TOPOLOGY PROCEEDINGS
Volume 2, 1977
Pages 213–217
http://topology.auburn.edu/tp/

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Topology Proceedings

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ISSN: 0146-4124

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1. Introduction

Freely decomposable mappings were recently introduced by G. R. Gordh, Jr. and the author in [1] as a generalization of monotone mappings. It was shown that every inverse sequence of locally connected (semi-locally connected) continua with freely decomposable bonding mappings has a locally connected (semi-locally connected) limit. Other basic properties of freely decomposable mappings were established in [1]. For example, every freely decomposable mapping of a unicoherent continuum onto a locally connected continuum is monotone. This paper continues the study of freely decomposable mappings.

A *continuum* is a compact connected metric space and a *mapping* is a continuous surjection between continua. If X is a continuum, then $X = A \cup B$ is a *decomposition* provided that A and B are proper subcontinua of X.

A mapping $f: X \to Y$ is said to be *freely decomposable* (denoted FD) if for each decomposition $Y = A \cup B$ there exists a decomposition $X = A' \cup B'$ such that $f(A') \subseteq A$ and $f(B') \subseteq B$.

The continuum X is *freely decomposable* if for each pair of distinct points a and b in X, there exists a decomposition $X = A \cup B$ such that $a \in A \setminus B$ and $b \in B \setminus A$. It is known that a continuum is semi-locally connected if and only if it is freely decomposable [2].

2. A Certain Class of Continua

In [1] it was shown that every FD mapping onto a locally connected continuum which contains no separating points is monotone. The following problem naturally arises.

Problem. Characterize those continua onto which every FD mapping is monotone.

This class of continua then includes all locally connected continua without separating points. The figure eight is an example of a continuum in this class which contains a separating point. As the next two theorems show any continuum Y in this class is locally connected and contains no arc each point of which separates Y.

Theorem 1. *If* Y *is a non-locally connected continuum, then there exist a continuum* X *and a non-monotone* FD *mapping* $f: X \to Y$.

Proof. Since Y is not locally connected, there exist a point $p \in Y$, an open set $U \subseteq Y$ containing p, a continuum K such that $p \in K \subseteq c\ell(U)$ and $K \cap bd(U) \neq \emptyset$, and a sequence $\{C_n\}$ of distinct components of U disjoint from K such that $K = \lim\{C_n\}$. Let K' be a topological copy of K and let $h: K \to K'$ be a homeomorphism. Let $X = Y \cup K'$ be the continuum obtained by attaching Y to K' along $K \cap bd(U)$ by the restriction of h to $K \cap bd(U)$. Now define $f: X \to Y$ by

$$f(x) = \begin{cases} x \text{ if } x \in Y, \text{ and} \\ h^{-1}(x) \text{ if } x \in K'. \end{cases}$$

It is clear that f is a mapping, and f is not monotone since

$f^{-1}(p) = \{p,h(p)\}$. To see that f is an FD mapping, let $Y = A \cup B$ be a decomposition. Let A' be the component of $f^{-1}(A)$ which contains $f^{-1}(A) \cap Y = A$, and let B' be the component of $f^{-1}(B)$ which contains $f^{-1}(B) \cap Y = B$. It suffices to show that $X = A' \cup B'$. To this end let $x \in K' \setminus Y$, and choose a sequence $\{x_n\}$ converging to $h^{-1}(x)$ such that $x_n \in C_n$. Without loss of generality and by passing to subsequences we may assume that $x_n \in A$ for each n. Let C_n' denote the component of $U \cap A$ which contains x_n. Since $\lim\inf\{C_n'\} \neq \emptyset$, it follows that $\lim\sup\{C_n'\} \subseteq K$ is a continuum. Note that $U \cap A \neq A$, for otherwise A would be a subcontinuum of U meeting distinct components of U. Hence, $c\ell(C_n') \cap bd_A(U \cap A) \neq \emptyset$ where $bd_A(M)$ denotes the boundary of M relative to A. It follows that $(\lim\sup\{C_n'\}) \cap K \cap bd(U) \cap A \neq \emptyset$. Since $\lim\sup\{C_n'\} \subseteq A$, it follows that $h(\lim\sup\{C_n'\})$ is a subcontinuum of $f^{-1}(A)$ which meets A and contains x. Thus $x \in A'$ and $X = A' \cup B'$.

Theorem 2. *If* Y *is a locally connected continuum which contains an arc* A *such that each point of* A *separates* Y, *then there exist a continuum* X *and a non-monotone* FD *mapping* $f: X \to Y$.

Proof. Let a and b be the endpoints of A. Let A' be an arc with endpoints a' and b', and let $X = Y \cup A'$ be the continuum obtained by attaching a' to a and b' to b. Let $f: X \to Y$ be a mapping so that f is the identity on Y and $f|A'$ is a homeomorphism of A' onto A. Then f is clearly non-monotone. To see that f is an FD mapping let $Y = P \cup Q$ be a decomposition. By a result of Whyburn [3, p. 51] all but countably many points of A separate a from b in Y. It follows that if a and b are both in P or Q, then $A \subseteq P$ or $A \subseteq Q$, respectively. Otherwise, without loss of generality, assume that $a \in P$ and $b \in Q \setminus P$. Again using Whyburn's result we see that $P \cap A$ is connected. In either case it is clear that there is a decomposition $X = P' \cup Q'$ such that $f(P') \subseteq P$ and $f(Q') \subseteq Q$.

3. Freely Decomposable Mappings on Irreducible Continua

The continuum X is said to be *irreducible* if there exist points a and b of X such that no proper subcontinuum of X contains a and b. Every FD mapping of an irreducible continuum onto a locally connected continuum is monotone [1]. The following theorem is a generalization of that result.

Theorem 3. *If* X *is irreducible,* Y *is semi-locally connected, and* $f: X \to Y$ *is an* FD *mapping, then* f *is monotone. Consequently, if* Y *is nondegenerate,* Y *is an arc.*

Proof. Suppose f is not monotone. Then there exist $y \in Y$ and x_1 and x_2 in distinct components of $f^{-1}(y)$. Let $I \subseteq X$ be a continuum irreducible from x_1 to x_2 and let $p \in I \setminus f^{-1}(y)$. Because Y is semi-locally connected, there is a decomposition $Y = A \cup B$ with $f(p) \in A \setminus B$ and $y \in B \setminus A$. Choose a decomposition $X = A' \cup B'$ such that $f(A') \subseteq A$ and $f(B') \subseteq B$. Let $a \in A'$ and $b \in B'$ such that X is irreducible from a to b.

Let $J \subseteq B'$ be a continuum irreducible from x_1 to x_2. Choose $q \in J \setminus (A' \cup f^{-1}(y))$. Let $Y = C \cup D$ be a decomposition such that $f(q) \in C \setminus D$ and $y \in D \setminus C$, and choose a decomposition $X = C' \cup D'$ such that $f(C') \subseteq C$ and $f(D') \subseteq D$.

Case I: $a \in C'$. Since $p \notin B'$, there exists an open set U such that $p \in U \subseteq c\ell(U) \subseteq A' \setminus B'$. For $i = 1,2$, let I_i be the closure of the component of $I \setminus c\ell(U)$ which contains x_i. Then $I_i \cap c\ell(U) \neq \emptyset$. If $q \in I_1 \cap I_2$, then $I_1 \cup I_2$ is a proper subcontinuum of I containing x_1 and x_2 which is a contradiction. Thus assume without loss of generality that $q \notin I_1$. It follows that $A' \cup I_1 \cup D'$ is a subcontinuum of X containing a and b but not q. This is a contradiction.

Case II: $a \in D'$. Let I' be the closure of the component of $I \setminus A'$ which contains x_1, and let J' be the closure of the component of $J \setminus C'$ which contains x_1. Then $I' \cap A' \neq \emptyset \neq J' \cap C'$. Since $p \notin I'$ and $q \notin J'$, it follows that $x_2 \notin I' \cup J'$. Thus $A' \cup I' \cup J' \cup C'$ is a subcontinuum of X containing a and b but not x_2. This is a contradiction.

Since f is monotone, Y is irreducible. If Y is nondegenerate, then Y is an arc by 6.3 of [4].

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