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1. Introduction

Let X denote a continuum (i.e., a compact, connected, nonvoid, metric space). The hyperspace of subcontinua of X, denoted C(X), is the space of all subcontinua of X endowed with the Hausdorff metric (e.g., [4]). A *Whitney map* on C(X) is a continuous function $\mu:C(X) \to [0,1]$ satisfying the following properties:

(i) $\mu(\{x\}) = 0$ for each $x \in X$,

(ii) $\mu(X) = 1$, and

(iii) if $A \subseteq B$ and $A \neq B$, then $\mu(A) < \mu(B)$.

Whitney [13] has shown that such functions always exist. Throughout this paper, μ will stand for an arbitrary Whitney map on C(X). It is known [2] that μ is monotone; that is, $\mu^{-1}(t)$ is a subcontinuum of C(X) for each t. The continua $\mu^{-1}(t)$ are called the *Whitney continua* of X.

In Section 2 we characterize the separating points of $\mu^{-1}(t)$ in terms of their separating properties as subcontinua of X. The rest of the paper contains applications of this result. In Section 3 we obtain some information about the Whitney continua of arc-like and circle-like continua. Section 4 establishes classes of continua which have the property that $\mu^{-1}(t)$ is an arc for t sufficiently close to 1.

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2. Separating points in $\mu^{-1}(t)$

If $G_1, G_2, \ldots, G_n$ are open subsets of X, then $N(G_1, \ldots, G_n)$

denotes the set of all points A in C(X) such that $A \subseteq \bigcup \{G_i : i = 1,2,\ldots,n\}$ and $A \cap G_i \neq \phi$ for each $i \leq n$. Recall that the collection of all such subsets of C(X) forms a basis for the Vietoris finite topology on C(X). It is well known that the Hausdorff metric and the Vietoris finite topology agree on C(X) (e.g., [8]).

If $t \in [0,1]$ and $x \in X$, then let $C_x^t = \{A \in \mu^{-1}(t) : x \in A\}$. Rogers [10, Theorem 4.2] has shown that C_x^t is an arcwise connected subcontinuum of C(X).

Theorem 2.1. *Let* A *be an element of* C(X) *with* $\mu(A) = t$. *Then* A *separates* $\mu^{-1}(t)$ *if and only if there exists a separation* $X - A = X_1 \cup X_2$ *such that for any* $B \in \mu^{-1}(t)$ *either* $B \subseteq X_1 \cup A$ *or* $B \subseteq X_2 \cup A$.

Proof. (only if) Let $\mu^{-1}(t) - \{A\} = \mathcal{S}_1 \cup \mathcal{S}_2$ be a separation. Let

$$X_1 = \bigcup \{B \in \mu^{-1}(t) : B \in \mathcal{S}_1\} - A \text{ and}$$
$$X_2 = \bigcup \{B \in \mu^{-1}(t) : B \in \mathcal{S}_2\} - A.$$

For each $p \in X$ there exists $P \in \mu^{-1}(t)$ with $p \in P$, thus $X - A = X_1 \cup X_2$. To show $X_1 \cap X_2 = \phi$ suppose on the contrary that $x \in X_1 \cap X_2$. Because $x \notin A$, it follows that $C_x^t \subseteq \mathcal{S}_1 \cup \mathcal{S}_2$. Since $x \in X_1 \cap X_2$, there exists $B_1 \in \mathcal{S}_1$ and $B_2 \in \mathcal{S}_2$ such that $x \in B_1$ and $x \in B_2$. The fact that B_1 and B_2 are in C_x^t implies $C_x^t \cap \mathcal{S}_1 \neq \phi \neq C_x^t \cap \mathcal{S}_2$. This contradicts the fact that $\mathcal{S}_1$ and $\mathcal{S}_2$ are separated because C_x^t is a continuum. To show that X_1 and X_2 are separated, by symmetry it suffices to show that no convergent sequence of points in X_1 converges to a point in X_2. To this end suppose $\{p_n\}$ is a sequence of points in X_1 which converges to some $p \in X$. For each n, choose $P_n \in \mathcal{S}_1$ such that $p_n \in P_n$. If P denotes the limit of a convergent subsequence of $\{P_n\}$, then $p \in P$. Since $\mu^{-1}(t)$ is a subcontinuum of C(X) and $\mathcal{S}_1$ and $\mathcal{S}_2$ are separated, it follows that $P \in \mathcal{S}_1 \cup \{A\}$. Hence,

$p \in P \subseteq X_1 \cup A$ and $X-A = X_1 \cup X_2$ is a separation. Finally, suppose $B \in \mu^{-1}(t)$ and that $B \in \mathcal{S}_1$. Then $B \subseteq \bigcup \{M \in \mu^{-1}(t): M \in \mathcal{S}_1\} \subseteq X_1 \cup A$. Hence, for any $B \in \mu^{-1}(t)$ either $B \subseteq X_1 \cup A$ or $B \subseteq X_2 \cup A$.

(if) Let $\mathcal{T}_1 = \{B \in \mu^{-1}(t): B \subseteq X_1 \cup A,\ B \neq A\}$ and

$$\mathcal{T}_2 = \{B \in \mu^{-1}(t): B \subseteq X_2 \cup A,\ B \neq A\}$$

To see that $\mu^{-1}(t)-\{A\} = \mathcal{T}_1 \cup \mathcal{T}_2$ is a separation, note that $N(X_1,X)$ and $N(X_2,X)$ are open subsets of $C(X)$ such that $\mathcal{T}_1 = N(X_1,X) \cap \mu^{-1}(t)$ and $\mathcal{T}_2 = N(X_2,X) \cap \mu^{-1}(t)$.

Using Theorem 2.1 we obtain a simple proof of the following well known result originally due to Krasinkiewicz [5] (see also [9], [10]).

Corollary 2.2. *If* X *is an arc, then* $\mu^{-1}(t)$ *is an arc for each* $t < 1$.

Proof. Let p and q be the non-separating points of X. If $t < 1$, then it is easily seen that there exist exactly one subcontinuum P of X and one subcontinuum Q of X such that $p \in P$ and $q \in Q$ and $P,Q \in \mu^{-1}(t)$. If $A \in \mu^{-1}(t)$ such that $P \neq A \neq Q$, then A separates X in the way required by Theorem 2.1. Thus, A separates $\mu^{-1}(t)$ and $\mu^{-1}(t)$ has exactly two non-separating points. It follows that $\mu^{-1}(t)$ is an arc.

Example 2.3. Let X be a simple triod (i.e., a continuum homeomorphic to the capital letter T). Let Y be a proper subcontinuum of X which is also a simple triod and which separates X. Let $\mu(Y) = t$. Then Y does not separate X in the way required by Theorem 2.1 and thus Y does not separate $\mu^{-1}(t)$.

3. Whitney continua of arc-like and circle-like continua

In this section we give sufficient conditions on $\mu^{-1}(t)$ to insure that X be decomposable. Information about the Whitney continua of arc-like and circle-like continua is obtained in the corollaries. Corollary 3.2 answers a question of J. T. Rogers, Jr. [10]. The proofs of Corollaries 3.3 and 3.4 were pointed out to the author by G. R. Gordh, Jr.

Theorem 3.1. *If* $\mu^{-1}(t)$ *is irreducible and decomposable for some* $t < 1$*, then* X *is decomposable.*

Proof. Let A and B be points in $\mu^{-1}(t)$ such that $\mu^{-1}(t)$ is irreducible from A to B. Let $\mathcal{S}$ and $\mathcal{T}$ be proper subcontinua of $\mu^{-1}(t)$ with $A \in \mathcal{S}$ and $B \in \mathcal{T}$ such that $\mu^{-1}(t) = \mathcal{S} \cup \mathcal{T}$. From [4, Lemma 1.1] it follows that $\bigcup \mathcal{S}$ and $\bigcup \mathcal{T}$ are subcontinua of X. It is clear that $X = (\bigcup \mathcal{S}) \cup (\bigcup \mathcal{T})$, so if $\bigcup \mathcal{S}$ and $\bigcup \mathcal{T}$ are proper subcontinua of X, then the theorem is proved. Assume for the purpose of this proof that $X = \bigcup \mathcal{T}$. Then $A \subseteq \bigcup \mathcal{T}$ so there exists $M \in \mathcal{T}$ such that $A \cap M \neq \phi$. This implies ([9] or [10]) that there is an arc $\mathcal{I}$ in $\mu^{-1}(t)$ with endpoints A and M. By the irreducibility of $\mu^{-1}(t)$, we have $\mathcal{S} - \mathcal{T} \subseteq \mathcal{I}$. It follows that a point N in $\mu^{-1}(t)$ can be choosen in $\mathcal{S} - \mathcal{T}$ such that N is different from A and N separates $\mu^{-1}(t)$. From Theorem 2.1, N is a subcontinuum of X which separates X and hence, X must be decomposable.

A continuum X is said to be *arc-like* if for each positive number ε, there is an ε-map (i.e., a map having point-inverses of diameter less than ε) of X onto an arc. *Circle-like* continua are defined in the same manner.

Corollary 3.2. *If* X *is indecomposable and arc-like, then* $\mu^{-1}(t)$ *is indecomposable and arc-like for each* $t < 1$.

Proof. Krasinkiewicz [5] has shown that $\mu^{-1}(t)$ must be arc-like for each $t < 1$. Since arc-like continua are unicoherent and are not triods, it follows from [11] that $\mu^{-1}(t)$ is irreducible for each $t < 1$. If $\mu^{-1}(t)$ were decomposable for some $t < 1$, then by Theorem 3.1 X would be decomposable also. Thus, $\mu^{-1}(t)$

is indecomposable and arc-like for each $t<1$.

Corollary 3.3. *Let* X *be arc-like and circle-like. Then* $\mu^{-1}(t)$ *is arc-like and circle-like for each* $t<1$ *if and only if* X *is indecomposable.*

Proof. (only if) Suppose X is arc-like, circle-like and decomposable. Rogers [10, Theorem 5.1] has shown that there exists $t<1$ such that $\mu^{-1}(t)$ is not circle-like. This is a contradiction.

(if) Since X is indecomposable and arc-like, it follows from Corollary 3.2 that $\mu^{-1}(t)$ is indecomposable and arc-like for each $t<1$. Burgess [1] has shown that such continua must also be circle-like.

Corollary 3.4. *Let* X *be circle-like. Then* $\mu^{-1}(t)$ *is circle-like for each* $t<1$ *if and only if* X *is indecomposable or* X *is not arc-like.*

Proof. (only if) Suppose X is decomposable and arc-like. Since X is decomposable, arc-like, and circle-like, it follows from [10, Theorem 5.1] that $\mu^{-1}(t)$ is not circle-like for some $t<1$. This is a contradiction.

(if) If X is circle-like and not arc-like, then $\mu^{-1}(t)$ is circle-like for each $t<1$ by [10, Theorem 4.7]. If X is indecomposable and arc-like, then by Corollary 3.2 $\mu^{-1}(t)$ is indecomposable and arc-like for each $t<1$. Burgess [1] proved that such continua are circle-like.

4. Whitney continua of certain irreducible continua

In this section we establish two classes of irreducible continua which have the property that $\mu^{-1}(t)$ is an arc for t sufficiently close to 1. It is also shown that when $\mu^{-1}(t)$ is an arc, $\mu^{-1}([t,1])$ is actually homeomorphic to the cone over an arc.

Let X be irreducible between a pair of points a and b. A decomposition $\mathcal{D}$ of X is said to be *admissible* if each element of $\mathcal{D}$ is a nonvoid proper subcontinuum of X, and each element of $\mathcal{D}$ which does not contain a or b separates X. It is known [3] that $X/\mathcal{D}$ is an arc whenever $\mathcal{D}$ is an admissible decomposition of X.

X is of *type* A provided that X is irreducible and has an admissible decomposition; X is of *type* A' if X is of type A and has an admissible decomposition each of whose elements has void interior. X is said to be *hereditarily of type* A' if every nondegenerate subcontinuum of X is of type A'. The reader is referred to [3] and [12] for general results concerning continua of type A. For example, an irreducible continuum X is of type A' if and only if each subcontinuum of X with nonvoid interior is decomposable ([3, Theorem 2.7] or [12, Theorem 10, p. 15]). It is also known that X is hereditarily of type A' if and only if X is arc-like and hereditarily decomposable [12, Theorem 13, pg. 50].

Theorem **4.1.** *If* X *is hereditarily of type* A', *then there exists* $t_o < 1$ *such that* $\mu^{-1}(t)$ *is an arc whenever* $t_o \leq t < 1$.

Proof. Let a and b be points in X such that X is irreducible between a and b, and let $\mathcal{D} = \{D(x)\}$ be an admissible decomposition of X each of whose elements has void interior. Let $t_o = \text{lub}\{\mu(D(x)):D(x) \in \mathcal{D}\}$. Clearly, $t_o < 1$. It follows from [3, Theorem 2.5] that $D(a) = \{x \in X:X$ is irreducible between x and b$\}$ and $D(b) = \{x \in X:X$ is irreducible between a and x$\}$. If $t_o \leq t < 1$, it will be shown that there exists a unique $A \in \mu^{-1}(t)$ such that $D(a) \cap A \neq \phi$. It is easy to see that there exists some $A \in \mu^{-1}(t)$ such that $D(a) \cap A \neq \phi$. To prove uniqueness, suppose there exists $P \in \mu^{-1}(t)$ with $D(a) \cap P \neq \phi$ and $A \neq P$. Since $D(a) = \{x \in X:X$ is irreducible between x and b$\}$, it follows that $D(a) \subseteq A$ and $D(a) \subseteq P$. Since $A \neq P$, pick $x \in A-P$ and $y \in P-A$. It follows that $x,y \notin D(a)$. Thus, let A' be a proper subcontinuum of X containing both x and b, and let P' be a proper subcontinuum of X containing both y and b. Since $A' \cup P'$ is a subcontinuum of X containing x and y but not a, A contains a and x but not y, and P contains a and y but not x, it follows that a,x,y are three points no one of which cuts between the other two. This is a contradiction to [3, Theorem 5.3]. Hence, A is unique and in a similar way there exists a unique $B \in \mu^{-1}(t)$ such that $D(b) \cap B \neq \phi$.

It will now be shown that if $M \in \mu^{-1}(t)$ with $A \neq M \neq B$, then M separates $\mu^{-1}(t)$. To apply Theorem 2.1 we must first show that M separates X. To this end it will be shown that there exists $x_o \in X$ such that $D(x_o) \subseteq M$, and it will then follow that M separates X since $a,b \notin M$. Suppose on the contrary that for each $x \in X$, $D(x) \not\subseteq M$. Since $\mu(M) \geq t_o$, there exist $x_1,x_2 \in M$ such that $D(x_1)$ and $D(x_2)$ are distinct elements of $\mathcal{D}$. It now follows from [3, Theorem 2.3] that there exists $x_o \in M$ such that $D(x_o) \subseteq M$. Since M separates X, let $X-M = X_1 \cup X_2$ be a separation and suppose there exists $N \in \mu^{-1}(t)$ such that $N \not\subseteq X_1 \cup M$ and $N \not\subseteq X_2 \cup M$. Pick $x \in X_1 \cap N$, $y \in X_2 \cap N$, and $z \in M-N$. It can be seen that no one of x,y,z cuts between the other two which contradicts [3, Theorem 5.3]. Therefore, M separates X in the way required by Theorem 2.1 and thus M separates $\mu^{-1}(t)$. It has been shown that $\mu^{-1}(t)$ contains at most two non-separating points A and B, and hence, $\mu^{-1}(t)$ is an arc.

Notation. Let X be a continuum of type A and let $\mathcal{D} = \{D(x)\}$ be an admissible decomposition of X. The following definitions of t_o, t_1, and t_2 will be used in Theorem 4.2:

$$t_o = \text{lub}\{\mu(D(x)) : D(x) \in \mathcal{D}\},$$

$t_1 = \text{lub}\{\mu(Y): Y \in C(X)$ and there exists $D(x) \in \mathcal{D}$ such that $D(x) \not\subseteq Y$ and $Y \cap D(x) \neq \phi \neq Y \cap (X-D(x))\}$; and,

$t_2 = \max\{t_o, t_1\}$.

Note that t_2 might not be less than 1. The continuum pictured in Figure 1 is a continuum of type A' such that t_2 is not less than 1. This continuum also has the property that for all t, $\mu^{-1}(t)$ is not an arc. If this continuum is modified in the obvious way so that it contains only finitely many circles, then it would be a continuum of type A' such that $t_2 < 1$. Neither of these continua is hereditarily of type A'. Another example of a continuum of type A' such that $t_2 < 1$ is a simple triod with a half ray spiraling down on it.

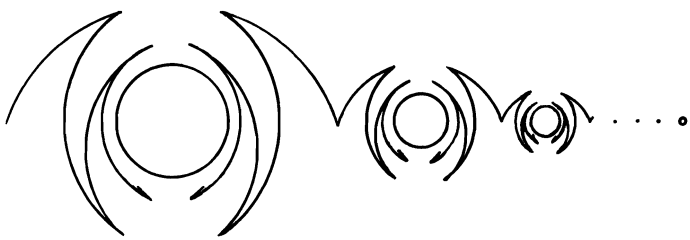

Figure 1

Theorem 4.2. *If* X *is a continuum of type* A *and* $t_2 < t < 1$, *then* $\mu^{-1}(t)$ *is an arc.*

Proof. Let a and b be points in X such that X is irreducible between a and b, let $\mathcal{D} = \{D(x)\}$ be an admissible decomposition of X, and let t be such that $t_2 < t < 1$. It will first be shown that there exists a unique $A \in \mu^{-1}(t)$ such that $a \in A$. It is easy to see that there exists some $A \in \mu^{-1}(t)$ such that $a \in A$. To prove uniqueness, suppose there exists $P \in \mu^{-1}(t)$ with $a \in P$ and $A \neq P$. Since $A \not\subseteq P$ and $P \not\subseteq A$, there exist $x \in A-P$ and $y \in P-A$. Since $t_2 < t$, $D(x) \subseteq A-P$. Let $X-D(x) = S \cup T$ be a separation and assume $P \subseteq S$. Since $a \in P$, $a \in S$ and $b \in T$. Because $D(x) \cup T$

is a continuum, so is $A \cup T$. But $a,b \in A \cup T$ and $y \in X-(A \cup T)$ which contradicts the fact that X is irreducible between a and b. Thus A is unique, and similarly there exists a unique $B \in \mu^{-1}(t)$ such that $b \in B$. It will now be shown that if $M \in \mu^{-1}(t)$ such that $A \neq M \neq B$, then M separates $\mu^{-1}(t)$. Pick $x \in M$. Then since $a,b \notin M$, $D(x) \subseteq M$, and $D(x)$ separates X, it follows that M separates X. Let $X-M = X_1 \cup X_2$ be a separation with $a \in X_1$ and $b \in X_2$. To apply Theorem 2.1 we must show that if $N \in \mu^{-1}(t)$, then either $N \subseteq X_1 \cup M$ or $N \subseteq X_2 \cup M$. Suppose on the contrary that there exists $N \in \mu^{-1}(t)$ such that $N \not\subseteq X_1 \cup M$ and $N \not\subseteq X_2 \cup M$. It follows that $X_1 \cap N \neq \phi \neq X_2 \cap N$ and $M-(X_1 \cup N \cup X_2) \neq \phi$. Pick $x_1 \in X_1 \cap N$ and $x_2 \in X_2 \cap N$ such that $D(x_1)$ and $D(x_2)$ separate X. Let $X-D(x_1) = S_1 \cup T_1$ and $X-D(x_2) = S_2 \cup T_2$ be separations with $a \in S_1 \cap S_2$ and $b \in T_1 \cap T_2$. It follows that $S_1 \cup D(x_1) \cup N \cup D(x_2) \cup T_2$ is a proper subcontinuum of X containing a and b, which contradicts the fact that X is irreducible between a and b. It has been shown that $\mu^{-1}(t)$ contains at most two non-separating points A and B, and hence, $\mu^{-1}(t)$ is an arc.

In [4] Kelley defined the function $\sigma:C(C(X)) \to C(X)$ by $\sigma(\mathfrak{M}) = \cup(\mathfrak{M})$ for each subcontinuum $\mathfrak{M}$ of $C(X)$. He showed that σ is a continuous function. The restriction of σ to $C(\mu^{-1}(t))$, is denoted σ_t. Krasinkiewicz [6] showed that σ_t is a function from $C(\mu^{-1}(t))$ onto $\mu^{-1}([t,1])$. In the next theorem it is shown that σ_t is also one-to-one whenever $\mu^{-1}(t)$ is an arc; hence in this case $\mu^{-1}([t,1])$ is a two cell.

Theorem 4.3. *If* $\mu^{-1}(t)$ *is an arc, then* σ_t *is one-to-one and hence,* $\mu^{-1}([t,1])$ *is homeomorphic to the cone over an arc.*

Proof. Let $\mathcal{H}$ and $\mathcal{K}$ be distinct subcontinua of $\mu^{-1}(t)$. Assume there exists $A \in \mathcal{H}-\mathcal{K}$. Then there exists a separating point M of $\mu^{-1}(t)$ such that $A \neq M$ and M separates A from $\mathcal{K}$ in

$\mu^{-1}(t)$. Let $\mu^{-1}(t)-\{M\} = \mathcal{S}_1 \cup \mathcal{S}_2$ be a separation with $A \in \mathcal{S}_1$ and $\mathcal{K} \subseteq \mathcal{S}_2$. Let

$$X_1 = \bigcup\{N \in \mu^{-1}(t) : N \in \mathcal{S}_1\} - M \text{ and}$$

$$X_2 = \bigcup\{N \in \mu^{-1}(t) : N \in \mathcal{S}_2\} - M.$$

From the proof of Theorem 2.1, it follows that $X_1 \cup X_2$ is a separation of X-M. Clearly, $\bigcup(\mathcal{K}) \subseteq X_2 \cup M$ and $A \cap X_1 \neq \phi$, so $\bigcup(\mathcal{H}) \neq \bigcup(\mathcal{K})$ and $\sigma(\mathcal{H}) \neq \sigma(\mathcal{K})$. Hence, σ_t is a homeomorphism of $C(\mu^{-1}(t))$ onto $\mu^{-1}([t,1])$. Since $\mu^{-1}(t)$ is an arc, $C(\mu^{-1}(t))$ is homeomorphic to the cone over an arc and thus, $\mu^{-1}([t,1])$ is homeomorphic to the cone over an arc.

Corollary 4.4. *If* X *is arc-like and hereditarily decomposable, then for some* $t < 1$, $\mu^{-1}([t,1])$ *is a two cell.*

Remark. In a recent preprint [7] J. Krasinkiewicz and Sam B. Nadler, Jr. have proven Corollary 3.2 and have shown that if X is arc-like and decomposable, then there exists $t_o < 1$ such that $\mu^{-1}(t)$ is an arc whenever $t_o \leq t < 1$. Since continua hereditarily of type A' are arc-like and hereditarily decomposable, Theorem 4.1 follows immediately from their results.

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