dimensional face of Δ. By using sliced Z-set unkotting we can further assume that $k=\mathrm{id}$ on $E\times\{0\}\times\Delta$.

Use Theorem 2.3 (with Addendum 2.5) to find an f.p. homeomorphism $H:E\times[0,1]\times\Delta\to E\times[0,1]\times\Delta$ such that $H=\mathrm{id}$ on $(E\times\{0\}\times\Delta)\cup(E\times[0,1]\times\partial_n\Delta)$ and $(p\times\mathrm{id})fkH$ is δ-close to $(p\times\mathrm{id})\pi$. Here $\delta>0$ is chosen small with respect to ε.

Let $\partial\Delta\times[0,1]\subset\Delta$ be a boundary collar so that $\partial\Delta$ is identified with $\partial\Delta\times\{0\}$. Define $j:E\times[0,1]\times\partial\Delta\times[0,1]\to\rho^{-1}(\partial\Delta\times[0,1])$ by setting $j=kH\circ[(H^{-1}k^{-1}|\rho^{-1}(\partial\Delta)\times\mathrm{id}_{[0,1]}]$. Note that $j|E\times[0,1]\times\partial\Delta=\mathrm{id}$ and that $(p\times\mathrm{id})fj$ is (2δ)-close to $(p\times\mathrm{id})\pi$. Think of j as giving a 'controlled' collar on $\rho^{-1}(\partial\Delta)$ in M.

Define an f.p. homotopy $G_t:M\to M$, $0\leq t\leq 1$, as follows. First, set $G_t=\mathrm{id}$ on $M\backslash\rho^{-1}(\partial\Delta\times[0,1])$. Then for (w,x,y,z) in $E\times[0,1]\times\partial\Delta\times[0,1]$, set $G_tj(w,x,y,z)=j(w,(1-t)x+txz,y,z)$. Note that $G_t=\mathrm{id}$ on $E\times\Delta$, $(p\times\mathrm{id})fG_t$, $0\leq t\leq 1$, is a (2δ)-homotopy, and $fG_t|E\times[0,1]\times\partial\Delta=\pi$.

Since f is an n-simplex of $\mathrm{Wh}(p: E \to B)$, there exists an f.p. homotopy $K_t: M \to M$, $0 \le t \le 1$, such that

(1) $K_0 = \mathrm{id}$ and $K_1 = f$,

(2) $K_t | E \times \Delta = \mathrm{id}$ for each t,

(3) fK_t, $0 \le t \le 1$, is a $(p \times \mathrm{id})^{-1}(\delta)$-homotopy.

Finally, define $F_t: M \to M$, $0 \le t \le 1$, by

$$F_t = \begin{cases} G_{3t} & \text{if } 0 \le t \le \frac{1}{3}, \\ K_{3t-1} \circ G_1 & \text{if } \frac{1}{3} \le t \le \frac{2}{3}, \\ fG_{3-3t} & \text{if } \frac{2}{3} \le t \le 1. \end{cases} \qquad \square$$

We are now ready for the first main result of this section. It gives a geometric interpretation of the homotopy relation in $\mathrm{Wh}(p: E \to B)$.

Theorem 3.2. *Let $f: M \to E \times \Delta$ and $g: N \to E \times \Delta$ represent the classes $[f]$ and $[g]$ in $\pi_n \mathrm{Wh}(p: E \to B)$, respectively. There exists an $\varepsilon_0 > 0$, $\varepsilon_0 = \varepsilon_0(B, n)$, so that the following are equivalent:*

(i) $[f] = [g]$;

(ii) *for every $\varepsilon > 0$ there exists an f.p. homeomorphism $h: M \to N$ such that $h = \mathrm{id}$ on $(E \times \Delta) \cup (E \times [0, 1] \times \partial\Delta)$ and gh is f.p. $(p \times \mathrm{id})^{-1}(\varepsilon)$-homotopic to $f \,\mathrm{rel}(E \times \Delta) \cup (E \times [0, 1] \times \partial\Delta)$;*

(iii) *there exists an f.p. homeomorphism $h: M \to N$ such that $h = \mathrm{id}$ on $E \times \Delta$, $gh = f$ on $E \times [0, 1] \times \partial\Delta$, and $(p \times \mathrm{id})gh$ is ε_0-close to $(p \times \mathrm{id})f$.*

Proof. We first show that condition (i) implies condition (ii). Since $[f] = [g]$, there is an $(n+1)$-simplex in $\mathrm{Wh}(p: E \to B)$ which we can represent by $\tilde{f}: \tilde{M} \to E \times \Delta \times [0, 1]$ where we have a fibering $\rho: \tilde{M} \to \Delta \times [0, 1]$ and an embedding of $\tilde{M}$ in $Q \times \Delta \times [0, 1]$ with the following properties (among others):

(1) $\tilde{f} = f$ on $\rho^{-1}(\Delta \times \{0\}) = M$;

(2) $\tilde{f} = g$ on $\rho^{-1}(\Delta \times \{1\}) = N$;

(3) $\tilde{f} = \pi$ on $\rho^{-1}(\partial\Delta \times [0, 1]) = E \times [0, 1] \times \partial\Delta \times [0, 1]$.

Trivialize ρ by finding an f.p. homeomorphism $k: E \times [0, 1] \times \Delta \times [0, 1] \to \tilde{M}$ such that $k | E \times [0, 1] \times \partial\Delta \times [0, 1] = (k | E \times [0, 1] \times \partial\Delta \times \{0\}) \times \mathrm{id}$ and use sliced Z-set unknotting to get $k = \mathrm{id}$ on $E \times \Delta \times [0, 1]$. (Here and throughout the proof we are assuming $n \ge 1$. Then $n = 0$ case is similar, but easier.)

Now use Theorem 2.3 to find an f.p. homeomorphism $H: E \times [0, 1] \times \Delta \times [0, 1] \to E \times [0, 1] \times \Delta \times [0, 1]$ such that $H = \mathrm{id}$ on $(E \times \Delta \times [0, 1]) \cup (E \times [0, 1] \times \Delta \times \{0\}) \cup (E \times [0, 1] \times \partial\Delta \times [0, 1])$ and $(p \times \mathrm{id})\tilde{f}kH$ is ε-close to $(p \times \mathrm{id})((\tilde{f}k | E \times [0, 1] \times \Delta \times \{0\}) \times \mathrm{id})$.

Let h be the homeomorphism given by the composition

$$M \xrightarrow{(k|)^{-1}} E \times [0, 1] \times \Delta \times \{0\} = E \times [0, 1] \times \Delta \times \{1\}$$

$$\xrightarrow{H|} E \times [0, 1] \times \Delta \times \{1\} \xrightarrow{k|} N.$$

The asserted homotopy from f to gh at time t, $0 \leq t \leq 1$, is given by the composition

$$M \xrightarrow{(k|)^{-1}} E \times [0,1] \times \Delta \times \{0\} = E \times [0,1] \times \Delta \times \{t\}$$

$$\xrightarrow{H|} E \times [0,1] \times \Delta \times \{t\} \xrightarrow{k|} \rho^{-1}(\Delta \times \{t\}) \xrightarrow{\bar{f}|} E \times \Delta \times \{t\} = E \times \Delta.$$

Since condition (ii) obviously implies condition (iii), we are left with showing that condition (iii) implies condition (i).

To this end use Lemma 3.1 to find an f.p. homotopy $F_t : M \to M$, $0 \leq t \leq 1$, such that

(1) $F_0 = \mathrm{id}$ and $F_1 = f$,

(2) $F_t | E \times \Delta = \mathrm{id}$,

(3) fF_t, $0 \leq t \leq 1$, is a $(p \times \mathrm{id})^{-1}(\delta)$-homotopy where $\delta > 0$ is small,

(4) $fF_t | E \times [0,1] \times \partial\Delta = \pi$.

Consider the homotopy $ghF_t : gh \simeq f$, $0 \leq t \leq 1$. It is rel $E \times \Delta$ and it is constantly equal to π to $E \times [0,1] \times \partial\Delta$. Moreover, it is a $(p \times \mathrm{id})^{-1}(2\varepsilon_0 + \delta)$-homotopy since $(p \times \mathrm{id})ghF_t$ is ε_0-close to $(p \times \mathrm{id})fF_t$.

Now let $j : M \times [0,1] \to Q \times \Delta \times [0,1]$ be a sliced Z-embedding such that $j = \mathrm{id}$ on $(M \times \{0\}) \cup (E \times \Delta \times [0,1]) \cup (E \times [0,1] \times \partial\Delta \times [0,1])$ and $j | M \times \{1\} = h$.

Define $\tilde{F} : M \times [0,1] \to M \times [0,1]$ by setting $\tilde{F}(x,t) = (F_{1-t}(x), t)$. Define a homotopy $\tilde{F}_s : \mathrm{id} \simeq \tilde{F}$, $0 \leq s \leq 1$, by setting $\tilde{F}_s(x,t) = (F_{(1-t)s}(x), t)$. Finally, define $G : j(M \times [0,1)] \to E \times \Delta \times [0,1]$ by setting $G = (gh \times \mathrm{id})\tilde{F}j^{-1}$. Note that

(1) $G = f$ on $j(M \times \{0\}) = M$,

(2) $G = g$ on $j(M \times \{1\}) = N$,

(3) $G | E \times \Delta \times [0,1] = \mathrm{id}$,

(4) $G | E \times [0,1] \times \partial\Delta \times [0,1] = \pi$.

We now want to use Theorem 2.1 to deform G to an $(n+1)$-simplex of $\mathrm{Wh}(p : E \to B)$ showing $[f] = [g]$. For this, we need to show that G is an f.p. sdr with small control in $B \times \Delta \times [0,1]$. To this end, let $g_s : \mathrm{id}_N \simeq g$, $0 \leq s \leq 1$, be an f.p. homotopy with small control in $B \times \Delta$ coming from the fact that g is an n-simplex in $\mathrm{Wh}(p : E \to B)$. Then define $G_s : j(M \times [0,1)] \to j(M \times [0,1])$, $0 \leq s \leq 1$, by $G_s = j(h^{-1}g_s h \times \mathrm{id})\tilde{F}_s j^{-1}$. Then $G_s : \mathrm{id} \simeq G$, $0 \leq s \leq 1$, and the homotopy GG_s has small (depending on δ and ε_0) diameter in $B \times \Delta \times [0,1]$. □

The following result gives a way to represent certain homotopy equivalences to $E \times \Delta$ by elements of $\pi_n \mathrm{Wh}(p : E \to B)$.

Theorem 3.3. *There exists an $\varepsilon_0 > 0$, $\varepsilon_0 = \varepsilon_0(B, n)$, such that whenever we are given the following data*:

(i) *a locally trivial fiber bundle projection $\rho : M \to \Delta$ with compact Q-manifold fibers*;

(ii) *an f.p. $(p \times \mathrm{id})^{-1}(\varepsilon_0)$-equivalence $f : M \to E \times \Delta$ for which there is an f.p. homeomorphism $k : E \times [0,1] \times \partial\Delta \to \rho^{-1}(\partial\Delta)$ such that $fk = \pi$*;

then there exists a well-defined torsion $\tau(f)$ *in* $\pi_n \mathrm{Wh}(p: E \to B)$ *with the following two properties*:

(i) *if* $[f]$ *is in* $\pi_n \mathrm{Wh}(p: E \to B)$, *then* $\tau(f) = [f]$;

(ii) *there exists an* $\varepsilon_1 > 0$ *such that for every* $\varepsilon > 0$, $\varepsilon < \varepsilon_1$, *there exists a* $\delta > 0$ *so that if* $f: M \to E \times \Delta$ *and* $g: N \to E \times \Delta$ *are f.p.* $(p \times \mathrm{id})^{-1}(\delta)$*-equivalence for which* $\tau(f)$ *and* $\tau(g)$ *are defined, then* $\tau(f) = \tau(g)$ *if and only if there exists an f.p. homeomorphism* $h: M \to N$ *such that* $h | E \times \partial\Delta = \mathrm{id}$, $gh | \rho^{-1}(\partial\Delta) = f|$, *and gh is f.p.* $(p \times \mathrm{id})^{-1}(\varepsilon)$*-homotopic* rel $\rho^{-1}(\partial\Delta)$ *to f.*

Proof. We will just show how to construct $\tau(f)$ and then appeal to the proofs of Proposition 3.5 and 3.6 in [14] for the proofs of the properties.

Given the data, let $g: E \times \Delta \to M$ be an f.p. $(p \times \mathrm{id})^{-1}(\varepsilon_0)$-homotopy inverse for f. Approximate g by a sliced Z-embedding $\tilde{g}: E \times \Delta \to M$ such that $\tilde{g} | E \times \partial\Delta = \mathrm{id}$. By standard techniques [18, p. 31] we can regard $\tilde{g}$ as an inclusion map and find an f.p. homotopy of f to $\tilde{f}$ via a $(p \times \mathrm{id})^{-1}(\varepsilon')$-homotopy (where $\varepsilon' > 0$ is small if ε_0 is) so that $\tilde{f}$ is an f.p. $(p \times \mathrm{id})^{-1}(\varepsilon')$-sdr. Now embed M into $Q \times \Delta$ with a sliced Z-embedding j such that $j | \rho^{-1}(\partial\Delta) = k^{-1}$ and $j | E \times \Delta = \mathrm{id}$. Then define $\tau(f) = [\tilde{f}j^{-1}]$ in $\pi_n \mathrm{Wh}(p: E \to B)$.

We just mention briefly what [14, Proposition 3.5] says about the well-definedness of $\tau(f)$. The definition does depend on $k | E \times \{0\} \times \Delta$, but is otherwise independent of k. The definition of $\tau(f)$ is also independent of the choice of g. □.

In the case $n = 0$, Theorem 3.3 is extremely simplified because we are not concerned with what happens over $\partial\Delta$. Theorem 1 in the introduction follows immediately from the next result.

Corollary 3.4. *If M is a compact Q-manifold and* $f: M \to E$ *is a controlled homotopy equivalence, then* $\tau(f)$ *vanishes in* $\pi_0 \mathrm{Wh}(p: E \to B)$ *if and only if f is* $p^{-1}(\varepsilon)$*-homotopic to a homeomorphism for every* $\varepsilon > 0$.

Proof. Suppose $\tau(f)$ vanishes. This means $\tau(f) = [\pi]$ in $\pi_0 \mathrm{Wh}(p: E \to B)$ where $\pi: E \times [0, 1] \to E$ is projection. Let $k: E \times [0, 1] \to E$ be a homeomorphism close to π. By Theorem 3.3 there is a homeomorphism $h: M \to E \times [0, 1]$ such that f is $p^{-1}(\delta)$-homotopic to πh where $\delta > 0$ is small. Then kh is the desired homeomorphism approximating f.

On the other hand, if f is $p^{-1}(\varepsilon)$-homotopic to a homeomorphism g for small $\varepsilon > 0$ and k is as before, then $k^{-1}g: M \to E \times [0, 1]$ is a homenomorphism which shows, via Theorem 3.3, that $\tau(f) = [\pi]$. □

We are now ready to discuss two semi-simplicial complexes which turn out to be homotopy equivalent to $\mathrm{Wh}(p: E \to B)$ but are easier to work with in some circumstances.

The first of these complexes is denoted by $\widehat{\mathrm{Wh}}(p: E \to B)$. It has a typical n-simplex of the form $f: M \to E \times \Delta \times [0, \infty)$ where:

(1) we are a given a projection $\rho: M \to \Delta \times [0, \infty)$ of a locally trivial fiber bundle with compact Q-manifold fiber;
(2) M is embedded in $Q \times \Delta \times [0, \infty)$ as a sliced (over $\Delta \times [0, \infty)$) Z-set;
(3) M contains $E \times \Delta \times [0; \infty)$ as a sliced z-set;
(4) there is a decreasing sequence $\varepsilon_0, \varepsilon_1, \varepsilon_2, \ldots$ of positive real numbers converging to 0 (called a *controlling sequence* for f) and an f.p. homotopy $\alpha: M \times [0, 1] \to M$ such that $\alpha_t: \mathrm{id} \simeq f$, $0 \leq t \leq 1$, rel $E \times \Delta \times [0, \infty)$ and the diameter of $(p \times \mathrm{id})f\{\alpha(\{x\} \times [0, 1])\}$ is less than ε_i whenever the $[0, \infty)$-coordinate of $\rho(x)$ is greater than or equal to i.

There is a natural way to define a map $i: \mathrm{Wh}(p: E \to B) \to \widehat{\mathrm{Wh}}(p: E \to B)$. Let $f: M \to E \times \Delta$ be an n-simplex in $\mathrm{Wh}(p: E \to B)$. It is not hard to see that $f \times \mathrm{id}: M \times [0, \infty) \to E \times \Delta \times [0, \infty)$ is an n-simplex in $\widehat{\mathrm{Wh}}(p: E \to B)$, and we define $i(f) = f \times \mathrm{id}$.

The next lemma shows that $i_*: \pi_n \mathrm{Wh}(p: E \to B) \to \pi_n \widehat{\mathrm{Wh}}(p: E \to B)$ is an epimorphism, and it will also be useful in the proof of Theorem 2. For notation, let $f: M \to E \times \Delta \times [0, \infty)$ be an n-simplex in $\widehat{\mathrm{Wh}}(p: E \to B)$ as above and for each t in $[0, \infty)$ let $f_t = f|: \rho^{-1}(\Delta \times \{t\}) \to E \times \Delta \times \{t\}$.

Lemma 3.5. *There exists an integer N such that if $t \geq N$, then $\tau(f_t)$ is defined and $i_* \tau(f_t) = [f]$ in $\pi_n \widehat{\mathrm{Wh}}(p: E \to B)$.*

Proof. Choose N large so that ε_N (in the controlling sequence or f) is small with respect to ε_0, ε_1 in Theorem 3.3. Using Theorem 2.1, we can find an f.p. homotopy of $f|\rho^{-1}(\Delta \times [t, \infty))$ rel$(E \times \Delta \times [t, \infty)) \cup \rho^{-1}(\partial\Delta \times [t, \infty))$ to a map g such that g is an f.p. $(p \times \mathrm{id})^{-1}(\varepsilon)$-sdr for every $\varepsilon > 0$ and the homotopy from f to g becomes arbitrarily small near infinity (when measured in $B \times \Delta \times [0, \infty)$).

Note that $[g_t] = \tau(f_t)$ in $\pi_n \mathrm{Wh}(p: E \to B)$, so $i_* \tau(f_t) = [g_t \times \mathrm{id}_{[0,\infty)}]$. Once g is extended over $[0, t)$, it is clear that $[g] = [g_t \times \mathrm{id}]$ and $[g] = [f]$. The lemma follows. □

Proposition 3.6. *The map $i: \widehat{\mathrm{Wh}}(p: E \to B) \to \mathrm{Wh}(p: E \to B)$ is a homotopy equivalence.*

Proof. In light of Lemma 3.5, it only remains to show that $i_*: \pi_n \mathrm{Wh}(p: E \to B) \to \pi_n \widehat{\mathrm{Wh}}(p: E \to B)$ is a monomorphism. Let $[f]$ and $[g]$ be classes in $\pi_n \mathrm{Wh}(p: E \to B)$ such that $i_*([f]) = i_*([g])$. By going sufficiently close to infinity in an $(n+1)$-simplex of $\widehat{\mathrm{Wh}}(p: E \to B)$ which connects $i_*([f])$ and $i_*([g])$, we can find a map which can be deformed using Theorem 2.1 to an $(n+1)$-simplex of $\mathrm{Wh}(p: E \to B)$ connecting $[f]$ and $[g]$. □

The next result shows that we can essentially disregard the sliced Z-embedding of M in $Q \times \Delta \times [0, \infty)$ in the definition of $\widehat{\mathrm{Wh}}(p: E \to B)$. A similar result holds for $\mathrm{Wh}(p: E \to B)$ by Theorem 3.2.

Lemma 3.7. *Let* $f: M \to E\times\Delta\times[0,\infty)$ *and* $g: N\to E\times\Delta\times[0,\infty)$ *be two n-simplices in* $\widehat{\mathrm{Wh}}(p:E\to B)$ *for which there is an f.p. homeomorphism* $h: M\to N$ *such that* $h\,|\,E\times\Delta\times[0,\infty)=\mathrm{id}$ *and* $gh=f$. *If f and g represent classes in* $\pi_n\widehat{\mathrm{Wh}}(p:E\to B)$, *then they represent the same class.*

Proof. Recall that M and N are embedded in $Q\times\Delta\times[0,\infty)$, and since f and g represent homotopy classes, we have $f=\pi=g$ on

$$M\cap(Q\times\partial\Delta\times[0,\infty))=E\times[0,1]\times\partial\Delta\times[0,\infty)=N\cap(Q\times\partial\Delta\times[0,\infty)).$$

It follows that $h|: E\times[0,1]\times\partial\Delta\times[0,\infty)\to E\times[0,1]\times\partial\Delta\times[0,\infty)$ is an f.p. homeomorphism which affects only the $[0,1]$-coordinate of any point and $h\,|\,E\times\{0\}\times\partial\Delta\times[0,\infty)=\mathrm{id}$. It follows from an Alexander trick that there is an f.p. isotopy $h_s:\mathrm{id}\simeq h|$, $0\leq s\leq 1$, rel $E\times\{0\}\times\partial\Delta\times[0,\infty)$.

Use relative sliced Z-set unknotting to find an f.p. isotopy $H_s: Q\times\Delta\times[0,\infty)\to Q\times\Delta\times[0,\infty)$, $0\leq s\leq 1$, such that $H_0=\mathrm{id}$, $H_1\,|\,M=h$, $H_s\,|\,E\times\Delta\times[0,\infty)=\mathrm{id}$, and $H_s\,|\,E\times[0,1]\times\partial\Delta\times[0,\infty)=h_s$.

Define a sliced Z-embedding $j: M\times[0,1]\to Q\times\Delta\times[0,\infty)\times[0,1]$ by setting $j(x,s)=(H_s(x),s)$. Let $\tilde{M}=j(M\times[0,1])$ and define $\tilde{f}:\tilde{M}\to E\times\Delta\times[0,\infty)\times[0,1]$ by setting $\tilde{f}=(f\times\mathrm{id})j^{-1}$. This gives an $(n+1)$-simplex in $\widehat{\mathrm{Wh}}(p:E\to B)$ which shows that f and g represent the same class in $\pi_n\widehat{\mathrm{Wh}}(p:E\to B)$. □

We now define a semi-simplicial complex $\overline{\mathrm{Wh}}(p:E\to B)$ which incorporates the freedom granted by Proposition 3.7 into its definition. A typical n-simplex of $\overline{\mathrm{Wh}}(p:E\to B)$ is an equivalence class $[f]$ represented by an n-simplex $f: M\to E\times\Delta\times[0,\infty)$ of $\widehat{\mathrm{Wh}}(p:E\to B)$. Another such n-simplex $g: N\to E\times\Delta\times[0,\infty)$ of $\widehat{\mathrm{Wh}}(p:E\to B)$ is *equivalent* to f if there is an f.p. homeomorphism $h: M\to N$ such that $h\,|\,E\times\Delta\times[0,\infty)=\mathrm{id}$ and $gh=f$. There is a natural 'quotient' map $q:\widehat{\mathrm{Wh}}(p:E\to B)\to\overline{\mathrm{Wh}}(p:E\to B)$ defined by $q(f)=[f]$. Lemma 3.7 implies that the induced map g_* on homotopy groups is a monomorphism. Since g_* is obviously an epimorphism, we have the following result.

Proposition 3.8. *The map* $q:\widehat{\mathrm{Wh}}(p:E\to B)\to\overline{\mathrm{Wh}}(p:E\to B)$ *is a homotopy equivalence.*

The careful reader will have observed that we have only shown that the induced maps, i_* and q_*, of Propositions 3.6 and 3.8 induce bijections between path components and isomorphisms between higher homotopy groups when these homotopy groups are based at the projection π. However, we need to show that these maps are isomorphisms for arbitrary basepoints. There are at least two ways around this. First, one can rework our proofs with minor modifications to allow for arbitrary basepoints. Second, one can observe that in the sequel we only need that the induced maps, Ωi and Ωq, between loop spaces are homotopy equivalences. Since these loops are based at π, this follows from the proofs we have already given.

4. Controlled pseudo-isotopy theory

In this section we will define the space $\mathscr{P}(p:E\to B)$ of controlled pseudo-isotopies on $p:E\to B$, the fibration of the previous sections. We also show how to represent certain pseudo-isotopies on $E\times\Delta$ by elements of $\pi_n\mathscr{P}(p:E\to B)$. This is analogous to the torsion construction in the previous section. Finally, we will discuss the group structure on $\pi_0\mathscr{P}(p:E\to B)$.

Let $h:E\times[0,1]\times\Delta\to E\times[0,1]\times\Delta$ be an f.p. (over Δ) homeomorphism. For each t in Δ, let $h_t=h|:E\times[0,1]\times\{t\}\to E\times[0,1]\times\{t\}$ and continue to let $\pi:E\times[0,1]\to E$ denote projection (we also use π to denote projection $E\times[0,1]\times\Delta\to E\times\Delta$). If $\varepsilon>0$, we say h is an *n-parameter ε-pseudo-isotopy* on $p:E\to B$ provided $h|E\times\{0\}\times\Delta=\text{id}$ and $d(p\pi h_t, p\pi)<\varepsilon$ for each t in Δ.

An *n-parameter controlled pseudo-isotopy* on $p:E\to B$ is a homeomorphism $h:E\times[0,1]\times\Delta\times[0,\infty)\to E\times[0,1]\times\Delta\times[0,\infty)$ such that

(1) h is f.p. over $\Delta\times[0,\infty)$;

(2) $h|E\times\{0\}\times\Delta\times[0,\infty)=\text{id}$;

(3) there is a decreasing sequence $\varepsilon_0,\varepsilon_1,\varepsilon_2,\ldots$ of positive real numbers converging to 0 (called a *controlling sequence* for h) such that for each integer $i\geq 0$, $h|:E\times[0,1]\times\Delta\times\{u\}\to E\times[0,1]\times\Delta\times\{u\}$ is an n-parameter ε_i-pseudo-isotopy on $p:E\to B$ whenever $u\geq i$.

Let $\mathscr{P}(p:E\to B)$ denote the semi-simplicial complex of controlled pseudo-isotopies on $p:E\to B$; that is, the n-simplices are n-parameter controlled pseudo-isotopies on $p:E\to B$. This complex satisfies the Kan condition and the homotopy groups willl be based at the identity $E\times[0,1]\to E\times[0,1]$.

The next proposition shows how to turn an ε-pseudo-isotopy into a controlled pseudo-isotopy if ε is small enough.

Proposition 4.1. *There exists an $\varepsilon>0$, $\varepsilon=\varepsilon(B,n)$, so that if $h:E\times[0,1]\times\Delta\to E\times[0,1]\times\Delta$ is an n-parameter ε-pseudo-isotopy, then there is a homeomorphism $H:E\times[0,1]\times\Delta\times[0,\infty)\to E\times[0,1]\times\Delta\times[0,\infty)$ such that:*

(i) *H is f.p. over $\Delta\times[0,\infty)$;*

(ii) *$H=\text{id}$ on $(E\times\{0\}\times\Delta\times[0,\infty))\cup(E\times[0,1]\times\Delta\times\{0\})$;*

(iii) *$(h\times\text{id})H$ is an n-parameter controlled pseudo-isotopy.*

Proof. Given $h:E\times[0,1]\times\Delta\to E\times[0,1]\times\Delta$ as above, note that $(p\times\text{id})\pi h$ is ε-close to $(p\times\text{id})\pi$ and $(p\times\text{id})\pi h|E\times\{0\}\times\Delta=p\times\text{id}=(p\times\text{id})\pi|$. It follows from [16] that there is an f.p. (over $\Delta\times[0,1]$) approximate fibration g: $E\times[0,1]\times\Delta\times[0,1]\to B\times\Delta\times[0,1]$ such that:

(1) $g|E\times[0,1]\times\Delta\times\{0\}=(p\times\text{id})\pi h$;

(2) $g|E\times[0,1]\times\Delta\times\{1\}=(p\times\text{id})\pi$;

(3) $g|E\times\{0\}\times\Delta\times[0,1]=p\times\text{id}$;

(4) $g|E\times[0,1]\times\Delta\times\{u\}$ is ε'-close to $(p\times\text{id})\pi h$ for each u in $[0,\infty)$ where the size of $\varepsilon'>0$ depends on the size of ε.

It follows Theorem 2.3 that there exists a homeomorphism $\tilde{H}: E\times[0,1]\times\Delta\times[0,1]\to E\times[0,1]\times\Delta\times[0,1]$ such that:

(1) $\tilde{H}$ is f.p. over $\Delta\times[0,1]$;

(2) $\tilde{H}=\mathrm{id}$ on $(E\times\{0\}\times\Delta\times[0,1])\cup(E\times[0,1]\times\Delta\times\{0\})$;

(3) $((p\times\mathrm{id})\pi h\times\mathrm{id})\tilde{H}$ is μ-close to g where $\mu>0$ is as small as we need.

The homeomorphism H is defined to be $\tilde{H}$ on $E\times[0,1]\times\Delta\times[0,1]$ and we continue this construction to define H on $E\times[0,1]\times\Delta\times[1,2]$, then $E\times[0,1]\times\Delta\times[2,3]$, etc. □

There are two addenda that we will need. Their proofs follow from the proof of Proposition 4.1.

Addendum 4.2. *If $h|E\times[0,1]\times\partial\Delta=\mathrm{id}$, then H can be constructed so that $H|E\times[0,1]\times\partial\Delta\times[0,\infty)=\mathrm{id}$.*

Addendum 4.3. *For every $\varepsilon>0$ there exists a $\delta>0$, $\delta=\delta(B,n,\varepsilon)$, such that if h is an n-parameter δ-pseudo-isotopy, then H can be constructed so that $(h\times\mathrm{id})H$ has a controlling sequence beginning with ε.*

The following proposition shows that the construction in Proposition 4.1 is well-defined.

Proposition 4.4. *There exists an $\varepsilon>0$, $\varepsilon=\varepsilon(B,n)$, so that if $h: E\times[0,1]\times\Delta\to E\times[0,1]\times\Delta$ is an n-parameter ε-pseudo-isotopy with $h|E\times[0,1]\times\partial\Delta=\mathrm{id}$ and $H: E\times[0,1]\times\Delta\times[0,\infty)\to E\times[0,1]\times\Delta\times[0,\infty)$ is a homeomorphism such that:*

(i) *H is f.p. over $\Delta\times[0,\infty)$;*

(ii) *$H=\mathrm{id}$ on $(E\times\{0\}\times\Delta\times[0,\infty))\cup(E\times[0,1]\times\partial\Delta\times[0,\infty))\cup(E\times[0,1]\times\Delta\times\{0\})$;*

(iii) *$(h\times\mathrm{id})H$ is an n-parameter controlled pseudo-isotopy with a controlling sequence beginning with ε;*

then the class of $(h\times\mathrm{id})H$ in $\pi_n\mathcal{P}(p:E\to B)$ is independent of H.

Proof. If H' is another homeomorphism satisfying the three conditions above, then we can use the techniques in the proof of Proposition 4.1 to find an $(n+1)$-simplex in $\mathcal{P}(p:E\to B)$ connecting $(h\times\mathrm{id})H$ and $(h\times\mathrm{id})H'$ which will show $[(h\times\mathrm{id})H]=[(h\times\mathrm{id})H']$ in $\pi_n\mathcal{P}(p:E\to B)$. □

We will now define the torsion of certain pseudo-isotopies. For a fixed n, let $\varepsilon=\varepsilon(B,n)$ be given by Proposition 4.4 and let $\delta=\delta(B,n,\varepsilon)$ be given by Addendum 4.3 and assume $\delta\leq\varepsilon$. Let $h: E\times[0,1]\times\Delta\to E\times[0,1]\times\Delta$ be an n-parameter δ-pseudo-isotopy such that $h|E\times[0,1]\times\partial\Delta=\mathrm{id}$. By the propositions above we can find a homeomorphism $H: E\times[0,1]\times\Delta\times[0,\infty)\to E\times[0,1]\times\Delta\times[0,\infty)$ such that:

(1) H is f.p. over $\Delta\times[0,\infty)$;

(2) $H = \mathrm{id}$ on $(E \times \{0\} \times \Delta \times [0, \infty)) \cup (E \times [0, 1] \times \partial\Delta \times [0, \infty)) \cup (E \times [0, \infty) \times \Delta \times \{0\})$;

(3) $(h \times \mathrm{id})H$ is an n-parameter controlled pseudo-isotopy with a controlling sequence beginning with ε.

Define the *torsion* $\tau(h)$ of h to be the class of $(h \times \mathrm{id})H$ in $\pi_n\mathscr{P}(p: E \to B)$. Note that $\tau(h)$ is well-defined (i.e., independent of H) by Proposition 4.4.

The next result of this section shows that $\tau(h)$ is invariant under a small (measured in B) isotopy of h. The proof uses the techniques of the propositions above and is left to the reader.

Proposition 4.5. *There exists a $\gamma > 0$, $\gamma = \gamma(B, n)$, such that if $h, h': E \times [0, 1] \times \Delta \to E \times [0, 1] \times \Delta$ are two n-parameter δ-pseudo-isotopies (where δ comes from the definition of torsion) such that $h = \mathrm{id} = h'$ on $E \times [0, 1] \times \partial\Delta$ and h and h' are γ-isotopic* $\mathrm{rel}(E \times \{0\} \times \Delta) \cup (E \times [0, 1] \times \partial\Delta)$ *when measured in B, then $\tau(h) = \tau(h')$.*

Composition of maps induces a group structure on $\pi_0\mathscr{P}(p: E \to B)$. The final result of this section shows that this group is abelian.

Proposition 4.6. *$\pi_0\mathscr{P}(p: E \to B)$ is an abelian group where the group operation is induced by composition of pseudo-isotopies.*

Proof. It suffices to show that if $g, h: E \times [0, 1] \to E \times [0, 1]$ are two ε-pseudo-isotopies, then $\tau(gh) = \tau(hg)$ if $\varepsilon > 0$ is small enough. To this end, let $S: E \times [0, 1] \to E$ be a homeomorphism quite close to projection and let $E_1 = S(E \times [0, \frac{1}{2}])$ and $E_2 = S(E \times [\frac{1}{2}, 1])$.

Use Z-set unknotting to find a small (measured in E) isotopy of h rel $E \times \{0\}$ to h' where $h' | S(E \times \{0\}) \times [0, 1] = \mathrm{id}$. Then slide along the interval direction in $S(E \times [0, 1])$ to find a small isotopy of h' rel $E \times \{0\}$ to h'' where $h'' | E_1 \times [0, 1] = \mathrm{id}$. If the isotopies are small enough, then $\tau(h'') = \tau(h)$.

Likewise, find g'' such that $\tau(g) = \tau(g'')$ and $g'' | E_2 \times [0, 1] = \mathrm{id}$. Then $g''h'' = h''g''$ and $\tau(gh) = \tau(hg)$. □

5. The homotopy equivalence $\mathscr{P}(p: E \to B) \simeq \Omega\mathrm{Wh}(p: E \to B)$

In this section we establish Theorem 2 by defining a map $\alpha: \mathscr{P}(p: E \to B) \to \Omega\overline{\mathrm{Wh}}(p: E \to B)$ and proving that α is a homotopy equivalence. Since $\overline{\mathrm{Wh}}(p: E \to B)$ and $\mathrm{Wh}(p: E \to B)$ are homotopy equivalent by Section 3, it will follow that $\mathscr{P}(p: E \to B)$ and $\Omega\mathrm{Wh}(p: E \to B)$ are homotopy equivalent.

We begin by defining α. Let $\tilde{E} = E \times [0, 1] \times \Delta \times [0, \infty)$ and let $h: \tilde{E} \to \tilde{E}$ be an n-simplex of $\mathscr{P}(p: E \to B)$. We first need a canonical homotopy $h_s: \mathrm{id} \simeq h$, $0 \leq s \leq 1$. To this end, extend h via the identity to get $\hat{h}: E \times [-1, 1] \times \Delta \times [0, \infty) \to E \times [-1, 1] \times \Delta \times [0, \infty)$. Let $r: E \times \mathbb{R} \times \Delta \times [0, \infty) \to \tilde{E}$ be the retraction induced by the retraction $\mathbb{R}\to[0,1]$ which collapses $(-\infty,0]$ to 0 and $[1,+\infty)$ to 1. For $0\le s\le 1$, let $\Theta_s: E\times\mathbb{R}\times\Delta\times[0,\infty)\to E\times\mathbb{R}\times\Delta\times[0,\infty)$ be the homeomorphism induced by the homeomorphism $\mathbb{R}\to\mathbb{R}$ which takes x to $x-1+s$. Then define $h_s:\tilde{E}\to\tilde{E}$ by setting $h_s=r\Theta_s^{-1}\hat{h}\Theta_s|\tilde{E}$. Note that $h_s:\mathrm{id}\simeq h$, $0\le s\le 1$, rel $E\times\{0\}\times\Delta\times[0,\infty)$ and that h_s is f.p. over $\Delta\times[0,\infty)$. This is called the *canonical homotopy* from id to h.

Recall that we have fixed a Z-embedding of $E\times[0,1]$ into Q. Define a map $\tilde{h}:\tilde{E}\times[0,1]\to Q\times\Delta\times[0,\infty)\times[0,1]$ by setting $\tilde{h}(x,s)=(h_s(x),s)$. Note that $\tilde{h}$ is f.p. over $\Delta\times[0,\infty)\times[0,1]$, $\tilde{h}=\mathrm{id}$ on $(E\times\{0\}\times\Delta\times[0,\infty)\times[0,1]\cup(\tilde{E}\times\{0\})$, and $\tilde{h}=h$ on $\tilde{E}\times\{1\}$.

Approximate $\tilde{h}$ by a sliced Z-embedding $j:\tilde{E}\times[0,1]\to Q\times\Delta\times[0,\infty)\times[0,1]$ rel$(E\times\{0\}\times\Delta\times[0,\infty)\times[0,1])\cup(\tilde{E}\times\{0,1\})$. Let $M=j(\tilde{E}\times[0,1])$ and define $f:M\to E\times\Delta\times[0,\infty)\times[0,1]$ by setting $f=\pi\tilde{h}j^{-1}$ where we are using $\pi:\tilde{E}\times[0,1]\to E\times\Delta\times[0,\infty)$ to denote projection as usual.

Finally, define $\alpha:\mathscr{P}(p:E\to B)\to\Omega\overline{\mathrm{Wh}}(p:E\to B)$ by setting $\alpha(h)=f$.

Proposition 5.1. *$\alpha:\mathscr{P}(p:E\to B)\to\Omega\overline{\mathrm{Wh}}(p:E\to B)$ is well-defined.*

Proof. The only arbitrary choice made in the defintion of $\alpha(h)$ was the embedding j. But this is already allowed in the definition of $\Omega\overline{\mathrm{Wh}}(p:E\to B)$. Therefore, we are only left with showing that f is a loop in $\overline{\mathrm{Wh}}(p:E\to B)$. Since $f|\tilde{E}\times\{0,1\}=\pi$, we only have to show that f has the correct controlled sdr property.

For this, define $\tilde{h}_u:\tilde{E}\times[0,1]\to\tilde{E}\times[0,1]$, $0\le u\le 1$, by setting $\tilde{h}_u(x,s)=(h_{us}(x),s)$. And define $\pi_u:\tilde{E}\times[0,1]\to\tilde{E}\times[0,1]$ by setting $\pi_u(w,x,y,z,s)=(w,(1-u)x,y,z,s)$ for (w,x,y,z,s) in $E\times[0,1]\times\Delta\times[0,\infty)\times[0,1]$. Finally, define $f_u:M\to M$ by $f_u=j\pi_u\tilde{h}_uj^{-1}$. Then $f_u:\mathrm{id}\simeq f$, $0\le u\le 1$, shows that f is an n-simplex in $\Omega\overline{\mathrm{Wh}}(p:E\to B)$. □

The next result deals with the group properties at the level of path components.

Proposition 5.2. *$\alpha_*:\pi_0\mathscr{P}(p:E\to B)\to\pi_0\Omega\overline{\mathrm{Wh}}(p:E\to B)$ is a group homomorphism.*

Proof. Represent two classes in $\pi_0\mathscr{P}(p:E\to B)$ by $\tau(h)$ and $\tau(g)$ where h and g are ε-pseudo-isotopies, $\varepsilon>0$ small. Using the notation in the proof of Proposition 4.6, we can assume that $h|E_1\times[0,1]=\mathrm{id}$ and $g|E_2\times[0,1]=\mathrm{id}$. In definining $\alpha_*\tau(h)$ and $\alpha_*\tau(g)$, the canonical homotopies are used to construct maps $\tilde{h},\tilde{g}:E\times[0,1]\times[0,1]\to E\times[0,1]\times[0,1]$ (these are the 0-level of maps $E\times[0,1]\times[0,\infty)\times[0,1]\to E\times[0,1]\times[0,\infty)\times[0,1]$). These maps satisfy $\tilde{h}|E_1\times[0,1]\times[0,1]=\mathrm{id}$ and $\tilde{g}|E_2\times[0,1]\times[0,1]=\mathrm{id}$.

The relevant fact is that $\tilde{g}\tilde{h}(=\tilde{h}\tilde{g})$ is the map arising from the canonical homotopy $\mathrm{id}\simeq gh$ when defining $\alpha_*\tau(gh)$. We will show that

$$\alpha_*\tau(gh)=\alpha_*\tau(h)\cdot\alpha_*\tau(g)$$

where the multiplication on the right hand side arises from the loop space structure.

Find a sliced Z-embedding $j: E \times [0,1] \times [0,1] \to Q \times [0,1]$ with image M such that $j = \mathrm{id}$ on $(E \times \{0\} \times [0,1]) \cup (E \times [0,1] \times \{0\}) \cup ((E_1 \cap E_2) \times [0,1] \times [0,1])$ and $j = gh$ on $E \times [0,1] \times \{1\}$. Then $\alpha_* \tau(gh) = \tau(f)$ where $f: M \to E \times [0,1]$ is defined by $f = \pi \tilde{g} \tilde{h} j^{-1}$.

Define $k: E \times [0,1] \times [0,1] \to E \times [0,1] \times [0,1]$ by

$$k(x, y, t) = \begin{cases} (\mathrm{proj}\, \tilde{h}(x, y, 2t), t) & \text{if } 0 \leq t \leq \frac{1}{2}, \\ (\mathrm{proj}\, \tilde{g}(h(x, y), 2t-1), t) & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Define $\tilde{f}: M \to E \times [0,1]$ by $\tilde{f} = \pi k j^{-1}$. It is clear that $\tau(f) = \tau(\tilde{f})$ and $\tau(\tilde{f}) = \alpha_* \tau(h) \cdot \alpha_* \tau(g)$. □

Our goal is to show that $\alpha: \mathscr{P}(p: E \to B) \to \Omega \overline{\mathrm{Wh}}(p: E \to B)$ is a homotopy equivalence. For this we need to show that $\alpha_*: \pi_n \mathscr{P}(p: E \to B) \to \pi_n \Omega \overline{\mathrm{Wh}}(p: E \to B)$ is an isomorphism for each choice of basepoint. However, the proof of the proposition above shows that we can just concern ourselves with the usual basepoints: the identitiy for $\mathscr{P}(p: E \to B)$ and the projection for $\Omega \overline{\mathrm{Wh}}(p: E \to B)$. We will do this in the next two propositions to complete the proof of Theorem 2.

Proposition 5.3. *$\alpha_*: \pi_n \mathscr{P}(p: E \to B) \to \pi_n \Omega \overline{\mathrm{Wh}}(p: E \to B)$ is a monomorphism.*

Proof. Let $h: \tilde{E} \to \tilde{E}$ be an n-parameter controlled pseudo-isotopy representing the j. But this is already allowed in the definition of $\Omega \overline{\mathrm{Wh}}(p: E \to B)$. Therefore, we are only left with showing that f is a loop in $\overline{\mathrm{Wh}}(p: E \to B)$. Since $f \mid \tilde{E} \times \{0,1\} = \pi$, we only have to show that f has the correct controlled sdr property.

Since $[f] = [\pi]$ there is an $(n+1)$-simplex in $\Omega \overline{\mathrm{Wh}}(p: E \to B)$ given by $G: \tilde{M} \to E \times \Delta \times [0, \infty) \times [0,1]^2$ where (among other properties):

(1) there is a bundle projection $\tilde{\rho}: \tilde{M} \to \Delta \times [0, \infty) \times [0,1]^2$;
(2) $E \times \Delta \times [0, \infty) \times [0,1]^2 \subset \tilde{M} \subset Q \times \Delta \times [0, \infty) \times [0,1]^2$;
(3) if $A = (\partial \Delta \times [0, \infty) \times [0,1]^2) \cup (\Delta \times [0, \infty) \times \{0,1\} \times [0,1]) \cup (\Delta \times [0, \infty) \times [0,1] \times \{0\})$, then $\tilde{\rho}^{-1}(A) = E \times [0,1] \times A$;
(4) $\tilde{\rho}^{-1}(\Delta \times [0, \infty) \times [0,1] \times \{1\} = M$;
(5) $G = \mathrm{id}$ on $E \times \Delta \times [0, \infty) \times [0,1]^2$;
(6) $G = \pi$ on $\tilde{\rho}^{-1}(\mathring{A})$;
(7) $G = f$ on M.

Since $\tilde{\rho}$ is trivial, we can use sliced Z-set unknotting to find an f.p. homeomorphism $k: \tilde{E} \times [0,1]^2 \to \tilde{M}$ such that:

(1) $k = \mathrm{id}$ on $(E \times \{0\} \times \Delta \times [0, \infty) \times [0,1]^2) \cup (E \times [0,1] \times \partial \Delta \times [0, \infty) \times [0,1]^2) \cup (E \times [0,1] \times \Delta \times [0, \infty) \times \{0\} \times [0,1]) \cup (E \times [0,1] \times \Delta \times [0, \infty) \times [0,1] \times \{0\})$;
(2) $k = j$ on $E \times [0,1] \times \Delta \times [0, \infty) \times [0,1] \times \{1\}$.

Use Theorem 2.3 to find an f.p. homeomorphism $g: \tilde{E} \times [0,1]^2 \to \tilde{E} \times [0,1]^2$ such that:

(1) $g = \mathrm{id}$ on the sets listed above where $k = \mathrm{id}$ and $k = j$;
(2) the map $(p \times \mathrm{id}) Gkg: \tilde{E} \times [0,1]^2 \to B \times \Delta \times [0, \infty) \times [0,1]^2$ is close to $(p \times \mathrm{id})\pi$ with the 'closeness' becoming arbitrarily small near infinity.

Finally, consider the homeomorphism $H:\tilde{E}\times[0,1]\to\tilde{E}\times[0,1]$ which is defined to be the restriction of kg to $\tilde{E}\times\{1\}\times[0,1]$. Note that:

(1) $H=\mathrm{id}$ on $(E\times\{0\}\times\Delta\times[0,\infty)\times[0,1])\cup(E\times[0,1]\times\partial\Delta\times[0,\infty)\times[0,1])\cup(E\times[0,1]\times\Delta\times[0,\infty)\times\{0\})$;

(2) $H=h$ on $E\times[0,1]\times\Delta\times[0,\infty)\times\{1\}$;

(3) $(p\times\mathrm{id})\pi H=(p\times\mathrm{id})\pi(kg|)=(p\times\mathrm{id})G(kg|)$ which is close to $(p\times\mathrm{id})\pi$.

It follows that H is an $(n+1)$-simplex in $\mathscr{P}(p:E\to B)$ showing $[h]=[\mathrm{id}]$. □

Proposition 5.4. *$\alpha_*:\pi_n\mathscr{P}(p:E\to B)\to\pi_n\Omega\overline{\mathrm{Wh}}(p:E\to B)$ is an epimorphism.*

Proof. Recall from Section 3 that there are isomorphisms Ωi_*: $\pi_n\Omega\mathrm{Wh}(p:E\to B)\to\pi_n\Omega\widehat{\mathrm{Wh}}(p:E\to B)$ and Ωq_*: $\pi_n\Omega\widehat{\mathrm{Wh}}(p:E\to B)\to\pi_n\Omega\overline{\mathrm{Wh}}(p:E\to B)$. Thus, let $f:M\to E\times\Delta\times[0,1]$ represent the class $[f]$ in $\pi_n\Omega\mathrm{Wh}(p:E\to B)$ where:

(1) $M\subset Q\times\Delta\times[0,1]$ as a sliced Z-set;

(2) $f=\mathrm{id}$ on $E\times\Delta\times[0,1]\subset M$;

(3) $f=\pi$ on $M\cap(Q\times\partial(\Delta\times[0,1]))=E\times[0,1]\times\partial(\Delta\times[0,1])$;

(4) f is an f.p. $(p\times\mathrm{id})^{-1}(\varepsilon)$-sdr for every $\varepsilon>0$.

As we have done before, we can find a trivializing homeomorphism $k:E\times[0,1]\times\Delta\times[0,1]\to M$ such that $k=\mathrm{id}$ on $(E\times\{0\}\times\Delta\times[0,1])\cup(E\times[0,1]\times\partial\Delta\times[0,1])\cup(E\times[0,1]\times\Delta\times\{0\})$.

Also, use Theorem 2.3 to find an f.p. homeomorphism $g:E\times[0,1]\times\Delta\times[0,1]\to E\times[0,1]\times\Delta\times[0,1]$ such that $g=\mathrm{id}$ on the set indicated above where $k=\mathrm{id}$ and $(p\times\mathrm{id})fkg:E\times[0,1]\times\Delta\times[0,1]\to B\times\Delta\times[0,1]$ is ε-close to $(p\times\mathrm{id})\pi$ for a given small $\varepsilon>0$.

Consider the homeomorphism $h:E\times[0,1]\times\Delta\to E\times[0,1]\times\Delta$ which is defined to be the restriction $kg|E\times[0,1]\times\Delta\times\{1\}$. Note that $h=\mathrm{id}$ on $(E\times\{0\}\times\Delta)\cup(E\times[0,1]\times\partial\Delta)$ and h is an n-parameter ε-pseudo-isotopy. If ε is small enough, then $\tau(h)$ is defined in $\pi_n\mathscr{P}(p:E\to B)$. We will complete the proof by showing that $\alpha_*\tau(h)=(\Omega q_*)(\Omega i_*)([f])$.

Recall that $\tau(h)$ is defined to be $[(h\times\mathrm{id})H]$ where H is a certain homeomorphism on $\tilde{E}$. Next recall the notation used for defining $\alpha_*\tau(h)$. The canonical homotopy $\mathrm{id}\simeq(h\times\mathrm{id})H$ is used to define a map $\tilde{h}:\tilde{E}\times[0,1]\to\tilde{E}\times[0,1]$. We then need a sliced Z-embedding $j:\tilde{E}\times[0,1]\to Q\times\Delta\times[0,\infty)\times[0,1]$. To this end let $\tilde{H}:\tilde{E}\times[0,1]\to\tilde{E}\times[0,1]$ be the f.p. homeomorphism such that $\tilde{H}|\tilde{E}\times\{0\}=\mathrm{id}$ and $\tilde{H}|\tilde{E}\times\{1\}=H$ which arises by phasing H out to the identity. Then set $j=(kg\times\mathrm{id})\tilde{H}$. (Note that $[0,\infty)$ and $[0,1]$ have been interchanged.)

Let $\tilde{f}=\pi\tilde{h}j^{-1}:M\times[0,\infty)\to E\times\Delta\times[0,1]\times[0,\infty)$. Then $\alpha_*\tau(h)=[\tilde{f}]$. Let $f'=\tilde{f}|M\times\{0\}$. We will complete the proof by showing $\tau(f')=[f]$. This suffices because $[\tilde{f}]=(\Omega q_*)(\Omega i_*)\tau(f')$ by Lemma 3.5 and therefore $\alpha_*\tau(h)=[\tilde{f}]=(\Omega q_*)(\Omega i_*)[f]$.

In order to show that $\tau(f')=[f]$, we will show that f' is f.p. homotopic to $f\,\mathrm{rel}(E\times\Delta\times[0,1])\cup(E\times[0,1]\times\partial(\Delta\times[0,1]))$ via a homotopy which is small when measured in $B\times\Delta\times[0,1]$. To this end, let $F_t:\mathrm{id}\simeq f$, $0\leqslant t\leqslant 1$, be the homotopy given by Lemma 3.1. Let $\tilde{h}_0=\tilde{h}|E\times[0,1]\times\Delta\times\{0\}\times[0,1]$. It is the map which arises from the canonical homotopy from id to h. Note that $f' = \pi\tilde{h}_0 g^{-1}k^{-1}$. Our desired homotopy is $\pi\tilde{h}_0 g^{-1}k^{-1}F_t : f' \simeq f$, $0 \leq t \leq 1$. Note that $(p\times \mathrm{id})\pi\tilde{h}_0 g^{-1}k^{-1}F_t$ is close to $(p\times \mathrm{id})\pi g^{-1}k^{-1}F_t$ because of the nature of the canonical homotopy. And $(p\times \mathrm{id})\pi g^{-1}k^{-1}F_t$ is close to $(p\times \mathrm{id})fF_t$ because of the way g was chosen. Finally, the homotopy $(p\times \mathrm{id})fF_t$ is small because of the way F_t was chosen. □

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