SOME PROPERTIES OF WHITNEY CONTINUAS IN THE HYPERSPACE $C(X)$

by

C. BRUCE HUGHES
SOME PROPERTIES OF WHITNEY CONTINUA IN THE HYPERSPACE $C(X)$

C. Bruce Hughes

1. Introduction

Let $X$ denote a continuum (i.e., a compact, connected, non-void, metric space). The hyperspace of subcontinua of $X$, denoted $C(X)$, is the space of all subcontinua of $X$ endowed with the Hausdorff metric (e.g., [4]). A Whitney map on $C(X)$ is a continuous function $\mu : C(X) \to [0, 1]$ satisfying the following properties:

(i) $\mu(\{x\}) = 0$ for each $x \in X$,
(ii) $\mu(X) = 1$, and
(iii) if $A \subseteq B$ and $A \neq B$, then $\mu(A) < \mu(B)$.

Whitney [13] has shown that such functions always exist. Throughout this paper, $\mu$ will stand for an arbitrary Whitney map on $C(X)$. It is known [2] that $\mu$ is monotone; that is, $\mu^{-1}(t)$ is a subcontinuum of $C(X)$ for each $t$. The continua $\mu^{-1}(t)$ are called the Whitney continua of $X$.

In Section 2 we characterize the separating points of $\mu^{-1}(t)$ in terms of their separating properties as subcontinua of $X$. The rest of the paper contains applications of this result. In Section 3 we obtain some information about the Whitney continua of arc-like and circle-like continua. Section 4 establishes classes of continua which have the property that $\mu^{-1}(t)$ is an arc for $t$ sufficiently close to 1.

The author would like to express his appreciation to G. R. Gordh, Jr. for many lengthy discussions and helpful comments about the contents of this paper.

2. Separating points in $\mu^{-1}(t)$

If $G_1, G_2, \ldots, G_n$ are open subsets of $X$, then $N(G_1, \ldots, G_n)$
denotes the set of all points $A$ in $C(X)$ such that $A \subseteq U \{G_i : i = 1, 2, \ldots, n\}$ and $A \cap G_i \neq \emptyset$ for each $i \leq n$. Recall that the collection of all such subsets of $C(X)$ forms a basis for the Vietoris finite topology on $C(X)$. It is well known that the Hausdorff metric and the Vietoris finite topology agree on $C(X)$ (e.g., [81]).

If $t \in [0,1]$ and $x \in X$, then let $C^t_x = \{A \in \mu^{-1}(t) : x \in A\}$. Rogers [10, Theorem 4.2] has shown that $C^t_x$ is an arcwise connected subcontinuum of $C(X)$.

**Theorem 2.1.** Let $A$ be an element of $C(X)$ with $\mu(A) = t$. Then $A$ separates $\mu^{-1}(t)$ if and only if there exists a separation $X-A = X_1 U X_2$ such that for any $B \in \mu^{-1}(t)$ either $B \subseteq X_1 U A$ or $B \subseteq X_2 U A$.

**Proof.** (only if) Let $\mu^{-1}(t) - \{A\} = \bar{S}_1 U \bar{S}_2$ be a separation. Let

$$X_1 = U \{B \in \mu^{-1}(t) : B \in \bar{S}_1\} - A \text{ and } X_2 = U \{B \in \mu^{-1}(t) : B \in \bar{S}_2\} - A.$$  

For each $p \in X$ there exists $P \in \mu^{-1}(t)$ with $p \in P$, thus $X-A = X_1 U X_2$. To show $X_1 \cap X_2 = \emptyset$ suppose on the contrary that $x \in X_1 \cap X_2$. Because $x \not\in A$, it follows that $C^t_x \subseteq \bar{S}_1 U \bar{S}_2$. Since $x \in X_1 \cap X_2$, there exists $B_1 \in \bar{S}_1$ and $B_2 \in \bar{S}_2$ such that $x \in B_1$ and $x \in B_2$. The fact that $B_1$ and $B_2$ are in $C^t_x$ implies $C^t_x \cap \bar{S}_1 \neq \emptyset \neq C^t_x \cap \bar{S}_2$. This contradicts the fact that $\bar{S}_1$ and $\bar{S}_2$ are separated because $C^t_x$ is a continuum. To show that $X_1$ and $X_2$ are separated, by symmetry it suffices to show that no convergent sequence of points in $X_1$ converges to a point in $X_2$. To this end suppose $\{p_n\}$ is a sequence of points in $X_1$ which converges to some $p \in X$. For each $n$, choose $p_n \in \bar{S}_1$ such that $p_n \in P_n$. If $P$ denotes the limit of a convergent subsequence of $\{p_n\}$, then $p \in P$. Since $\mu^{-1}(t)$ is a subcontinuum of $C(X)$ and $\bar{S}_1$ and $\bar{S}_2$ are separated, it follows that $p \in \bar{S}_1 U \{A\}$. Hence,
p \in P \subseteq X_1 \cup A \text{ and } X-A = X_1 \cup X_2 \text{ is a separation. Finally, }

\text{suppose } B \in \mathcal{U}^{-1}(t) \text{ and that } B \in S_1. \text{ Then } B \subseteq \bigcup \{M \in \mathcal{U}^{-1}(t): M \in S_1\} \subseteq X_1 \cup A. \text{ Hence, for any } B \in \mathcal{U}^{-1}(t) \text{ either } B \subseteq X_1 \cup A \text{ or } B \subseteq X_2 \cup A.

\text{(if) Let } J_1 = \{B \in \mathcal{U}^{-1}(t): B \subseteq X_1 \cup A, B \neq A\} \text{ and }

J_2 = \{B \in \mathcal{U}^{-1}(t): B \subseteq X_2 \cup A, B \neq A\}.

\text{To see that } \mathcal{U}^{-1}(t) - \{A\} = J_1 \cup J_2 \text{ is a separation, note that }

N(X_1, X) \text{ and } N(X_2, X) \text{ are open subsets of } C(X) \text{ such that }

J_1 = N(X_1, X) \cap \mathcal{U}^{-1}(t) \text{ and } J_2 = N(X_2, X) \cap \mathcal{U}^{-1}(t).

\text{Using Theorem 2.1 we obtain a simple proof of the following well known result originally due to Krasinkiewicz [5] (see also [9], [10]).}

\text{Corollary 2.2. If } X \text{ is an arc, then } \mathcal{U}^{-1}(t) \text{ is an arc for each } t < 1.

\text{Proof. Let } p \text{ and } q \text{ be the non-separating points of } X. \text{ If } t < 1, \text{ then it is easily seen that there exist exactly one sub-continuum } P \text{ of } X \text{ and one subcontinuum } Q \text{ of } X \text{ such that } p \in P \text{ and } q \in Q \text{ and } P, Q \in \mathcal{U}^{-1}(t). \text{ If } A \in \mathcal{U}^{-1}(t) \text{ such that } P \neq A \neq Q, \text{ then } A \text{ separates } X \text{ in the way required by Theorem 2.1. Thus, } A \text{ separates } \mathcal{U}^{-1}(t) \text{ and } \mathcal{U}^{-1}(t) \text{ has exactly two non-separating points. It follows that } \mathcal{U}^{-1}(t) \text{ is an arc.}

\text{Example 2.3. Let } X \text{ be a simple triod (i.e., a continuum homeomorphic to the capital letter T). Let } Y \text{ be a proper sub-continuum of } X \text{ which is also a simple triod and which separates } X. \text{ Let } \nu(Y) = t. \text{ Then } Y \text{ does not separate } X \text{ in the way required by Theorem 2.1 and thus } Y \text{ does not separate } \mathcal{U}^{-1}(t).

3. Whitney continua of arc-like and circle-like continua

In this section we give sufficient conditions on \( \mathcal{U}^{-1}(t) \) to insure that \( X \) be decomposable. Information about the Whitney continua of arc-like and circle-like continua is obtained in
the corollaries. Corollary 3.2 answers a question of J. T. Rogers, Jr. [10]. The proofs of Corollaries 3.3 and 3.4 were pointed out to the author by G. R. Gordh, Jr.

**Theorem 3.1.** If \( \mu^{-1}(t) \) is irreducible and decomposable for some \( t < 1 \), then \( X \) is decomposable.

**Proof.** Let \( A \) and \( B \) be points in \( \mu^{-1}(t) \) such that \( \mu^{-1}(t) \) is irreducible from \( A \) to \( B \). Let \( \mathcal{S} \) and \( \mathcal{T} \) be proper subcontinua of \( \mu^{-1}(t) \) with \( A \in \mathcal{S} \) and \( B \in \mathcal{T} \) such that \( \mu^{-1}(t) = \mathcal{S} \cup \mathcal{T} \). From [4, Lemma 1.1] it follows that \( \mathcal{U} \mathcal{S} \) and \( \mathcal{U} \mathcal{T} \) are subcontinua of \( X \). It is clear that \( X = (\mathcal{U} \mathcal{S}) \cup (\mathcal{U} \mathcal{T}) \), so if \( \mathcal{U} \mathcal{S} \) and \( \mathcal{U} \mathcal{T} \) are proper subcontinua of \( X \), then the theorem is proved. Assume for the purpose of this proof that \( X = \mathcal{U} \mathcal{T} \). Then \( A \subseteq \mathcal{U} \mathcal{T} \) so there exists \( M \in \mathcal{T} \) such that \( A \cap M \neq \emptyset \). This implies ([9] or [10]) that there is an arc \( \mathcal{J} \) in \( \mu^{-1}(t) \) with endpoints \( A \) and \( M \). By the irreducibility of \( \mu^{-1}(t) \), we have \( \mathcal{S} \mathcal{-} \mathcal{J} \subseteq \mathcal{J} \). It follows that a point \( N \) in \( \mu^{-1}(t) \) can be choosen in \( \mathcal{S} \mathcal{-} \mathcal{J} \) such that \( N \) is different from \( A \) and \( N \) separates \( \mu^{-1}(t) \). From Theorem 2.1, \( N \) is a subcontinuum of \( X \) which separates \( X \) and hence, \( X \) must be decomposable.

A continuum \( X \) is said to be arc-like if for each positive number \( \varepsilon \), there is an \( \varepsilon \)-map (i.e., a map having point-inverses of diameter less than \( \varepsilon \)) of \( X \) onto an arc. Circle-like continua are defined in the same manner.

**Corollary 3.2.** If \( X \) is indecomposable and arc-like, then \( \mu^{-1}(t) \) is indecomposable and arc-like for each \( t < 1 \).

**Proof.** Krasinkiewicz [5] has shown that \( \mu^{-1}(t) \) must be arc-like for each \( t < 1 \). Since arc-like continua are unicoherent and are not triods, it follows from [11] that \( \mu^{-1}(t) \) is irreducible for each \( t < 1 \). If \( \mu^{-1}(t) \) were decomposable for some \( t < 1 \), then by Theorem 3.1 \( X \) would be decomposable also. Thus, \( \mu^{-1}(t) \)
is indecomposable and arc-like for each t < 1.

**Corollary 3.3.** Let X be arc-like and circle-like. Then \( \mu^{-1}(t) \) is arc-like and circle-like for each t < 1 if and only if X is indecomposable.

**Proof.** (only if) Suppose X is arc-like, circle-like and decomposable. Rogers [10, Theorem 5.1] has shown that there exists t < 1 such that \( \mu^{-1}(t) \) is not circle-like. This is a contradiction.

(if) Since X is indecomposable and arc-like, it follows from Corollary 3.2 that \( \mu^{-1}(t) \) is indecomposable and arc-like for each t < 1. Burgess [1] has shown that such continua must also be circle-like.

**Corollary 3.4.** Let X be circle-like. Then \( \mu^{-1}(t) \) is circle-like for each t < 1 if and only if X is indecomposable or X is not arc-like.

**Proof.** (only if) Suppose X is decomposable and arc-like. Since X is decomposable, arc-like, and circle-like, it follows from [10, Theorem 5.1] that \( \mu^{-1}(t) \) is not circle-like for some t < 1. This is a contradiction.

(if) If X is circle-like and not arc-like, then \( \mu^{-1}(t) \) is circle-like for each t < 1 by [10, Theorem 4.7]. If X is indecomposable and arc-like, then by Corollary 3.2 \( \mu^{-1}(t) \) is indecomposable and arc-like for each t < 1. Burgess [1] proved that such continua are circle-like.

4. **Whitney continua of certain irreducible continua**

In this section we establish two classes of irreducible continua which have the property that \( \mu^{-1}(t) \) is an arc for t sufficiently close to 1. It is also shown that when \( \mu^{-1}(t) \) is an arc, \( \mu^{-1}([t,1]) \) is actually homeomorphic to the cone over an arc.
Let $X$ be irreducible between a pair of points $a$ and $b$. A decomposition $\mathcal{D}$ of $X$ is said to be *admissible* if each element of $\mathcal{D}$ is a nonvoid proper subcontinuum of $X$, and each element of $\mathcal{D}$ which does not contain $a$ or $b$ separates $X$. It is known [3] that $X/\mathcal{D}$ is an arc whenever $\mathcal{D}$ is an admissible decomposition of $X$.

$X$ is of *type A* provided that $X$ is irreducible and has an admissible decomposition; $X$ is of *type A' if $X$ is of type A and has an admissible decomposition each of whose elements has void interior. $X$ is said to be *hereditarily of type A'* if every nondegenerate subcontinuum of $X$ is of type A'. The reader is referred to [3] and [12] for general results concerning continua of type A. For example, an irreducible continuum $X$ is of type A' if and only if each subcontinuum of $X$ with nonvoid interior is decomposable ([3, Theorem 2.7] or [12, Theorem 10, p. 15]). It is also known that $X$ is hereditarily of type A' if and only if $X$ is arc-like and hereditarily decomposable [12, Theorem 13, pg. 50].

**Theorem 4.1.** If $X$ is hereditarily of type A', then there exists $t_0 < 1$ such that $\mu^{-1}(t)$ is an arc whenever $t_0 < t < 1$.

**Proof.** Let $a$ and $b$ be points in $X$ such that $X$ is irreducible between $a$ and $b$, and let $\mathcal{D} = \{D(x)\}$ be an admissible decomposition of $X$ each of whose elements has void interior. Let $t_0 = \text{lub}\{u(D(x)) : D(x) \in \mathcal{D}\}$. Clearly, $t_0 < 1$. It follows from [3, Theorem 2.5] that $D(a) = \{x \in X : X$ is irreducible between $x$ and $b\}$ and $D(b) = \{x \in X : X$ is irreducible between $a$ and $x\}$. If $t_0 < t < 1$, it will be shown that there exists a unique $A \in \mu^{-1}(t)$ such that $D(a) \cap A \neq \emptyset$. It is easy to see that there exists some $A \in \mu^{-1}(t)$ such that $D(a) \cap A \neq \emptyset$. To prove uniqueness, suppose there exists $P \in \mu^{-1}(t)$ with $D(a) \cap P \neq \emptyset$ and $A \neq P$. Since $D(a) = \{x \in X : X$ is irreducible between $x$ and $b\}$,
it follows that $D(a) \subseteq A$ and $D(a) \subseteq P$. Since $A \neq P$, pick $x \in A-P$ and $y \in P-A$. It follows that $x, y \notin D(a)$. Thus, let $A'$ be a proper subcontinuum of $X$ containing both $x$ and $b$, and let $P'$ be a proper subcontinuum of $X$ containing both $y$ and $b$. Since $A' \cup P'$ is a subcontinuum of $X$ containing $x$ and $y$ but not $a$, $A$ contains $a$ and $x$ but not $y$, and $P$ contains $a$ and $y$ but not $x$, it follows that $a, x, y$ are three points no one of which cuts between the other two. This is a contradiction to [3, Theorem 5.3]. Hence, $A$ is unique and in a similar way there exists a unique $B \in \mu^{-1}(t)$ such that $D(b) \cap B \neq \emptyset$.

It will now be shown that if $M \in \mu^{-1}(t)$ with $A \neq M \neq B$, then $M$ separates $\mu^{-1}(t)$. To apply Theorem 2.1 we must first show that $M$ separates $X$. To this end it will be shown that there exists $x_0 \in X$ such that $D(x_0) \subseteq M$, and it will then follow that $M$ separates $X$ since $a, b \notin M$. Suppose on the contrary that for each $x \in X$, $D(x) \not\subseteq M$. Since $\mu(M) > t_o$, there exist $x_1, x_2 \in M$ such that $D(x_1)$ and $D(x_2)$ are distinct elements of $\mathcal{D}$. It now follows from [3, Theorem 2.3] that there exists $x_0 \in M$ such that $D(x_0) \subseteq M$. Since $M$ separates $X$, let $X-M = X_1 \cup X_2$ be a separation and suppose there exists $N \in \mu^{-1}(t)$ such that $N \not\subseteq X_1 \cup M$ and $N \not\subseteq X_2 \cup M$. Pick $x \in X_1 \cap N$, $y \in X_2 \cap N$, and $z \in M-N$. It can be seen that no one of $x, y, z$ cuts between the other two which contradicts [3, Theorem 5.3]. Therefore, $M$ separates $X$ in the way required by Theorem 2.1 and thus $M$ separates $\mu^{-1}(t)$. It has been shown that $\mu^{-1}(t)$ contains at most two non-separating points $A$ and $B$, and hence, $\mu^{-1}(t)$ is an arc.

Notation. Let $X$ be a continuum of type $A$ and let $\mathcal{D} = \{D(x)\}$ be an admissible decomposition of $X$. The following definitions of $t_o, t_1$, and $t_2$ will be used in Theorem 4.2:

$$t_o = \text{lub}\{\mu(D(x)) : D(x) \in \mathcal{D}\},$$
$t_1 = \text{lub}\{\mu(Y) : Y \in C(X) \text{ and there exists } D(x) \subseteq \mathcal{D} \text{ such that } D(x) \nsubseteq Y \text{ and } Y \cap D(x) \neq \emptyset \neq Y \cap (X - D(x))\}$;

and,

$t_2 = \max\{t_0, t_1\}$.

Note that $t_2$ might not be less than 1. The continuum pictured in Figure 1 is a continuum of type A' such that $t_2$ is not less than 1. This continuum also has the property that for all $t$, $\mu^{-1}(t)$ is not an arc. If this continuum is modified in the obvious way so that it contains only finitely many circles, then it would be a continuum of type A' such that $t_2 < 1$. Neither of these continua is hereditarily of type A'. Another example of a continuum of type A' such that $t < 1$ is a simple triod with a half ray spiraling down on it.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{continuum.png}
\caption{Figure 1}
\end{figure}

**Theorem 4.2.** If $X$ is a continuum of type A and $t_2 < t < 1$, then $\mu^{-1}(t)$ is an arc.

**Proof.** Let $a$ and $b$ be points in $X$ such that $X$ is irreducible between $a$ and $b$, let $\mathcal{D} = \{D(x)\}$ be an admissible decomposition of $X$, and let $t$ be such that $t_2 < t < 1$. It will first be shown that there exists a unique $A \in \mu^{-1}(t)$ such that $a \in A$. It is easy to see that there exists some $A \in \mu^{-1}(t)$ such that $a \in A$. To prove uniqueness, suppose there exists $P \in \mu^{-1}(t)$ with $a \in P$ and $A \neq P$. Since $A \nsubseteq P$ and $P \nsubseteq A$, there exist $x \in A - P$ and $y \in P - A$. Since $t_2 < t$, $D(x) \subseteq A - P$. Let $X - D(x) = S \cup T$ be a separation and assume $P \subseteq S$. Since $a \in P$, $a \in S$ and $b \in T$. Because $D(x) \cup T$
is a continuum, so is $A \cup T$. But $a, b \in A \cup T$ and $y \in X-(A \cup T)$ which contradicts the fact that $X$ is irreducible between $a$ and $b$. Thus $A$ is unique, and similarly there exists a unique $B \in \mu^{-1}(t)$ such that $b \in B$. It will now be shown that if $M \in \mu^{-1}(t)$ such that $A \neq M \neq B$, then $M$ separates $\mu^{-1}(t)$.

Pick $x \in M$. Then since $a, b \notin M$, $D(x) \subseteq M$, and $D(x)$ separates $X$, it follows that $M$ separates $X$. Let $X-M = X_1 \cup X_2$ be a separation with $a \in X_1$ and $b \in X_2$. To apply Theorem 2.1 we must show that if $N \in \mu^{-1}(t)$, then either $N \subseteq X_1 \cup M$ or $N \subseteq X_2 \cup M$.

Suppose on the contrary that there exists $N \in \mu^{-1}(t)$ such that $N \not\subseteq X_1 \cup M$ and $N \not\subseteq X_2 \cup M$. It follows that $X_1 \cap N \neq \emptyset \neq X_2 \cap N$ and $M-(X_1 \cup N \cup X_2) \neq \emptyset$. Pick $x_1 \in X_1 \cap N$ and $x_2 \in X_2 \cap N$ such that $D(x_1)$ and $D(x_2)$ separate $X$. Let $X-D(x_1) = S_1 \cup T_1$ and $X-D(x_2) = S_2 \cup T_2$ be separations with $a \in S_1 \cap S_2$ and $b \in T_1 \cap T_2$.

It follows that $S_1 \cup D(x_1) \cup N \cup D(x_2) \cup T_2$ is a proper subcontinuum of $X$ containing $a$ and $b$, which contradicts the fact that $X$ is irreducible between $a$ and $b$. It has been shown that $\mu^{-1}(t)$ contains at most two non-separating points $A$ and $B$, and hence, $\mu^{-1}(t)$ is an arc.

In [4] Kelley defined the function $\sigma: C(C(X)) \rightarrow C(X)$ by $\sigma(\mathcal{M}) = U(\mathcal{M})$ for each subcontinuum $\mathcal{M}$ of $C(X)$. He showed that $\sigma$ is a continuous function. The restriction of $\sigma$ to $C(\mu^{-1}(t))$, is denoted $\sigma_t$. Krasinkiewicz [6] showed that $\sigma_t$ is a function from $C(\mu^{-1}(t))$ onto $\mu^{-1}([t,1])$. In the next theorem it is shown that $\sigma_t$ is also one-to-one whenever $\mu^{-1}(t)$ is an arc; hence in this case $\mu^{-1}([t,1])$ is a two cell.

Theorem 4.3. If $\mu^{-1}(t)$ is an arc, then $\sigma_t$ is one-to-one and hence, $\mu^{-1}([t,1])$ is homeomorphic to the cone over an arc.

Proof. Let $\mathcal{K}$ and $\mathcal{K}'$ be distinct subcontinua of $\mu^{-1}(t)$. Assume there exists $A \in \mathcal{K}'-\mathcal{K}$. Then there exists a separating point $M$ of $\mu^{-1}(t)$ such that $A \neq M$ and $M$ separates $A$ from $\mathcal{K}$ in
Let $\mu^{-1}(t) - \{M\} = \mathcal{S}_1 \cup \mathcal{S}_2$ be a separation with $A \in \mathcal{S}_1$ and $\mathcal{K} \subseteq \mathcal{S}_2$. Let

$$X_1 = \bigcup \{N \in \mu^{-1}(t) : N \in \mathcal{S}_1\} - M$$

and

$$X_2 = \bigcup \{N \in \mu^{-1}(t) : N \in \mathcal{S}_2\} - M.$$

From the proof of Theorem 2.1, it follows that $X_1 \cup X_2$ is a separation of $X - M$. Clearly, $\bigcup (\mathcal{K}) \subseteq X_2 \cup M$ and $A \cap X_1 \neq \emptyset$, so

$$\bigcup (\mathcal{K}) \neq \bigcup (\mathcal{K})$$

and $\sigma(\mathcal{K}) \neq \sigma(\mathcal{K})$. Hence, $\sigma_t$ is a homeomorphism of $C(\mu^{-1}(t))$ onto $\mu^{-1}([t,1])$. Since $\mu^{-1}(t)$ is an arc, $C(\mu^{-1}(t))$ is homeomorphic to the cone over an arc and thus, $\mu^{-1}([t,1])$ is homeomorphic to the cone over an arc.

**Corollary 4.4.** If $X$ is arc-like and hereditarily decomposable, then for some $t < 1$, $\mu^{-1}([t,1])$ is a two cell.

**Remark.** In a recent preprint [7] J. Krasinkiewicz and Sam B. Nadler, Jr. have proven Corollary 3.2 and have shown that if $X$ is arc-like and decomposable, then there exists $t_0 < 1$ such that $\mu^{-1}(t)$ is an arc whenever $t_0 < t < 1$. Since continua hereditarily of type A' are arc-like and hereditarily decomposable, Theorem 4.1 follows immediately from their results.

**References**


University of Kentucky
Lexington, Kentucky 40506.