SOME REMARKS ON FREELY DECOMPOSABLE MAPPINGS

by

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1. Introduction

Freely decomposable mappings were recently introduced by G. R. Gordh, Jr. and the author in [1] as a generalization of monotone mappings. It was shown that every inverse sequence of locally connected (semi-locally connected) continua with freely decomposable bonding mappings has a locally connected (semi-locally connected) limit. Other basic properties of freely decomposable mappings were established in [1]. For example, every freely decomposable mapping of a unicoherent continuum onto a locally connected continuum is monotone. This paper continues the study of freely decomposable mappings.

A continuum is a compact connected metric space and a mapping is a continuous surjection between continua. If X is a continuum, then X = A U B is a decomposition provided that A and B are proper subcontinua of X.

A mapping f: X → Y is said to be freely decomposable (denoted FD) if for each decomposition Y = A U B there exists a decomposition X = A' U B' such that f(A') ⊆ A and f(B') ⊆ B.

The continuum X is freely decomposable if for each pair of distinct points a and b in X, there exists a decomposition X = A U B such that a ∈ A\B and b ∈ B\A. It is known that a continuum is semi-locally connected if and only if it is
freely decomposable [2].

2. A Certain Class of Continua

In [1] it was shown that every FD mapping onto a locally connected continuum which contains no separating points is monotone. The following problem naturally arises.

*Problem.* Characterize those continua onto which every FD mapping is monotone.

This class of continua then includes all locally connected continua without separating points. The figure eight is an example of a continuum in this class which contains a separating point. As the next two theorems show any continuum \( Y \) in this class is locally connected and contains no arc each point of which separates \( Y \).

**Theorem 1.** If \( Y \) is a non-locally connected continuum, then there exist a continuum \( X \) and a non-monotone FD mapping \( f: X \to Y \).

*Proof.* Since \( Y \) is not locally connected, there exist a point \( p \in Y \), an open set \( U \subseteq Y \) containing \( p \), a continuum \( K \) such that \( p \in K \subseteq \text{cl}(U) \) and \( K \cap \text{bd}(U) \neq \emptyset \), and a sequence \( \{C_n\} \) of distinct components of \( U \) disjoint from \( K \) such that \( K = \lim\{C_n\} \). Let \( K' \) be a topological copy of \( K \) and let \( h: K \to K' \) be a homeomorphism. Let \( X = Y \cup K' \) be the continuum obtained by attaching \( Y \) to \( K' \) along \( K \cap \text{bd}(U) \) by the restriction of \( h \) to \( K \cap \text{bd}(U) \). Now define \( f: X \to Y \) by

\[
    f(x) = \begin{cases} 
        x & \text{if } x \in Y, \\
        h^{-1}(x) & \text{if } x \in K'.
    \end{cases}
\]

It is clear that \( f \) is a mapping, and \( f \) is not monotone since
$f^{-1}(p) = \{p, h(p)\}$. To see that $f$ is an FD mapping, let $Y = A \cup B$ be a decomposition. Let $A'$ be the component of $f^{-1}(A)$ which contains $f^{-1}(A) \cap Y = A$, and let $B'$ be the component of $f^{-1}(B)$ which contains $f^{-1}(B) \cap Y = B$. It suffices to show that $X = A' \cup B'$. To this end let $x \in K \setminus Y$, and choose a sequence $\{x_n\}$ converging to $h^{-1}(x)$ such that $x_n \in C_n$. Without loss of generality and by passing to subsequences we may assume that $x_n \in A$ for each $n$. Let $C'_n$ denote the component of $U \cap A$ which contains $x_n$. Since $\lim \inf \{C'_n\} \neq \emptyset$, it follows that $\lim \sup \{C'_n\} \subseteq K$ is a continuum. Note that $U \cap A \neq A$, for otherwise $A$ would be a subcontinuum of $U$ meeting distinct components of $U$. Hence, $\overline{C'_n} \cap \overline{\text{bd}}_A(U \cap A) \neq \emptyset$ where $\overline{\text{bd}}_A(M)$ denotes the boundary of $M$ relative to $A$. It follows that $(\lim \sup \{C'_n\}) \cap K \cap \overline{\text{bd}}(U) \cap A \neq \emptyset$. Since $\lim \sup \{C'_n\} \subseteq A$, it follows that $h(\lim \sup \{C'_n\})$ is a subcontinuum of $f^{-1}(A)$ which meets $A$ and contains $x$. Thus $x \in A'$ and $X = A' \cup B'$.

**Theorem 2.** If $Y$ is a locally connected continuum which contains an arc $A$ such that each point of $A$ separates $Y$, then there exist a continuum $X$ and a non-monotone FD mapping $f: X \to Y$.

**Proof.** Let $a$ and $b$ be the endpoints of $A$. Let $A'$ be an arc with endpoints $a'$ and $b'$, and let $X = Y \cup A'$ be the continuum obtained by attaching $a'$ to $a$ and $b'$ to $b$. Let $f: X \to Y$ be a mapping so that $f$ is the identity on $Y$ and $f|A'$ is a homeomorphism of $A'$ onto $A$. Then $f$ is clearly non-monotone. To see that $f$ is an FD mapping let $Y = P \cup Q$ be a decomposition. By a result of Whyburn [3, p. 51] all
but countably many points of $A$ separate $a$ from $b$ in $Y$. It follows that if $a$ and $b$ are both in $P$ or $Q$, then $A \subseteq P$ or $A \subseteq Q$, respectively. Otherwise, without loss of generality, assume that $a \in P$ and $b \in Q \setminus P$. Again using Whyburn's result we see that $P \cap A$ is connected. In either case it is clear that there is a decomposition $X = P' \cup Q'$ such that $f(P') \subseteq P$ and $f(Q') \subseteq Q$.

3. Freely Decomposable Mappings on Irreducible Continua

The continuum $X$ is said to be irreducible if there exist points $a$ and $b$ of $X$ such that no proper subcontinuum of $X$ contains $a$ and $b$. Every FD mapping of an irreducible continuum onto a locally connected continuum is monotone [1]. The following theorem is a generalization of that result.

Theorem 3. If $X$ is irreducible, $Y$ is semi-locally connected, and $f: X \rightarrow Y$ is an FD mapping, then $f$ is monotone. Consequently, if $Y$ is nondegenerate, $Y$ is an arc.

Proof. Suppose $f$ is not monotone. Then there exist $y \in Y$ and $x_1$ and $x_2$ in distinct components of $f^{-1}(y)$. Let $I \subseteq X$ be a continuum irreducible from $x_1$ to $x_2$ and let $p \in I \setminus f^{-1}(y)$. Because $Y$ is semi-locally connected, there is a decomposition $Y = A \cup B$ with $f(p) \in A \setminus B$ and $y \in B \setminus A$. Choose a decomposition $X = A' \cup B'$ such that $f(A') \subseteq A$ and $f(B') \subseteq B$. Let $a \in A'$ and $b \in B'$ such that $X$ is irreducible from $a$ to $b$.

Let $J \subseteq B'$ be a continuum irreducible from $x_1$ to $x_2$. Choose $q \in J \setminus (A' \cup f^{-1}(y))$. Let $Y = C \cup D$ be a decomposition such that $f(q) \in C \setminus D$ and $y \in D \setminus C$, and choose a decomposition $X = C' \cup D'$ such that $f(C') \subseteq C$ and $f(D') \subseteq D$. 

Case I: $a \in C'$. Since $p \notin B'$, there exists an open set $U$ such that $p \in U \subseteq \text{cl}(U) \subseteq A'B'$. For $i = 1, 2$, let $I_i$ be the closure of the component of $I \setminus \text{cl}(U)$ which contains $x_i$. Then $I_i \cap \text{cl}(U) \neq \emptyset$. If $q \in I_1 \cap I_2$, then $I_1 \cup I_2$ is a proper subcontinuum of $I$ containing $x_1$ and $x_2$ which is a contradiction. Thus assume without loss of generality that $q \notin I_1$. It follows that $A' \cup I_1 \cup D'$ is a subcontinuum of $X$ containing $a$ and $b$ but not $q$. This is a contradiction.

Case II: $a \in D'$. Let $I'$ be the closure of the component of $I \setminus A'$ which contains $x_1$, and let $J'$ be the closure of the component of $J \setminus C'$ which contains $x_1$. Then $I' \cap A' \neq \emptyset \neq J' \cap C'$. Since $p \notin I'$ and $q \notin J'$, it follows that $x_2 \notin I' \cup J'$. Thus $A' \cup I' \cup J' \cup C'$ is a subcontinuum of $X$ containing $a$ and $b$ but not $x_2$. This is a contradiction.

Since $f$ is monotone, $Y$ is irreducible. If $Y$ is non-degenerate, then $Y$ is an arc by 6.3 of [4].

References